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Research article

The existence of sign-changing solutions for Schrödinger-Kirchhoff problems in $\mathbb{R}^3$

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Abstract: In this paper, we consider the following Kirchhoff-type equation:

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^3,$$

where a, $b > 0$, $p \in (1, 5)$. By considering a minimization problem on a special constraint set, we prove that the above problem has at least one sign-changing solution for any $p \in (1, 5)$. Our results (especially $p \in (1, 3]$) can be regarded as an improvement on the existing results.

Keywords: Kirchhoff type equation; sign-changing solution; constraint variational method; Pohožaev identity
Mathematics Subject Classification: 35J20, 35J60

1. Introduction

In this paper, we study the existence of sign-changing solution to the following Kirchhoff equation by using a direct method

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where a, $b > 0$, $p \in (1, 5)$. In recent years, problem (1.1) has been extensively researched by many mathematicians. Therefore, there are a large number of results for the existence of nontrivial solutions, positive solutions, ground state solutions, sign-changing solutions, nodal solutions for problem (1.1). Please see [1–6] and the references therein. It is worth noting that Chen, Fu and Wu [4] established the existence of a positive ground state solution to problem (1.1) for any $b > 0$ and $p \in (1, 5)$. However, there is a question: whether problem (1.1) has sign-changing solutions for any $p \in (1, 5)$?

Recently, Wang, Zhang and Cheng [7] established the existence results of sign-changing solutions to the following problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u=f(u),\quad \text{in } \mathbb{R}^3, \tag{1.2}$$

where $f(t)$ satisfies the following crucial conditions:

(f1) $\lim_{t\to\infty}\frac{F(t)}{t^4}=\infty$, where $F(t)=\int_0^t f(s)ds$;

(f2) $\frac{f(t)}{t^3}$ is nondecreasing for $|t|>0$.

Obviously, when $p\in(1,3]$, $f(t)=|t|^{p-1}t$ does not satisfy (f1) and (f2). Qian [8] researched the existence of a ground state sign-changing solution to the following problem

$$\begin{cases} -(a-\lambda\int_\Omega|\nabla u|^2dx)\Delta u=|u|^{q-2}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where a is a positive constant, $q\in(2,2^*)(2^*=+\infty$ for $N=1,2$, $2^*=\frac{2N}{N-2}$ for $N\geq 3)$, $\Omega\subset\mathbb{R}^3$ is a bounded domain and $\lambda>0$ is a parameter. They mainly obtained that problem (1.3) has at least one sign-changing solution for small enough λ, thanks to truncated technique and constraint variational method. Besides, some similar problems have also been extensively researched. For more relevant results, please refer to [9, 10] and the references therein.

Motivated by the above mentioned results, our result is given in the following.

Theorem 1.1 *For any a, $b>0$ and $p\in(1,5)$, problem* (1.1) *has at least one sign-changing solution.*

Remark 1.2 When $p\in(3,5)$, the existence of one sign-changing solution to (1.1) is obtained by [7]. But when $p\in(1,3]$, it is difficult to prove the existence of sign-changing solutions. The main difficulty lies in proving the functional of problem (1.1) satisfies (PS)-conditions. To overcome this difficulty, we will apply some new tricks. Moreover, $f(t)\triangleq|t|^{p-1}t$ does not satisfy (f1)-(f2) when $p\in(1,3]$. We must point out that our result holds for any $b>0$. Therefore, our result can be seen as an improvement and extension of [7, 8]. Our result can also extent to more general $f(u)$.

In this paper, we shall work on the space

$$E=H^1_r(\mathbb{R}^3)\triangleq\left\{u\in H^1(\mathbb{R}^3):u(|x|)=u(x)\right\}$$

with the inner product and norm

$$\langle u,v\rangle=\int_{\mathbb{R}^3}(a\nabla u\nabla v+uv)dx,\quad \|u\|=\langle u,u\rangle^{\frac{1}{2}}.$$

$L^q(\mathbb{R}^3)(1\leq q<\infty)$ denotes Lebesgue space with norm $\|u\|_q=\left(\int_{\mathbb{R}^3}|u|^qdx\right)^{1/q}$. It is well known that the weak solution of problem (1.1) corresponds to the critical point of

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2dx+|u|^2)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^2-\frac{1}{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx. \tag{1.4}$$

Clearly, $I\in C^1(E,\mathbb{R})$ and we have

$$\langle I'(u),v\rangle=\int_{\mathbb{R}^3}(a\nabla u\nabla v+uv)dx+b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}\nabla u\nabla vdx-\int_{\mathbb{R}^3}|u|^{p-1}uvdx. \tag{1.5}$$

Setting $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$, $A(u^+, u^-) = \frac{b}{2}\int_{\mathbb{R}^3} |\nabla u^+|^2 dx \int_{\mathbb{R}^3} |\nabla u^-|^2 dx$. To state our result, we establish the following minimization problem

$$c \triangleq \inf\{I(u) : u \in \mathcal{M}\}, \tag{1.6}$$

where

$$\mathcal{M} \triangleq \left\{u \in E : u^\pm \neq 0, \frac{1}{2}\left\langle I'(u), u^+\right\rangle + P(u^+) + A(u^+, u^-) = \frac{1}{2}\left\langle I'(u), u^-\right\rangle + P(u^-) + A(u^+, u^-) = 0\right\}, \tag{1.7}$$

$$P(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} |u|^2 dx + \frac{b}{2}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \frac{3}{p+1}\int_{\mathbb{R}^3} |u|^{p+1} dx.$$

Obviously, the set $\mathcal{M}$ is a subset of the following special manifold:

$$\mathcal{N} \triangleq \left\{u \in E : \frac{1}{2}\langle I'(u), u\rangle + P(u) = 0\right\}. \tag{1.8}$$

Remark 1.3 Clearly, the manifold $\mathcal{M}$ has not been used in the existing literature. The usual manifold $\mathcal{M}_1$ has been used in previous literature is a subset of manifold $\mathcal{N}_1$, where

$$\mathcal{M}_1 = \{u \in E : u^\pm \neq 0, \left\langle I'(u), u^+\right\rangle = \left\langle I'(u), u^-\right\rangle = 0\}, \quad \mathcal{N}_1 = \{u \in E : \langle I'(u), u\rangle = 0\}.$$

As we all know, the manifold $\mathcal{N}_1$ is a commonly used manifold in the study of positive solutions. But the manifold $\mathcal{M}_1$ is not enough for us to prove our result when $p \in (1, 3]$. Thus, we need to find an another manifold. For researching positive solutions, one can also use a special manifold $\mathcal{N}$, which is a combination of the Nehari manifold and Pohožaev manifold for power $p \in (1, 5)$. In order to prove our result, we choose the manifold $\mathcal{M}$.

2. Preliminaries

Comparing with the 4-superlinear condition in [7], we meet some new difficulties. We need to show that the constraint set $\mathcal{M}$ is nonempty and the minimizing sequence on $\mathcal{M}$ is a (PS)-sequence of I in E by using some new tricks.

Lemma 2.1 *If $p \in (1, 5)$, then $\mathcal{M} \neq \emptyset$.*

Proof. For any $u \in E$ and $u^\pm \neq 0$, we set $u_t \triangleq t^{\frac{1}{2}} u(\frac{x}{t})$. In the following, we shall prove that there are positive constants s_1 and t_1 such that

$$\frac{1}{2}\left\langle I'(u^+_{s_1} + u^-_{t_1}), u^+_{s_1}\right\rangle + P(u^+_{s_1}) + A(u^+_{s_1}, u^-_{t_1}) = \frac{1}{2}\left\langle I'(u^+_{s_1} + u^-_{t_1}), u^-_{t_1}\right\rangle + P(u^-_{t_1}) + A(u^+_{s_1}, u^-_{t_1}) = 0, \tag{2.1}$$

which implies that $u^+_{s_1} + u^-_{t_1} \in \mathcal{M}$. Actually, equation (2.1) holds if and only if

$$\begin{cases} r(s,t) \triangleq as^2\alpha(u^+) + s^4[2\beta(u^+) + b\gamma(u^+)] + 2s^2t^2A(u^+, u^-) - \frac{p+7}{2(p+1)} s^{\frac{p+7}{2}} \xi(u^+) = 0, \\ l(s,t) \triangleq at^2\alpha(u^-) + t^4[2\beta(u^-) + b\gamma(u^-)] + 2s^2t^2A(u^+, u^-) - \frac{p+7}{2(p+1)} t^{\frac{p+7}{2}} \xi(u^-) = 0, \end{cases} \tag{2.2}$$

where

$$\alpha(u) \triangleq \int_{\mathbb{R}^3} |\nabla u|^2 dx,\ \beta(u) \triangleq \int_{\mathbb{R}^3} |u|^2 dx,\ \gamma(u) \triangleq \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2,\ \xi(u) \triangleq \int_{\mathbb{R}^3} |u|^{p+1} dx. \tag{2.3}$$

In the other words, we only need to show that there exists $m \in (0, M)$ such that

$$r(m,t) > 0, \quad r(M,t) < 0, \quad \forall\, t \in [m, M], \tag{2.4}$$

$$l(s,m) > 0, \quad l(s,M) < 0, \quad \forall\, s \in [m, M], \tag{2.5}$$

where M is a positive constant. Since $p \in (1,5)$, then $\frac{p+7}{2} > 4$. By (2.2), we can derive that $r(s,t) < 0$ as s enough large, $r(s,t) > 0$ as s enough small. And $l(s,t) < 0$ as t enough large, $l(s,t) > 0$ as t enough small. Consequently, (2.4)-(2.5) hold. Then from the Miranda's Theorem [11], there exist two positive constants s_1 and t_1 such that

$$r(s_1,t_1) = 0, \quad l(s_1,t_1) = 0. \tag{2.6}$$

Hence, (2.1) holds, which shows that $u^+_{s_1} + u^-_{t_1} \in \mathcal{M}$, i.e., $\mathcal{M} \neq \emptyset$. The proof is completed. □

Lemma 2.2 *The pair* (s_1, t_1) *with positive numbers in Lemma 2.1 is unique.*

Proof. In view of Lemma 2.1, there exists a pair (s_1, t_1) such that $u^+_{s_1} + u^-_{t_1} \in \mathcal{M}$ for any $u \in E$ and $u^\pm \neq 0$. Next, we shall prove the uniqueness of (s_1, t_1) by two steps.

Step 1. If $u \in \mathcal{M}$, then $(s_1, t_1) = (1, 1)$.

Since $u \in \mathcal{M}$, then we have

$$\begin{cases} r(1,1) \triangleq a\alpha(u^+) + 2\beta(u^+) + b\gamma(u^+) + 2A(u^+,u^-) - \frac{p+7}{2(p+1)}\xi(u^+) = 0, \\ l(1,1) \triangleq a\alpha(u^-) + 2\beta(u^-) + b\gamma(u^-) + 2A(u^+,u^-) - \frac{p+7}{2(p+1)}\xi(u^-) = 0. \end{cases} \tag{2.7}$$

Assume that $s_1 \leq t_1$. By (2.2), we have

$$\frac{1}{s_1^2} a\alpha(u^+) + 2\beta(u^+) + b\gamma(u^+) + 2A(u^+,u^-) \leq \frac{p+7}{2(p+1)} s_1^{\frac{p-1}{2}} \xi(u^+), \tag{2.8}$$

$$\frac{1}{t_1^2} a\alpha(u^-) + 2\beta(u^-) + b\gamma(u^-) + 2A(u^+,u^-) \geq \frac{p+7}{2(p+1)} t_1^{\frac{p-1}{2}} \xi(u^-). \tag{2.9}$$

It follows from (2.7) and (2.8) that

$$\left(\frac{1}{s_1^2} - 1\right) a \int_{\mathbb{R}^3} |\nabla u^+|^2 dx \leq \frac{p+7}{2(p+1)} \left[s_1^{\frac{p-1}{2}} - 1\right] \int_{\mathbb{R}^3} |u^+|^{p+1} dx. \tag{2.10}$$

If $s_1 < 1$, the negative right side of inequality (2.10) contradicts the positive left side. So $1 \leq s_1 \leq t_1$. Moveover, combining (2.7) and (2.9), $t_1 \leq 1$ can be also obtained. Then $(s_1, t_1) = (1, 1)$.

Step 2. If $u \notin \mathcal{M}$, then there exists a unique u_1 such that $u_1^+ + u_1^- \in \mathcal{M}$.

Suppose that there is an another pair (s_2, t_2) such that $u^+_{s_2} + u^-_{t_2} \in \mathcal{M}$. We set $v_1 \triangleq u^+_{s_1} + u^-_{t_1}$ and $v_2 \triangleq u^+_{s_2} + u^-_{t_2}$. By a simple calculation, we have

$$\int_{\mathbb{R}^3} \left[\frac{s_2^{7/2}}{s_1^{7/2}} v_1^+ + \frac{t_2^{7/2}}{t_1^{7/2}} v_1^-\right] dx = s_2^{7/2} \int_{\mathbb{R}^3} u^+ dx + t_2^{7/2} \int_{\mathbb{R}^3} u^- dx = \int_{\mathbb{R}^3} (v_2^+ + v_2^-) dx. \tag{2.11}$$

Thanks to $v_2 \in \mathcal{M}$ and step 1, we deduce that $(s_1, t_1) = (s_2, t_2)$. The proof is completed. □

Similar to [7], we can prove that $I(u_{s_1}^+ + u_{t_1}^-) = \max_{s,t\geq 0} I(u_s^+ + u_t^-)$. From Lemma 2.2, we consider the minimization problem

$$c_{\mathcal{M}} \triangleq \inf\{I(u) : u \in \mathcal{M}\}. \tag{2.12}$$

Lemma 2.3 *$c_{\mathcal{M}}$ is achieved.*

Proof. For each $u \in \mathcal{M}$, we have $G(u) \triangleq \frac{1}{2}\langle I'(u), u\rangle + P(u) = 0$. Then for any $p \in (1, 5)$, we have

$$\begin{aligned} I(u) =& I(u) - \frac{1}{4}G(u) \\ =& \frac{1}{4}a\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{p-1}{8(p+1)}\int_{\mathbb{R}^3} |u|^{p+1} dx \\ \geq& \frac{1}{4}a\int_{\mathbb{R}^3} |\nabla u|^2 dx > 0. \end{aligned} \tag{2.13}$$

That is $c_{\mathcal{M}} > 0$. Letting $\{u_n\} \subset \mathcal{M}$ such that $I(u_n) \to c_{\mathcal{M}}$. From (2.13), we know that $\{|\nabla u_n|_2\}$ is bounded in E. Since $G(u_n) = 0$, then

$$\begin{aligned} 2\int_{\mathbb{R}^3} |u_n|^2 dx =& \frac{p+7}{2(p+1)}\int_{\mathbb{R}^3} |u_n|^{p+1} dx - a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx - b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 \\ \leq& \frac{p+7}{2(p+1)}\|u_n\|_{p+1}^{p+1}. \end{aligned} \tag{2.14}$$

From Hölder and Sobolev inequalities, we have

$$\|u_n\|_{p+1}^{p+1} \leq \|u_n\|_2^{(p+1)\vartheta}\|u_n\|_6^{(p+1)(1-\vartheta)} \leq C\|u_n\|_2^{(p+1)\vartheta}\|\nabla u_n\|_2^{(p+1)(1-\vartheta)}, \tag{2.15}$$

where $\frac{1}{p+1} = \frac{\vartheta}{2} + \frac{1-\vartheta}{6}$. Then $(p+1)\vartheta < 2$. According to Young's inequality, we obtain that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\frac{p+7}{2(p+1)}\|u_n\|_{p+1}^{p+1} \leq \varepsilon\|u_n\|_2^2 + C_\varepsilon\|\nabla u_n\|_2^{\frac{2(p+1)(1-\vartheta)}{2-(p+1)\vartheta}}. \tag{2.16}$$

Set $\varepsilon = 1$, from (2.14) and (2.16), we have that $\{\|u_n\|_2\}$ is bounded. Hence, $\{u_n\}$ is bounded. Then, there exists u such that $u_n^\pm \rightharpoonup u^\pm$ in E. From (2.13), we can find a constant θ such that $\|u_n^\pm\| > \theta > 0$ for every $n \in \mathbb{N}$.

Since $\{u_n\} \subset \mathcal{M}$, we have that

$$a\int_{\mathbb{R}^3} |\nabla u_n^\pm|^2 dx + 2\int_{\mathbb{R}^3} |u_n^\pm|^2 dx + b\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\int_{\mathbb{R}^3} |\nabla u_n^\pm|^2 dx = \frac{p+7}{2(p+1)}\int_{\mathbb{R}^3} |u_n^\pm|^{p+1} dx. \tag{2.17}$$

Therefore, we have

$$\theta^2 \leq \|u_n^\pm\|^2 < C_1\int_{\mathbb{R}^3} |u_n^\pm|^{p+1} dx. \tag{2.18}$$

Then $\int_{\mathbb{R}^3} |u_n^\pm|^{p+1} dx > \frac{\theta^2}{C_1} > 0$. Since the embedding $E \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $2 < q < 6$, (2.18) shows that $u^\pm \neq 0$. Combining the compactness lemma of Strauss [11] and the weak semicontinuity of norm, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n^\pm|^{p+1} dx \to \int_{\mathbb{R}^3} |u^\pm|^{p+1} dx, \tag{2.19}$$

$$a\int_{\mathbb{R}^3}|\nabla u^{\pm}|^2dx+2\int_{\mathbb{R}^3}|u^{\pm}|^2dx\le\liminf_{n\to\infty}\left(a\int_{\mathbb{R}^3}|\nabla u_n^{\pm}|^2dx+2\int_{\mathbb{R}^3}|u_n^{\pm}|^2dx\right)\tag{2.20}$$

and

$$b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^{\pm}|^2dx\le\liminf_{n\to\infty}b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\int_{\mathbb{R}^3}|\nabla u_n^{\pm}|^2dx.\tag{2.21}$$

Then from (2.17) and (2.19)–(2.21), we have that

$$\frac{1}{2}\langle I'(u),u^{\pm}\rangle+p(u^{\pm})+A(u^{+}+u^{-})\le\liminf_{n\to\infty}\left\{\frac{1}{2}\langle I'(u_n),u_n^{\pm}\rangle+p(u_n^{\pm})+A(u_n^{+}+u_n^{-})\right\}=0.\tag{2.22}$$

Thus, there exists (s_u,t_u) such that $u_{s_u}^{+}+u_{t_u}^{-}\in\mathcal{M}$. Suppose that $0<t_u\le s_u$, then we obtain

$$\begin{aligned}&as_u^2\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx+2s_u^4\int_{\mathbb{R}^3}|u^{+}|^2dx+bs_u^4\left(\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx\right)^2+bs_u^4\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx\int_{\mathbb{R}^3}|\nabla u^{-}|^2dx\\&\ge s_u^2\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx+2s_u^4\int_{\mathbb{R}^3}|u^{+}|^2dx+bs_u^4\left(\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx\right)^2+bs_u^2t_u^2\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx\int_{\mathbb{R}^3}|\nabla u^{-}|^2dx\\&=\frac{p+7}{2(p+1)}s_u^{\frac{p+7}{2}}\int_{\mathbb{R}^3}|u^{+}|^{p+1}dx.\end{aligned}\tag{2.23}$$

From (2.19) and (2.22), we have

$$a\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx+2\int_{\mathbb{R}^3}|u^{+}|^2dx+b\int_{\mathbb{R}^3}|\nabla u|^2dx\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx\le\frac{p+7}{2(p+1)}\int_{\mathbb{R}^3}|u^{+}|^{p+1}dx.\tag{2.24}$$

By (2.23) and (2.24), we obtain

$$a\left(\frac{1}{s_u^2}-1\right)\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx\ge\frac{p+7}{2(p+1)}\left(s_u^{\frac{p-1}{2}}-1\right)\int_{\mathbb{R}^3}|u^{+}|^{p+1}dx,$$

which shows $s_u\le1$. Then $0<t_u\le s_u\le1$. Setting $\bar{u}=u_{s_u}^{+}+u_{t_u}^{-}$. Therefore, we can deduce that

$$\begin{aligned}c_{\mathcal{M}}\le&I(\bar{u})-\frac{1}{4}G(\bar{u})\\=&\frac{1}{4}as_u^2\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx+\frac{p-1}{8(p+1)}s_u^{\frac{p+7}{2}}\int_{\mathbb{R}^3}|u^{+}|^{p+1}dx+\frac{1}{4}at_u^2\int_{\mathbb{R}^3}|\nabla u^{-}|^2dx\\&+\frac{p-1}{8(p+1)}t_u^{\frac{p+7}{2}}\int_{\mathbb{R}^3}|u^{-}|^{p+1}dx\\\le&\frac{1}{4}a\int_{\mathbb{R}^3}|\nabla u^{+}|^2dx+\frac{p-1}{8(p+1)}\int_{\mathbb{R}^3}|u^{+}|^{p+1}dx+\frac{1}{4}a\int_{\mathbb{R}^3}|\nabla u^{-}|^2dx+\frac{p-1}{8(p+1)}\int_{\mathbb{R}^3}|u^{-}|^{p+1}dx\\=&I(u)-\frac{1}{4}G(u)\\\le&\liminf_{n\to\infty}\left(I(u_n)-\frac{1}{4}G(u_n)\right)=c_{\mathcal{M}}.\end{aligned}\tag{2.25}$$

(2.25) implies that $s_u=t_u=1$. That is $u=\bar{u}$ and $I(u)=c_{\mathcal{M}}$. The proof is completed. □

3. Proof of Theorem 1.1

Lemma 3.1. *Assume $c_{\mathcal{M}}$ attained in $\mathcal{M}$, then u is a critical point of I.*

Proof. Since $u \in \mathcal{M}$, $u^{\pm} \neq 0$. Then for any fixed $v \in H^1(\mathbb{R}^3)$, there exists $\varepsilon > 0$ such that $(u + wv)^{\pm} \neq 0$ for all $w \in (-\varepsilon, \varepsilon)$. Arguing by a contradiction, there is a sequence $\{w_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} w_i = 0, \quad u + w_i v = 0 \text{ a.e. on } \mathbb{R}^3.$$

Letting $i \to \infty$, we have $u = 0$ a.e. on $\mathbb{R}^3$. Which is a contradiction with $u^{\pm} \neq 0$.

From Lemma 2.1, there exists a unique pair $(s(w), t(w))$ such that $s(w)(u+wv)^+ + t(w)(u+wv)^- \in \mathcal{M}$. Next, we prove some standard properties of $(s(w), t(w))$ as Nehari manifold. For our purpose, we consider the function

$$\varphi(s, t, w) = G((u + wv)_s^+ + (u + wv)_t^-)$$

defined for $(s, t, w) \in (0, +\infty) \times (0, +\infty) \times (-\varepsilon, \varepsilon)$. Since $u \in \mathcal{M}$, we have $\varphi(1, 1, 0) = 0$. Moveover, φ is a C^1 function and

$$\begin{aligned}
&\left.\frac{\partial\varphi(s, t, w)}{\partial s}\right|_{(s,t,w)=(1,1,0)} \\
=&-\frac{p^2 + 6p - 7}{4(p + 1)}\int_{\mathbb{R}^3} |u^+|^{p+1} dx - 2a\int_{\mathbb{R}^3} |\nabla u^+|^2 dx \\
\leq&0.
\end{aligned}$$

Similarly, we can deduce that $\frac{\partial\varphi(s,t,w)}{\partial t}|_{(s,t,w)=(1,1,0)} \leq 0$. From the Implicit Function Theorem, the functions $s(w)$, $t(w)$ are C^1. And $t(0) = s(0) = 1$. Moveover, $s(w)$, $t(w) \neq 0$ near 0. We define

$$\Upsilon(w) = I\left((u + wv)_{s(w)}^+ + (u + wv)_{t(w)}^-\right).$$

Then we obtain that Υ is differentiable for small w and attains its minimum at $w = 0$. Hence, we derive that

$$\begin{aligned}
0 =&\Upsilon'(0) = \left.\frac{dI((u + wv)_{s(w)}^+ + (u + wv)_{t(w)}^-)}{dw}\right|_{w=0} \\
=&\left.\frac{\partial I((u + wv)_{s(w)}^+ + (u + wv)_{t(w)}^-)}{\partial s}\right|_{(s,t,w)=(1,1,0)} \left.\frac{ds}{dw}\right|_{w=0} + \left.\frac{dI((u + wv)_{s(w)}^+ + (u + wv)_{t(w)}^-)}{dw}\right|_{(s,t,w)=(1,1,0)} \\
&+ \left.\frac{\partial I((u + wv)_{s(w)}^+ + (u + wv)_{t(w)}^-)}{\partial t}\right|_{(s,t,w)=(1,1,0)} \left.\frac{dt}{dw}\right|_{w=0} \\
=&r(1, 1)s'(0) + l(1, 1)t'(0) + \langle I'(u), v\rangle \\
=&\langle I'(u), v\rangle .
\end{aligned}$$

Since $v \in E$ is arbitrary, we have that $I'(u) = 0$. □

Proof of Theorem 1.1. From Lemma 2.3 and 3.1, there is a $u \in \mathcal{M}$ such that $I(u) = c_{\mathcal{M}}$ and $I'(u) = 0$. Then problem (1.1) has at least one sign-changing solution. The proof is completed. □

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Conflict of interest

The authors declare that they have no competing interests.

References

1. J. Sun, S. B. Liu, Nontrivial solutions of Kirchhoff type problems, *Appl. Math. Lett.*, **25** (2012), 500–504.
2. Z. J. Guo, Ground states for Kirchhoff equations without compact condition, *J. Differ. Equations*, **259** (2015), 2884–2902.
3. Q. Q. Li, K. M. Teng, X. Wu, Ground states for Kirchhoff-type equations with critical or supercritical growth, *Math. Method. Appl. Sci.*, **40** (2017), 6732–6746.
4. W. Chen, Z. W. Fu, Y. Wu, Positive solutions for nonlinear Schrödinger-Kirchhoff equations in $\mathbb{R}^3$, *Appl. Math. Lett.*, **104** (2020), 106274.
5. Q. L. Xie, Least energy nodal solution for Kirchhoff type problem with an asymptotically 4-linear nonlinearity, *Appl. Math. Lett.*, **102** (2020), 106157.
6. G. B. Li, H. Y. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *J. Differ. Equations*, **257** (2014), 566–600.
7. L. Wang, B. L. Zhang, K. Cheng K, Ground state sign-changing solutions for the Schödinger-Kirchhoff equation in $\mathbb{R}^3$, *J. Math. Anal. Appl.*, **466** (2018), 1545–1569.
8. X. T. Qian, Ground state sign-changing solutions for a class of nonlocal problem, *J. Math. Anal. Appl.*, **2** (2021), 124753.
9. S. T. Chen, Y. B. Li, X. H. Tang, Sign-changing solutions for asymptotically linear Schrödinger equation in bounded domains, *Electron. J. Differ. Eq.*, **317** (2016), 1–9.
10. G. Q. Chai, W. M. Liu, Least energy sign-changing solutions for Kirchhoff-Poisson systems, *Bound. Value Probl.*, **2019** (2019), 1–25.
11. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.