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*Research article*

## **An iteration process for a general class of contractive-like operators: Convergence, stability and polynomiography**

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**Abstract:** In this paper, we propose a three step iteration process and analyze the performance of the process for a contractive-like operators. It is observed that this iterative procedure is faster than several iterative methods in the existing literature. To support the claim, a numerical example is presented using Maple 13. Some images are generated by using this iteration method for complex cubic polynomials. We believe that our presented work enrich the polynomiography software.

**Keywords:** iterative schemes; image generation; contractive-like operators; convergence; polynomials

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### **1. Introduction**

Fixed point theory provides a suitable framework to solve a system of linear and nonlinear functional equations, which are applicable in various areas of research such as engineering, chemistry, game theory, economics, etc. The Picard iteration procedure is one of the most simplest iterative procedures and is used for approximating unique fixed point (FP) for mapping/operators satisfying contractive type conditions. However, in case of slightly weaker contractive operators such as nonexpansive mappings, the Picard iterative procedure fails to converge.

So, it is important to consider iterative procedures which converge for the larger class of operators. Various iterative techniques were developed in literature to estimate the FPs of certain operators of

practical nature. Some famous iterative procedures are Mann [1], Ishikawa [2], Agarwal et al. [3], Noor [4], SP [5], CR [6], Abbas & Nazir [7], Picard-Mann [8], Picard-S [9] and Kadioglu & Yildirim [10].

The rate of convergence and stability are some of the most distinguishing features of an iterative procedure which are always desirable. These criteria play a vital role when one iterative procedure is compared with the other. For instance, Newton and quasi Newton methods are always preferred for solutions of the system of nonlinear Eqs on the basis of their rate of convergence given that an appropriate initial guess to the solution is known. Rhoades [11] proved that the convergence of the well-known Mann iteration is faster than that of Ishikawa iteration for the class of decreasing function and vice versa for the class of increasing functions. It is observed that [3], the convergence rate of Agarwal et al. iteration and Picard iteration is the same and both are faster than the Mann iteration for the class of contraction mappings. Authors in [7] introduced an iteration with better and faster convergence than Agarwal et al. iteration. The authors in [6] claimed that the CR iteration is either equivalent or faster than Mann, Picard, Ishikawa, Agarwal et al., SP, and Noor iterations for the class of quasi-contractive operators in the setting of Banach spaces. Karakaya with his coauthors in [12] proved that for contraction mappings, the CR iteration is faster than  $S^*$ -iteration. One other interesting result can be found in [9] and for other details, we refer [13–15] and references therein.

Motivated by the work quoted above, the authors [16] proposed a new iteration known as  $K^*$  iteration in Banach spaces and show that  $K^*$  iterative procedure is faster than many other iterative schemes. We compare the convergence of the  $K^*$  iterative procedure numerically with numerous iteration processes in the existing literature by using generalized mapping. The graphs of complex polynomials are also drawn using the  $K^*$ -iterative procedure. All the work done in this paper is for a general class of contractive-like operators.

## 2. Preliminaries

Consider a real normed space  $E$  and a mapping  $T : E \rightarrow E$ . If  $T(x^*) = x^*$  then  $x^* \in E$  is called FP of  $T$  and the set of FPs of  $T$  is represented by  $F(T)$ . Now, we give a brief description of existing iterations which are relevant to our work in this paper.

The well-known Picard-iteration sequence  $\{x_n\}$  is:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = Tx_n, \quad n \geq 0, \end{cases} \quad (2.1)$$

The Mann-iteration [1] is a one step iterative procedure which is described by the following sequence  $\{x_n\}$ :

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n^1)x_n + \alpha_n^1Tx_n, \quad n \geq 0, \end{cases} \quad (2.2)$$

where  $\{\alpha_n^1\}_{n=0}^\infty \in [0, 1]$ .

The Ishikawa iteration process [2] is a two step iterative process given by the following sequence  $\{x_n\}$ :

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n^1)x_n + \alpha_n^1Ty_n, \\ y_n = (1 - \alpha_n^2)x_n + \alpha_n^2Tx_n, \quad n \geq 0, \end{cases} \quad (2.3)$$

where  $\{\alpha_n^1\}_{n=0}^\infty, \{\alpha_n^2\}_{n=0}^\infty \in [0, 1]$ .

Motivated by [2], a two step iteration was proposed in [3], which is known as Agarwal et al. iteration;

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n^1)Tx_n + \alpha_n^1Ty_n, \\ y_n = (1 - \alpha_n^2)x_n + \alpha_n^2Tx_n, n \geq 0, \end{cases} \quad (2.4)$$

where  $\{\alpha_n^1\}_{n=0}^\infty, \{\alpha_n^2\}_{n=0}^\infty \in [0, 1]$ .

In 2013, Khan [8] proposed a new iteration namely ‘Picard-Mann-hybrid iteration’ which is given by the sequence

$$\begin{cases} x_0 \in E, \\ x_{n+1} = Ty_n, \\ y_n = (1 - \alpha_n^1)x_n + \alpha_n^1Tx_n, n \geq 0, \end{cases} \quad (2.5)$$

where  $\{\alpha_n^1\}_{n=0}^\infty \in [0, 1]$ .

Noor iteration [4] is a three step iterative procedure defined by the following scheme  $\{x_n\}$ :

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n^1)x_n + \alpha_n^1Ty_n, \\ y_n = (1 - \alpha_n^2)x_n + \alpha_n^2Tz_n, \\ z_n = (1 - \alpha_n^3)x_n + \alpha_n^3Tx_n, n \geq 0, \end{cases} \quad (2.6)$$

where  $\{\alpha_n^1\}_{n=0}^\infty, \{\alpha_n^2\}_{n=0}^\infty, \{\alpha_n^3\}_{n=0}^\infty \in [0, 1]$ .

The SP-iteration is a three step iteration introduced in 2011 [5] as;

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n^1)y_n + \alpha_n^1Ty_n, \\ y_n = (1 - \alpha_n^2)z_n + \alpha_n^2Tz_n, \\ z_n = (1 - \alpha_n^3)x_n + \alpha_n^3Tx_n, n \geq 0, \end{cases} \quad (2.7)$$

where  $\{\alpha_n^1\}_{n=0}^\infty, \{\alpha_n^2\}_{n=0}^\infty, \{\alpha_n^3\}_{n=0}^\infty \in [0, 1]$ .

The CR iteration was introduced Chugh et al. in 2012 [6] as;

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n^1)y_n + \alpha_n^1Ty_n, \\ y_n = (1 - \alpha_n^2)Tx_n + \alpha_n^2Tz_n, \\ z_n = (1 - \alpha_n^3)x_n + \alpha_n^3Tx_n, n \geq 0, \end{cases} \quad (2.8)$$

where  $\{\alpha_n^1\}_{n=0}^\infty, \{\alpha_n^2\}_{n=0}^\infty, \{\alpha_n^3\}_{n=0}^\infty \in [0, 1]$ .

The Picard-S-iteration [9] was introduced as;

$$\begin{cases} u_0 \in E, \\ u_{n+1} = Tv_n, \\ v_n = (1 - \alpha_n^1)Tu_n + \alpha_n^1Tw_n, \\ w_n = (1 - \alpha_n^2)u_n + \alpha_n^2Tu_n, n \geq 0, \end{cases} \quad (2.9)$$

where  $\{\alpha_n^1\}_{n=0}^\infty, \{\alpha_n^2\}_{n=0}^\infty \in [0, 1]$ .

Kadioglu & Yildirim [10] studied the following iteration procedure:

$$\begin{cases} u_0 \in E, \\ u_{n+1} = Tv_n, \\ v_n = (1 - \alpha_n^1)w_n + \alpha_n^1 Tw_n, \\ w_n = (1 - \alpha_n^2)u_n + \alpha_n^2 Tu_n, \quad n \geq 0, \end{cases} \quad (2.10)$$

where  $\{\alpha_n^1\}_{n=0}^\infty, \{\alpha_n^2\}_{n=0}^\infty \in [0, 1]$ .

Ullah and Arshad [16] introduce following a new modified iteration-process, known as  $K^*$  iteration process:

$$\begin{cases} x_0 \in E, \\ x_{n+1} = Ty_n, \\ y_n = T((1 - \alpha_n^1)z_n + \alpha_n^1 Tz_n), \\ z_n = (1 - \alpha_n^2)x_n + \alpha_n^2 Tx_n, \quad n \geq 0, \end{cases} \quad (2.11)$$

where  $\{\alpha_n^1\}_{n=0}^\infty, \{\alpha_n^2\}_{n=0}^\infty \in [0, 1]$ .

The iteration procedure (2.11) is stable and performs better than all above mentioned iterations.

The different stability results have been studied by Osilike in [17] by using the following definition of contractive

**Definition 1.** Let  $x, y \in X$ , then we have  $0 \leq \delta < 1$  and  $L \geq 0$  satisfying;

$$d(Tx, Ty) \leq \delta d(x, y) + Ld(x, Tx). \quad (2.12)$$

The authors in [18] proved the results of stability by using the following more general definition;

$$d(Tx, Ty) \leq \delta d(x, y) + \phi(d(x, Tx)). \quad (2.13)$$

Here  $x, y \in X$  and  $0 \leq \delta < 1$ . Moreover  $\phi : R^+ \rightarrow R^+$  is an increasing function satisfying  $\phi(0) = 0$ .

Recently, Bosede and Rhoades [19] pointed out that assumptions (2.12), (2.13) and its several variants are pointless. Indeed, if  $x = Tx^* = x^*$ , then (2.12) and (2.13) reduce to the following: For  $x, y \in X$ , there exist  $0 \leq \delta < 1$  satisfying

$$d(x^*, Ty) \leq \delta d(x^*, y) \quad (2.14)$$

and in the real normed spaces, for each  $x, y \in X$ , there exist  $0 \leq \delta < 1$  such that

$$\|x^* - Ty\| \leq \delta \|x^* - y\|. \quad (2.15)$$

For examples to illustrate that the (2.15) is more general than that of (2.12) and (2.13) see [20].

**Lemma 1.** [21] Consider the real sequences  $\{\psi_n\}_{n=0}^\infty$  and  $\{\varphi_n\}_{n=0}^\infty$  that are non-negative and

$$\psi_{n+1} \leq (1 - \phi_n)\psi_n + \varphi_n,$$

holds, where  $\phi_n \in (0, 1)$  for all  $n_0 \in \mathbb{N}$ ,  $\sum_{n=0}^\infty \phi_n = \infty$  and  $\frac{\varphi_n}{\phi_n} \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\lim_{n \rightarrow \infty} \psi_n = 0$ .

**Lemma 2.** [22] Consider a real non-negative sequence  $\{\psi_n\}_{n=0}^{\infty}$  and  $n_0 \in \mathbb{N}$  and

$$\psi_{n+1} \leq (1 - \phi_n)\psi_n + \phi_n\varphi_n,$$

holds for all  $n \geq n_0$ , where  $\phi_n \in (0, 1)$  for all  $n \in \mathbb{N}$ ,  $\sum_{n=0}^{\infty} \phi_n = \infty$  and  $\varphi_n \geq 0$  for all  $n \in \mathbb{N}$ . Then

$$0 \leq \limsup_{n \rightarrow \infty} \psi_n \leq \limsup_{n \rightarrow \infty} \varphi_n.$$

**Definition 2.** ([23]) Consider two convergent real sequences  $\{a_n\}_{n=0}^{\infty}$  and  $\{b_n\}_{n=0}^{\infty}$  that converges to  $a$  and  $b$ , respectively. The convergence rate of  $\{a_n\}_{n=0}^{\infty}$  is faster than that of  $\{b_n\}_{n=0}^{\infty}$  if we have

$$\lim_{n \rightarrow \infty} \frac{\|a_n - a\|}{\|b_n - b\|} = 0$$

**Definition 3.** ([23]) Consider two iterations  $\{u_n\}_{n=0}^{\infty}$  and  $\{v_n\}_{n=0}^{\infty}$  converge to the common FP  $p$ . If  $\|u_n - p\| \leq a_n$  and  $\|v_n - p\| \leq b_n$ , for all  $n \geq 0$ , where  $\{a_n\}_{n=0}^{\infty}$  and  $\{b_n\}_{n=0}^{\infty}$  two convergent sequences of real number that converges to 0. Then the convergence of  $\{u_n\}_{n=0}^{\infty}$  is faster than that of  $\{v_n\}_{n=0}^{\infty}$  if the convergence of  $\{a_n\}_{n=0}^{\infty}$  is faster than that of  $\{b_n\}_{n=0}^{\infty}$ .

**Definition 4.** ([24]) Let  $\{t_n\}_{n=0}^{\infty}$  be an arbitrary sequence in  $C$ . Then, an iteration procedure  $x_{n+1} = f(T, x_n)$ , converging to fixed point  $p$ , is said to be  $T$ -stable or stable with respect to  $T$ , if for  $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$ ,  $n = 0, 1, 2, 3, \dots$ , we have  $\lim_{n \rightarrow \infty} \epsilon_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} t_n = x^*$ .

### 3. Main results

**Theorem 3.1.** Consider a real normed  $(E, \|\cdot\|)$  and a mapping  $T : E \rightarrow E$  with a FP  $x^*$  satisfying (2.15). Let  $\{x_n\}_{n=0}^{\infty}$  be defined by (2.11), where  $\{\alpha_n^1\}_{n=0}^{\infty}$  and  $\{\alpha_n^2\}_{n=0}^{\infty}$  are in  $[0, 1]$  and  $\sum_{n=0}^{\infty} \alpha_n^1 \alpha_n^2 = \infty$ . Then  $\{x_n\}_{n=0}^{\infty}$  converges to  $x^*$  (strongly).

*Proof.* From (2.11), by means of simple calculation, we get

$$\|z_n - x^*\| = (1 - (1 - \delta)\alpha_n^2) \|x_n - x^*\|, \quad (3.1)$$

and

$$\begin{aligned} \|y_n - x^*\| &= \left\| T((1 - \alpha_n^1)z_n + \alpha_n^1 Tz_n) - x^* \right\| \\ &\leq \delta \left\| (1 - \alpha_n^1)z_n + \alpha_n^1 Tz_n - x^* \right\| \\ &\leq \delta \left\| (1 - \alpha_n^1)(z_n - x^*) + \alpha_n^1 (Tz_n - x^*) \right\| \\ &\leq \delta[(1 - \alpha_n^1) \|z_n - x^*\| + \delta\alpha_n^1 \|z_n - x^*\|] \\ &= \delta[(1 - \alpha_n^1) + \delta\alpha_n^1] \|z_n - x^*\|. \end{aligned} \quad (3.2)$$

By using (3.1) in (3.2) we obtain

$$\begin{aligned} \|y_n - x^*\| &\leq \delta[(1 - (1 - \delta)\alpha_n^1)(1 - (1 - \delta)\alpha_n^2)] \|x_n - x^*\| \end{aligned}$$

$$\begin{aligned}
&= \delta[1 - (1 - \delta)\alpha_n^1 - (1 - \delta)\alpha_n^2 + (1 - \delta)^2\alpha_n^1\alpha_n^2] \|x_n - x^*\| \\
&\leq \delta[1 - (1 - \delta)\alpha_n^1\alpha_n^2 \\
&\quad - (1 - \delta)\alpha_n^1\alpha_n^2 + (1 - \delta)\alpha_n^1\alpha_n^2] \|x_n - x^*\| \\
&= \delta[1 - (1 - \delta)\alpha_n^1\alpha_n^2] \|x_n - x^*\|.
\end{aligned} \tag{3.3}$$

Thus from (3.3)

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|Ty_n - x^*\| \\
&\leq \delta \|y_n - x^*\| \\
&\leq \delta^2(1 - (1 - \delta)\alpha_n^1\alpha_n^2) \|x_n - x^*\|.
\end{aligned} \tag{3.4}$$

Now, by (3.4) we inductively obtain,

$$\|x_{n+1} - x^*\| \leq \left(\delta^{2(n+1)}\right)_{k=0}^{\infty} (1 - (1 - \delta)\alpha_k^1\alpha_k^2) \|x_0 - x^*\|, n \geq 0. \tag{3.5}$$

Using  $0 < \delta < 1$ ,  $\{\alpha_n^1\}_{n=0}^{\infty}$  and  $\{\alpha_n^2\}_{n=0}^{\infty}$  are in  $[0, 1]$  and  $\sum_{n=0}^{\infty} \alpha_n^1\alpha_n^2 = \infty$ , we get

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x^*\| = 0$$

that is,  $\{x_n\}_{n=0}^{\infty}$  converges strongly to  $x^*$ .  $\square$

**Theorem 3.2.** Consider a real normed space  $(E, \|\cdot\|)$  and a mapping  $T : E \rightarrow E$  with a FP  $x^*$  satisfying (2.15). Let  $\{x_n\}_{n=0}^{\infty}$  be defined by (2.11), where  $\{\alpha_n^1\}_{n=0}^{\infty}$ ,  $\{\alpha_n^2\}_{n=0}^{\infty} \in [0, 1]$  such that  $\sum_{n=0}^{\infty} \alpha_n^1\alpha_n^2 = \infty$ . Then, the  $\{x_n\}_{n=0}^{\infty}$  is  $T$ -stable.

*Proof.* The sequence defined in (2.11) converges to  $x^*$  by Theorem 3.1 so consider real sequences  $\{t_n\}_{n=0}^{\infty}$ ,  $\{u_n\}_{n=0}^{\infty}$  and  $\{v_n\}_{n=0}^{\infty}$  in  $E$ , where

$$\begin{cases} t_{n+1} = Tu_n, \\ u_n = T((1 - \alpha_n^1)v_n + \alpha_n^1Tv_n), \\ v_n = (1 - \alpha_n^2)t_n + \alpha_n^2Tt_n, \quad n \geq 0, \end{cases} \tag{3.6}$$

Let  $\epsilon_n = \|t_{n+1} - Tu_n\|$ . We shall prove that  $\lim_{n \rightarrow \infty} \epsilon_n = 0 \Leftrightarrow \lim_{n \rightarrow \infty} t_n = x^*$ . Let  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ , Then we shall prove that  $\lim_{n \rightarrow \infty} t_n = x^*$  for mapping satisfying condition (2.15). That is,

$$\begin{aligned}
\|t_{n+1} - x^*\| &\leq \|t_{n+1} - Tu_n\| + \|Tu_n - x^*\| \\
&\leq \epsilon_n + \|Tu_n - x^*\|
\end{aligned} \tag{3.7}$$

Using condition (2.15), we have

$$\|Tu_n - x^*\| \leq \delta \|u_n - x^*\| \tag{3.8}$$

substituting (3.8) in (3.7), we get

$$\|t_{n+1} - x^*\| \leq \epsilon_n + \delta \|u_n - x^*\| \tag{3.9}$$

Then using (3.3)

$$\|t_{n+1} - x^*\| \leq \epsilon_n + \delta[\delta(1 - (1 - \delta)\alpha_n^1\alpha_n^2)]\|t_n - x^*\| \quad (3.10)$$

Since  $0 < \alpha^1\alpha^2 < \alpha_n^1\alpha_n^2$ ,

$$\|t_{n+1} - x^*\| \leq \epsilon_n + \delta^2(1 - (1 - \delta)\alpha^1\alpha^2)\|t_n - x^*\| \quad (3.11)$$

by mean of Lemma 1 in (3.11) we get

$$\lim_{n \rightarrow \infty} t_n = x^*.$$

Conversely, let  $\lim_{n \rightarrow \infty} t_n = x^*$ . We show that  $\lim_{n \rightarrow \infty} \epsilon_n = 0$

$$\begin{aligned} \epsilon_n &= \|t_{n+1} - Tu_n\| \\ &\leq \|t_{n+1} - x^*\| + \|x^* - Tu_n\| \\ &\leq \|t_{n+1} - x^*\| + \delta\|u_n - x^*\| \\ &\leq \|t_{n+1} - x^*\| + \delta^2(1 - (1 - \delta)\alpha_n^1\alpha_n^2)\|t_n - x^*\| \end{aligned}$$

Hence  $\lim_{n \rightarrow \infty} \epsilon_n = 0$  and the iteration (2.11) is  $T$ -stable.  $\square$

**Theorem 3.3.** Consider a real normed space  $(E, \|\cdot\|)$  and a mapping  $T : E \rightarrow E$  with FP  $x^*$  satisfying (2.15). Let the sequences  $\{x_n\}_{n=0}^\infty$  and  $\{u_n\}_{n=0}^\infty$  as in (2.11) and (2.10) respectively, where  $\{\alpha_n^1\}_{n=0}^\infty$  and  $\{\alpha_n^2\}_{n=0}^\infty$  are in  $[0, 1]$  and  $\sum_{n=0}^\infty \alpha_n^1\alpha_n^2 = \infty$ . Then the convergence of iteration scheme  $\{x_n\}_{n=0}^\infty$  to  $x^*$  of  $T$  is faster than that of  $\{u_n\}_{n=0}^\infty$ .

*Proof.* Using Theorem 3.1,

$$\|x_{n+1} - x^*\| \leq \delta^{2(n+1)} \|x_0 - x^*\|_{k=0}^\infty (1 - (1 - \delta)\alpha_k^1\alpha_k^2), \forall n \geq 0. \quad (3.12)$$

Let

$$a_n = \delta^{2(n+1)} \|x_0 - x^*\|_{k=0}^\infty (1 - (1 - \delta)\alpha_k^1\alpha_k^2), \forall n \geq 0, \quad (3.13)$$

From (2.10), we have

$$\begin{aligned} \|w_n - x^*\| &= \|(1 - \alpha_n^2)u_n + \alpha_n^2Tu_n - x^*\| \\ &= \|(1 - \alpha_n^2)(u_n - x^*) + \alpha_n^2(Tu_n - x^*)\| \\ &\leq (1 - \alpha_n^2)\|u_n - x^*\| + \delta\alpha_n^2\|u_n - x^*\| \\ &= (1 - (1 - \delta)\alpha_n^2)\|u_n - x^*\|. \end{aligned} \quad (3.14)$$

Similarly

$$\begin{aligned} \|v_n - x^*\| &= \|(1 - \alpha_n^1)w_n + \alpha_n^1Tw_n - x^*\| \\ &= \|(1 - \alpha_n^1)(w_n - x^*) + \alpha_n^1(Tw_n - x^*)\| \\ &\leq (1 - \alpha_n^1)\|w_n - x^*\| + \delta\alpha_n^1\|w_n - x^*\| \\ &= (1 - (1 - \delta)\alpha_n^1)\|w_n - x^*\|. \end{aligned} \quad (3.15)$$

From (3.14) and (3.15), we get

$$\begin{aligned}
 & \|v_n - x^*\| \\
 & \leq (1 - (1 - \delta)\alpha_n^1)(1 - (1 - \delta)\alpha_n^2) \|u_n - x^*\| \\
 & = (1 - (1 - \delta)\alpha_n^2 - (1 - \delta)\alpha_n^1 + (1 - \delta)^2\alpha_n^1\alpha_n^2) \|u_n - x^*\| \\
 & \leq (1 - (1 - \delta)\alpha_n^1\alpha_n^2 \\
 & \quad - (1 - \delta)\alpha_n^1\alpha_n^2 + (1 - \delta)\alpha_n^1\alpha_n^2) \|u_n - x^*\| \\
 & \leq (1 - (1 - \delta)\alpha_n^1\alpha_n^2) \|u_n - x^*\|.
 \end{aligned} \tag{3.16}$$

$$\begin{aligned}
 \|u_{n+1} - x^*\| & \leq \|Tv_n - x^*\| \\
 & \leq \delta \|v_n - x^*\| \\
 & \leq \delta(1 - \alpha_n^1\alpha_n^2(1 - \delta)) \|u_n - x^*\|.
 \end{aligned} \tag{3.17}$$

Now, by (3.17) we inductively obtain,

$$\|u_{n+1} - x^*\| \leq \delta^{n+1} \|u_0 - x^*\|_{k=0}^{\infty} (1 - (1 - \delta)\alpha_k^1\alpha_k^2), \forall n \geq 0. \tag{3.18}$$

Let

$$b_n = \delta^{n+1} \|u_0 - x^*\|_{k=0}^{\infty} (1 - (1 - \delta)\alpha_k^1\alpha_k^2), \forall n \geq 0. \tag{3.19}$$

Since  $0 < \delta < 1$ , then

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x^*\|}{\|u_{n+1} - x^*\|} & = \lim_{n \rightarrow \infty} \frac{a_n}{b_n} \\
 & = \lim_{n \rightarrow \infty} \delta^{n+1} \\
 & = 0.
 \end{aligned}$$

□

**Example 3.4.** Consider the usual normed space  $E = R$ ,  $S = [1, 100]$  and a mapping  $T : S \rightarrow S$  defined as  $Tx = \sqrt{x^2 - 8x + 40} \forall x \in S$ . Then,  $T$  satisfies the condition (2.15) with  $\delta \in [0.5222, 0.9987]$  and has a unique FP  $x^* = 5$ . Take  $\alpha_n^1 = \alpha_n^2 = \alpha_n^3 = \frac{1}{2}$  and initial guess  $x_0 = u_0 = 100$ . It can be observed from Tables 1 and 2 that the convergence of  $K^*$  iteration to  $x^* = 5$  is faster than that of SP, Noor, Ishikawa ARS, Picard-Mann, Abbas, CR, Picard-S and Kadioglu & Yildirim iterations.



**Table 1.** Comparison of convergence of different iterations.

Iter. No.	AS iteration	Picard-S	Kadioglu & Yildirim	Abbas	CR
0	100	100	100	100	100
1	88.3923	91.2889	92.2564	92.7394	93.2231
2	76.8413	82.6076	84.5359	85.4988	86.4638
3	65.3657	73.9626	76.8430	78.2820	79.7244
⋮	⋮	⋮	⋮	⋮	⋮
12	5.00000	5.88655	11.9615	16.4250	21.3496
⋮	⋮	⋮	⋮	⋮	⋮
16		5.00000	5.00188	5.02389	5.37398
⋮		⋮	⋮	⋮	⋮
19			5.00000	5.00001	5.00038
20			⋮	5.00000	5.00003
21				⋮	5.00000

**Table 2.** Comparison of convergence of different iterations.

Iter. No.	Picard-Maan	ARS	SP	Noor	Ishikawa	Maan
0	100	100	100	100	100	100
1	94.1899	95.1574	95.1580	96.6119	97.0950	98.0624
2	88.3923	90.3232	90.3246	93.2280	94.1930	96.1262
3	82.6088	85.4984	85.5007	89.8486	91.2942	94.1913
⋮	⋮	⋮	⋮	⋮	⋮	⋮
24	5.00000	5.01818	5.04720	21.6024	31.9246	54.0134
⋮	⋮	⋮	⋮	⋮	⋮	⋮
29		5.00000	5.00004	9.29078	19.0506	44.6622
⋮		⋮	⋮	⋮	⋮	⋮
31			5.00000	6.69608	14.4208	40.9632
⋮			⋮	⋮	⋮	⋮
52				5.00000	5.00018	7.57905
⋮				⋮	⋮	⋮
57					5.00000	5.32335
⋮					⋮	⋮
78					5.00000	

#### 4. Application of $K^*$ iteration

Polynomiography defined by Kalantari in 2005 as “the art and science of visualization in approximation of the zeros of complex polynomials, via fractal and non fractal images produced

using the convergence properties of iteration functions” [25]. The image obtained is named “polynomiograph”. Polynomiography gives a different approach to solve the old problem by utilizing some calculations and computer. Like fractals [26], the polynomiographs can be obtained by means of various iterations. A “polynomiographer” can control the shape in a predictable way by using various iteration for a variety of complex polynomials [27–30]. These patterns can be used for textures, carpet and tapestry designs etc.

#### 4.1. Modification of AS iteration

The famous Newton method is given by;

$$z_{n+1} = z_n - \frac{p(z_n)}{p'(z_n)}, \quad n = 0, 1, 2, \dots, \quad (4.1)$$

for complex polynomial  $p$ . Here  $z_o \in \mathbb{C}$  is an initial guess. Here, we modify Newton method by means of AS iteration the orbits for generation of polynomiograph is totally different than the orbits of Picard iteration and the iterations studied in [27–30]. Thus the obtained Basins of attraction are entirely different from the existing ones.

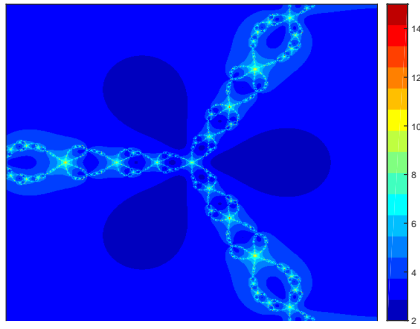
Consider a Banach space  $X = \mathbb{C}$  and  $v_o = (x_o, y_o)$  and  $\alpha_n^1 = \alpha^1$ ,  $\alpha_n^2 = \alpha^2$  such that  $0 < \alpha^1 \leq 1$  and  $0 \leq \alpha^2 \leq 1$ . Setting (2.11) in (4.1) for the Newton method, we obtain the following formula that is used to generate polinomiographs:

$$\begin{cases} z_{n+1} = T\left(u_n - \frac{p(u_n)}{p'(u_n)}\right), \\ u_n = T\left((1 - \alpha^1)v_n + \alpha^1\left(v_n - \frac{p(v_n)}{p'(v_n)}\right)\right), \\ v_n = (1 - \alpha^2)z_n + \alpha^2\left(z_n - \frac{p(z_n)}{p'(z_n)}\right), \quad n \geq 0, \end{cases} \quad (4.2)$$

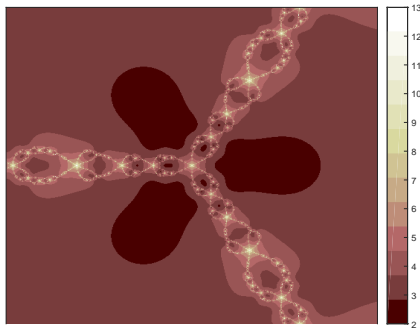
where  $0 < \alpha^1 \leq 1$  and  $0 \leq \alpha^2 \leq 1$ .

#### 4.2. Graphical examples

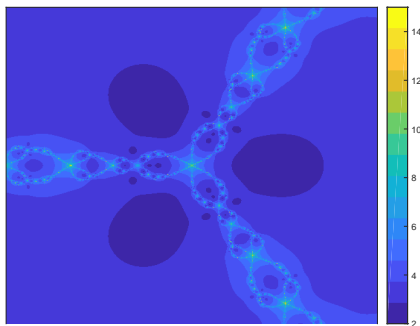
Polynomiographs for complex polynomial equation  $z^3 - 1 = 0$  are presented in Figures 1–6 and Figures 7–12 presented examples of complex polynomial equation  $z^3 - 2z + 2 = 0$  with the use of the following parameters: Resolution  $500 \times 500$  pixels and were generated with maximum number of iterations = 15, accuracy  $\varepsilon = 0.001$  and  $A = [-2, 2]^2$ . For polynomiographs presented in the following figures, one can observe that for changing parameters  $\alpha^1$  and  $\alpha^2$  images have different basins of attraction.



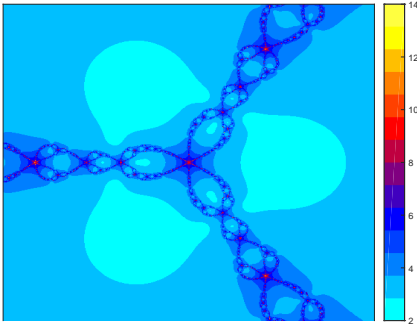
**Figure 1.** Polynomiograph for  $\alpha^1 = 0.75, \alpha^2 = 0.15$ .



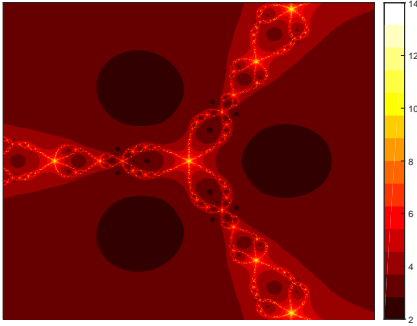
**Figure 2.** Polynomiograph for  $\alpha^1 = 0.5, \alpha^2 = 0.2$ .



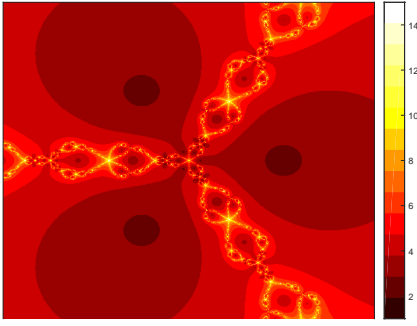
**Figure 3.** Polynomiograph for  $\alpha^1 = 0.3, \alpha^2 = 0.5$ .



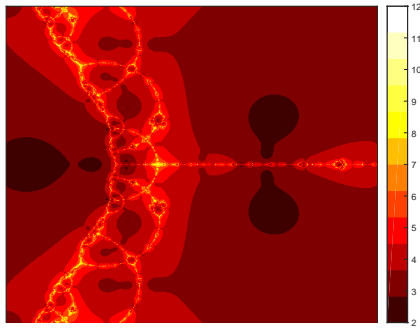
**Figure 4.** Polynomiograph for  $\alpha^1 = 0.45, \alpha^2 = 0.85$ .



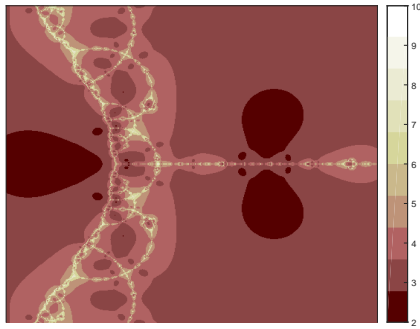
**Figure 5.** Polynomiograph for  $\alpha^1 = 0.2, \alpha^2 = 0.8$ .



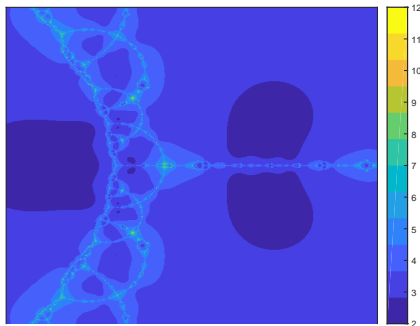
**Figure 6.** Polynomiograph for  $\alpha^1 = 0.05, \alpha^2 = 0.04$ .



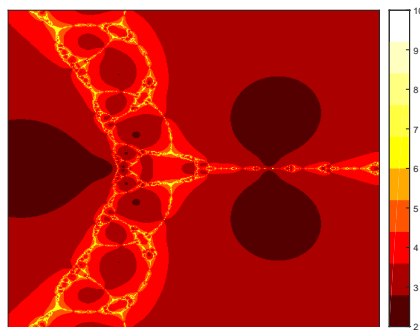
**Figure 7.** Polynomiograph for  $\alpha^1 = 0.1, \alpha^2 = 0.3$ .



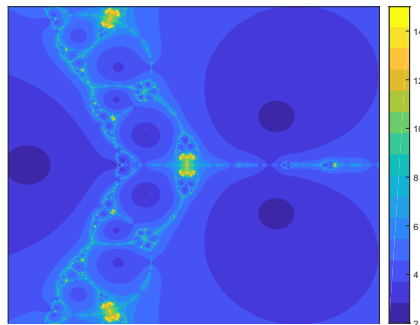
**Figure 8.** Polynomiograph for  $\alpha^1 = 0.3, \alpha^2 = 0.3$ .



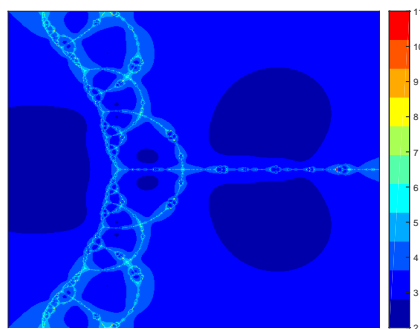
**Figure 9.** Polynomiograph for  $\alpha^1 = 0.5, \alpha^2 = 0.4$ .



**Figure 10.** Polynomiograph for  $\alpha^1 = 0.85, \alpha^2 = 0.15$ .



**Figure 11.** Polynomiograph for  $\alpha^1 = 0.02, \alpha^2 = 0.05$ .



**Figure 12.** Polynomiograph for  $\alpha^1 = 0.6, \alpha^2 = 0.7$ .

## 5. Conclusions

We have introduced  $K^*$  iteration for finding FP of a general class of contractive-like operators. Theorem 3.1 shows that our iterative method converges strongly to the FP of contractive-like operators.

In Theorem 3.3 it is concluded that  $K^*$  iteration procedure performs faster than the leading Kadioglu & Yildirim iteration process (2.10). Example 3.4 is given to verify our claim. We have presented polynomiographs for complex cubic polynomials via  $K^*$  iteration process. A large variety of nicely looking aesthetic patterns can be obtained by changing parameters  $\alpha^1$  and  $\alpha^2$  and  $\varepsilon$  involved in our iterative procedure.

### Conflict of interest

The authors declare that they do not have any competing interests.

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