Research article

Bifurcation phenomena in a single-species reaction-diffusion model with spatiotemporal delay

Gaoxiang Yang[1,*] **and Xiaoyu Li**[2]

[1] School of Mathematics and Statistics, Ankang University, Ankang, Shaanxi 725000, China
[2] School of Computer Science and Engineering, Xi'an University of Technology, Xi'an, Shaanxi 710048, China

* **Correspondence:** Email: gaoxiang7661@126.com; Tel: +8609153206532.

Abstract: In this paper we investigate bifurcation phenomena in a single-species reaction-diffusion model with spatiotemporal delay under the conditions of the weak and strong kernel functions. We have found that when the weak kernel function is introduced there is Hopf bifurcation but no Turing bifurcation and wave bifurcation to occur, but when the strong kernel function is introduced there exist Hopf bifurcation and wave bifurcation but no Turing bifurcation to occur. Especially, taking the inverse of the average time delay as a bifurcation parameter, we investigate influences of the time delay on the formation of spatiotemporal patterns through the numerical method. Some spatiotemporal patterns induced by Hopf bifurcation and wave bifurcation are respectively shown to illustrate the mechanism of the complexity of spatiotemporal dynamics.

Keywords: spatiotemporal delay; kernel function; Hopf bifurcation; wave bifurcation; spatiotemporal patterns
Mathematics Subject Classification: 35K57, 35B32

1. Introduction

A single-species reaction-diffusion model with spatiotemporal delay which was firstly proposed by Britton [1] is considered as follows:

$$\frac{\partial u}{\partial t} = d\Delta u + ru\left(1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)(\psi * u)\right), \tag{1.1}$$

where $u = u(x, t)$ for $(x, t) \in \mathbb{R} \times [0, +\infty)$ presents the density of a single-species at space x and time t, ψ is a given function and $\psi * u$ means a convolution in space x and time t, the parameters d, r, α and β are all positive constants and $1 + \alpha - \beta > 0$. For the biological explanations of all terms in the model (1.1),

according to references [1, 2], we have known that $d > 0$ is the diffusive coefficient, the term αu is a measure of the advantage to individuals in aggregating or grouping and βu^2 represents competition for space. Besides, the spatio-temporal convolution $\psi * u$, which represents competition between the individuals for food resources, is interpreted as a weighted average of the values of u over all past time and over all points in space and has the following form

$$(\psi * u)(x, t) = \int_{\mathbb{R}} \int_{-\infty}^{t} \psi(x - y, t - s)u(y, s)\mathrm{d}y\mathrm{d}s,$$

where $\psi(x, t)$ is a given positive kernel function which satisfies

$$\int_{\mathbb{R}} \int_{0}^{+\infty} \psi(x, t)\mathrm{d}x\mathrm{d}t = 1.$$

For model (1.1), a lot of work had been done by many researchers Britton [1], Gourley [3] and Billingham [4] who had shown that there exist the travelling wave front solution and the periodic travelling wave solution.

However, in the recent several decades there has been an increasing interest in the two species reaction-diffusion equations that incorporate spatially and temporally nonlocal terms in the form of the convolution of a kernel function, see [5–9]. Their studied results demonstrated that the predator-prey models with the nonlocal term have many bifurcation phenomena which lead to the complex spatio-temporal patterns to occur. For example, the work of [5] had shown that when the nonlocal term was introduced the more bifurcation phenomena would occur than the system without the nonlocal term. Besides, in order to explore the mechanism of emergence of complex spatiotemporal patterns, the bifurcation behaviours including Hopf bifurcation, Turing bifurcation, wave bifurcation, Turing-Hopf bifurcation an so on, are also investigated in reaction-diffusion equations by many researchers [10–15]. The definitions of these bifurcations are introduced in the references [16, 17] and have the important relations with roots of the characteristic equation of the studied system. The complex spatiotemporal patterns which include the spatially inhomogeneous periodic solution, homogeneous solution [10], stationary spatial patterns [11] and quasi-periodic solution [12] are also pointed out by them, respectively.

In this paper, according to [1], we take the particular kernel functions on the basis of a random walk argument as follows

$$\psi(x, t) = \frac{1}{\sqrt{4d\pi t}} \exp\left(-\frac{|x|^2}{4dt}\right)\tau \exp\left(-\tau t\right)$$

and

$$\psi(x, t) = \frac{1}{\sqrt{4d\pi t}} \exp\left(-\frac{|x|^2}{4dt}\right)\tau^2 t \exp\left(-\tau t\right),$$

which are respectively called the weak kernel and strong kernel, where x is the spatial variable, t is the temporal variable, $\tau = \frac{1}{\tilde{\tau}}$ and $\tilde{\tau}$ called the average time delay means the competition time for food resources. We mainly consider all kinds of bifurcation phenomena including Hopf bifurcation, Turing bifurcation and wave bifurcation for Eq (1.1) with the weak and strong kernel functions. We take the inverse of the average time delay in the kernel function as a bifurcation parameter and explore influences of the time delay on the emergence of the different bifurcation behaviours including Hopf bifurcation, Turing bifurcation and wave bifurcation. The studied results have found that there only

exists Hopf bifurcation in Eq (1.1) with the weak kernel and there are Hopf bifurcation and wave bifurcation in Eq (1.1) with the strong kernel. Furthermore, we suppose that Eq (1.1) has the positive initial condition $\phi(x)$ for $x \in \mathbb{R}$. And because there is zero population flux across the boundary tending to the infinity the following homogeneous Neumman boundary condition $\frac{\partial u(x,t)}{\partial n} = 0$ is supposed for Eq (1.1), where n is the outward unit normal vector of the boundary which tends to infinity.

The rest of this paper is organized as follows. In Section 2, the bifurcation analyses of a single-species reaction-diffusion model with spatiotemporal delay including the weak kernel and the strong kernel is given, respectively. The conditions of occurrence of Hopf and wave bifurcation are also obtained. In Section 3, we investigate the effect of the time delay on the spatiotemporal patterns through the numerical method. We finally give some conclusions in Section 4.

2. Bifurcation analysis under the different kernel functions

In this section we will employ the linearized analysis to study bifurcation phenomena of Eq (1.1) as the parameter τ varies. The conditions of the emergence of Hopf bifurcation, Turing bifurcation and wave bifurcation in Eq (1.1) under the weak kernel and strong kernel are also given, respectively.

2.1. The case of the weak kernel function

When the kernel function has the following weak form

$$\psi(x, t) = \frac{1}{\sqrt{4d\pi t}} \exp\left(-\frac{|x|^2}{4dt}\right)\tau \exp\left(-\tau t\right),$$

according to the results in the references [1, 18], we let

$$v(x, t) = \psi * u, \tag{2.1}$$

which leads to transform Eq (1.1) into the following form

$$\begin{cases} \frac{\partial u}{\partial t} = d\Delta u + f_1(u, v), \\ \frac{\partial v}{\partial t} = d\Delta v + f_2(u, v), \end{cases} \tag{2.2}$$

where $f_1(u, v) = ru(1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)v)$, $f_2(u, v) = \tau(u - v)$. Eq (2.2) has three equilibrium points $(0, 0)$, $(1, 1)$ and $(-\frac{1}{\beta}, -\frac{1}{\beta})$. From the viewpoints of population biology, we pay our attention to the equilibrium point $E^+ = (1, 1)$ which corresponds to the maximum state of the studied species. The linearized equations of system (2.2) are as follows

$$\begin{cases} \frac{\partial u}{\partial t} = d\Delta u + a_{11}u + a_{12}v, \\ \frac{\partial v}{\partial t} = d\Delta v + a_{21}u + a_{22}v, \end{cases} \tag{2.3}$$

where

$$a_{11} = \frac{\partial f_1(u,v)}{\partial u}\Big|_{u=1,v=1} = r(\alpha - 2\beta),$$
$$a_{12} = \frac{\partial f_1(u,v)}{\partial v}\Big|_{u=1,v=1} = -r(1 + \alpha - \beta),$$
$$a_{21} = \frac{\partial f_2(u,v)}{\partial u}\Big|_{u=1,v=1} = \tau,$$
$$a_{22} = \frac{\partial f_2(u,v)}{\partial v}\Big|_{u=1,v=1} = -\tau.$$

Let

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \exp(\lambda t + i\mathbf{k} \cdot r), \tag{2.4}$$

where c_1 and c_2 are coefficients, k denotes the wave numbers in the X direction, and λ is the growth rate of perturbation in time t, i is the imaginary unit and $i^2 = -1$, r is the spatial vector in one-dimensional space. Thus, substituting Eq (2.4) into Eq (2.3), we get the characteristic equation which is

$$\lambda^2 + Tr(k^2)\lambda + \Delta(k^2) = 0, \tag{2.5}$$

where $Tr(k^2) = 2dk^2 - a_{11} - a_{22}$ and $\Delta(k^2) = (dk^2 - a_{11})(dk^2 - a_{22}) - a_{12}a_{21}$.

Assume that $\alpha > 2\beta$, from the equation $Tr(0) = a_{11} + a_{22} = 0$, we obtain

$$\tau = \tau_{Hc} = r(\alpha - 2\beta). \tag{2.6}$$

That is to say, when $\tau = \tau_{Hc}$, the characteristic Eq (2.5) for $k^2 = 0$ becomes

$$\lambda^2 + a_{11}a_{22} - a_{12}a_{21} = 0,$$

which has a pair of purely imaginary roots $\pm \sqrt{a_{11}a_{22} - a_{12}a_{21}} = \pm \sqrt{r\tau_{Hc}(1 + \beta)}$. Besides, the following transversality condition

$$\frac{\mathrm{d}\mathrm{Re}\lambda}{\mathrm{d}\tau}\bigg|_{\tau=\tau_{Hc}} = -\frac{1}{2} < 0$$

is also satisfied.

When $k^2 = 0$, the roots of Eq (2.5) will determine the local stability of the equilibrium point E^+. If the equilibrium point E^+ is stable for $k^2 = 0$, then for $k^2 \neq 0$, it is also stable because $Tr(k^2) = 2dk^2 + Tr(0)$ and $\Delta(k^2) = d^2k^4 + Tr(0)k^2 + \Delta(0)$ are both positive under the conditions of $Tr(0)$ and $\Delta(0) > 0$ through using the Routh-Hurwtiz criterion. Connecting the above analyses, the following results are obtained immediately.

Theorem 2.1 Assume that f_1 and $f_2 \in C^1(\mathbb{R}^2, \mathbb{R})$, $\alpha > 2\beta$, $Tr_0 = \tau - r(\alpha - 2\beta)$, $\Delta_0 = r\tau(1 + \beta)$ and $\tau = \tau_{Hc}$ is defined by (2.6).

(i) When $k^2 = 0$, the equilibrium point E^+ is asymptotically stable for $\tau > \tau_{Hc}$ and unstable for $0 \leq \tau < \tau_{Hc}$, and Eq (2.2) undergoes Hopf bifurcation at $\tau = \tau_{Hc}$ which leads to the spatially homogeneous periodic solution to occur;

(ii) When $k^2 \neq 0$, the equilibrium point E^+ of Eq (2.2) is also stable because roots of the characteristic Eq (2.5) have the negative real parts under the conditions of $Tr(k^2) = 2dk^2 + Tr_0$, $\Delta(k^2) = d^2k^4 + Tr_0k^2 + \Delta_0$, $Tr_0 > 0$ and $\Delta_0 > 0$. That is to say, there are not Turing bifurcation and wave bifurcation to occur.

2.2. The case of the srong kernel function

When the kernel function has the following strong form

$$\psi(x, t) = \frac{1}{\sqrt{4d\pi t}} \exp\left(-\frac{|x|^2}{4dt}\right)\tau^2 t \exp(-\tau t),$$

we use the same symbols as the subsection 2.1 for the convenience of our studies and let

$$\begin{cases} v(x,t) = \int_{\mathbb{R}} \int_{-\infty}^{t} \frac{1}{\sqrt{4d\pi(t-s)}} \exp\left(-\frac{|x-y|^2}{4d(t-s)}\right) \tau^2 (t-s) \exp\left(-\tau(t-s)\right) u(y,s) \mathrm{d}y \mathrm{d}s, \\ w(x,t) = \int_{\mathbb{R}} \int_{-\infty}^{t} \frac{1}{\sqrt{4d\pi(t-s)}} \exp\left(-\frac{|x-y|^2}{4d(t-s)}\right) \tau \exp\left(-\tau(t-s)\right) u(y,s) \mathrm{d}y \mathrm{d}s, \end{cases} \tag{2.7}$$

according to the results in the references [1, 18], which lead to (1.1) to transform into the following form

$$\begin{cases} \frac{\partial u}{\partial t} = d\Delta u + g_1(u,v,w), \\ \frac{\partial v}{\partial t} = d\Delta v + g_2(u,v,w), \\ \frac{\partial w}{\partial t} = d\Delta w + g_3(u,v,w), \end{cases} \tag{2.8}$$

where $g_1(u,v,w) = ru(1 + \alpha u - \beta u^2 - (1 + \alpha - \beta)v)$, $g_2(u,v,w) = \tau(w - v)$ and $g_3(u,v,w) = \tau(u - w)$. Similar to Eq (2.2), Eq (2.8) has three equilibrium points $(0,0,0)$, $(1,1,1)$ and $(-\frac{1}{\beta}, -\frac{1}{\beta}, -\frac{1}{\beta})$.

The linearized equations of (2.8) at $E_+ = (1,1,1)$ are

$$\begin{cases} \frac{\partial u}{\partial t} = d\Delta u + b_{11}u + b_{12}v + b_{13}w, \\ \frac{\partial v}{\partial t} = d\Delta v + b_{21}u + b_{22}v + b_{23}w, \\ \frac{\partial w}{\partial t} = d\Delta w + b_{31}u + b_{32}v + b_{33}w, \end{cases} \tag{2.9}$$

where

$$\begin{aligned} b_{11} &= \frac{\partial g_1(u,v,w)}{\partial u}\big|_{u=1,v=1,w=1} = r(\alpha - 2\beta), \\ b_{12} &= \frac{\partial g_1(u,v,w)}{\partial v}\big|_{u=1,v=1,w=1} = -r(1 + \alpha - \beta), \\ b_{13} &= \frac{\partial g_1(u,v,w)}{\partial w}\big|_{u=1,v=1,w=1} = 0, \\ b_{21} &= \frac{\partial g_2(u,v,w)}{\partial u}\big|_{u=1,v=1,w=1} = 0, \\ b_{22} &= \frac{\partial g_2(u,v,w)}{\partial v}\big|_{u=1,v=1,w=1} = -\tau, \\ b_{23} &= \frac{\partial g_2(u,v,w)}{\partial v}\big|_{u=1,v=1,w=1} = \tau, \\ b_{31} &= \frac{\partial g_3(u,v,w)}{\partial u}\big|_{u=1,v=1,w=1} = \tau, \\ b_{32} &= \frac{\partial g_3(u,v,w)}{\partial v}\big|_{u=1,v=1,w=1} = 0, \\ b_{33} &= \frac{\partial g_3(u,v,w)}{\partial v}\big|_{u=1,v=1,w=1} = -\tau. \end{aligned}$$

Similarly, we can get the characteristic equation as follows

$$\lambda^3 + m_1(k^2)\lambda^2 + m_2(k^2)\lambda + m_3(k^2) = 0, \tag{2.10}$$

where

$$\begin{aligned} m_1(k^2) &= 3dk^2 - b_{11} + 2\tau, \\ m_2(k^2) &= (dk^2 + \tau)^2 + 2(dk^2 + \tau)(dk^2 - b_{11}), \\ m_3(k^2) &= (dk^2 + \tau)^2(dk^2 - b_{11}) - \tau^2 b_{12}. \end{aligned}$$

According to the Routh-Hurwitz criterion, the conditions of the emergence of Hopf bifurcation, Turing bifurcation and wave bifurcation have the relation with the distribution of roots of Eq (2.10) at $k^2 = 0$ and $k^2 \neq 0$, respectively. That is to say, we need to discuss whether the following results hold.

For $k^2 = 0$, if $m_1(0) > 0$, $m_3(0) > 0$ and $m_1(0)m_2(0) - m_3(0) > 0$ are satisfied, then the roots of Eq (2.10) have the negative real parts which lead the equilibrium point E_+ to be asymptotically stable.

For $k^2 = 0$, if $m_1(0) > 0$ and $m_1(0)m_2(0) - m_3(0) = 0$ hold, then Eq (2.10) has a pair of purely imaginary roots which lead to Hopf bifurcation to occur at the equilibrium point E_+.

For $k^2 \neq 0$, when the characteristic Eq (2.10) has a zero root and the other roots with negative real parts under the conditions of $m_1(0) > 0, m_2(0) > 0$ and $m_3(0) > 0$, there is possible Turing bifurcation to occur in Eq (2.8).

For $k^2 \neq 0$, when the characteristic Eq (2.10) has a pair of purely imaginary roots and the another root with negative real part under the conditions of $m_1(0) > 0, m_2(0) > 0$ and $m_3(0) > 0$, there possibly exist wave bifurcation in Eq (2.8).

According to the above discussions of the root distributions of Eq (2.10) the following results are obtained immediately.

Theorem 2.2 Assume that g_1, g_2 and $g_3 \in C^1(\mathbb{R}^3, \mathbb{R})$, $17\beta - 9\alpha - 1 < 0$, $m_1(0) > 0, m_3(0) > 0$ and $\tau = \tau_{shc\pm}, \tau = \tau_{swc}$ are defined by (2.13) and (2.19), respectively.

(i) For $k^2 = 0$, the constant equilibrium point E_+ is asymptotically stable for $0 \leq \tau < \tau_{shc-}$ and $\tau > \tau_{shc+}$ and unstable for $\tau_{shc-} < \tau < \tau_{shc+}$, and Eq (2.8) undergoes the Hopf bifurcation at $\tau = \tau_{shc\pm}$;

(ii) For $k^2 \neq 0$, under the conditions of $m_1(0) > 0$, $m_3(0) > 0$ and $m_1(0)m_2(0) - m_3(0) > 0$, there does not exist the Turing bifurcation in Eq (2.8);

(iii) For $k^2 \neq 0$, under the conditions of conditions of $m_1(0) > 0$, $m_3(0) > 0$ and $m_1(0)m_2(0) - m_3(0) > 0$, Eq (2.8) undergoes the wave bifurcation at $\tau = \tau_{swc}$ at the critical wave number $k = k_{wc} = \frac{b_{11} - \tau_{swc}}{2d}$.

Proof: (i) For $k^2 = 0$, the characteristic equation becomes into

$$\lambda^3 + m_1(0)\lambda^2 + m_2(0)\lambda + m_3(0) = 0, \tag{2.11}$$

where $m_1(0) = 2\tau - b_{11}$, $m_2(0) = \tau^2 - 2\tau b_{11}$ and $m_3(0) = -\tau^2(b_{11} + b_{12})$. If Eq (2.8) exists the Hopf bifurcation, the sufficient conditions are that Eq (2.11) has a pair of purely imaginary roots $\pm i\omega_0(\omega_0 > 0)$ and the another root has the negative real part. Substituting $\lambda = \pm i\omega_0$ into the Eq (2.11), we get

$$m_1(0)m_2(0) - m_3(0) = 0. \tag{2.12}$$

Through the detailed calculations about Eq (2.12), under the condition of $17\beta - 9\alpha - 1 < 0$ the critical value of the bifurcation parameter τ is as follows

$$\tau = \begin{cases} \tau_{shc+} = \frac{4b_{11} - b_{12} + \sqrt{b_{12}^2 - 8b_{11}b_{12}}}{4}, \\ \tau_{shc-} = \frac{4b_{11} - b_{12} - \sqrt{b_{12}^2 - 8b_{11}b_{12}}}{4}. \end{cases} \tag{2.13}$$

And the another root of Eq (2.11) is $\lambda = -m_1(0)$. In order to obtain the transversality condition, we need to differentiate about the parameter τ on the two sides of Eq (2.11) and get

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \frac{-\lambda^2 \frac{\mathrm{d}m_1(0)}{\mathrm{d}\tau} - \lambda \frac{\mathrm{d}m_2(0)}{\mathrm{d}\tau} - \frac{\mathrm{d}m_3(0)}{\mathrm{d}\tau}}{3\lambda^2 + 2m_1(0)\lambda + m_2(0)}. \tag{2.14}$$

Connecting

$$\frac{\mathrm{d}m_1(0)}{\mathrm{d}\tau} = 2, \frac{\mathrm{d}m_2(0)}{\mathrm{d}\tau} = 2\tau - 2b_{11} \text{ and } \frac{\mathrm{d}m_3(0)}{\mathrm{d}\tau} = -2\tau(b_{11} + b_{12}),$$

we have

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \frac{-2\lambda^2 - 2\lambda(\tau - b_{11}) + 2\tau(b_{11} + b_{12})}{3\lambda^2 + 2m_1(0)\lambda + m_2(0)}.$$

So when $\tau = \tau_{shc\pm}$,

$$\frac{d\lambda}{d\tau}\bigg|_{\tau=\tau_{shc\pm}} = \frac{\omega_0^2 + \tau_{shc}(b_{11} + b_{12}) - \omega_0(\tau_{shc} - b_{11})i}{-\omega_0^2 + \omega_0(2\tau_{shc} - b_{11})i}.$$

And the transversality condition of occurrence of Hopf bifurcation

$$\frac{d\mathrm{Re}\lambda}{d\tau}\bigg|_{\tau=\tau_{shc\pm}} = \frac{3(\tau_{shc} - b_{11})^2}{\omega_0^2 + (2\tau_{shc} - b_{11})^2} > 0$$

is also satisfied.

(ii) For $k^2 \neq 0$, the conditions of Eq (2.8) undergoing Turing bifurcation are that Eq (2.10) has a zero root and the equilibrium point E_+ is asymptotically stable for $k^2 = 0$. These conditions need the following results simultaneously hold

$$\begin{cases} m_3(k^2) = 0, \\ m_1(0) > 0, \\ m_3(0) > 0, \\ m_1(0)m_2(0) - m_3(0) > 0. \end{cases} \tag{2.15}$$

Let $z = k^2$, then

$$m_3(k^2) = m_3(z) = (dz + \tau)^2(dz - b_{11} - \tau^2 b_{12}) = d^3 z^3 + d^2(2\tau - b_{11})z^2 + d\tau(\tau - 2b_{11})z - \tau^2(b_{11} + b_{12}).$$

The derivative function of $m_3(z)$ about the variable z is

$$\frac{dm_3(z)}{dz} = 3d^3 z^2 + 2d^2(2\tau - b_{11})z + d\tau(\tau - 2b_{11}) = 3d^3(z + \frac{2\tau - b_{11}}{3d})^2 - \frac{d(\tau + b_{11})^2}{3}. \tag{2.16}$$

Because the zero points of the derivative function $\frac{dm_3(z)}{dz}$ correspond the local extreme value of $m_3(z)$, we get the two zero points $z_1 = -\frac{\tau}{d}$ and $z_2 = \frac{2b_{11} - \tau}{3d}$ from $\frac{dm_3(z)}{dz} = 0$. Because $z = k^2$ is positive, we can obtain the only zero point $z = z_2 = \frac{2b_{11} - \tau}{3d} = k^2 > 0$ which leads to $2b_{11} - \tau > 0$. And connecting the equality $m_2(0) = \tau(\tau - 2b_{11})$, we get $m_2(0) < 0$. Then combining with the inequalities $m_1(0) > 0$ and $m_3(0) > 0$, we have $m_1(0)m_2(0) - m_3(0) < 0$. There exists a contradiction with $m_1(0)m_2(0) - m_3(0) > 0$. This contradiction demonstrates that for $k^2 \neq 0$ there does not exist Turing bifurcation in Eq (2.8).

(iii) For $k^2 \neq 0$, the conditions of Eq (2.8) undergoing wave bifurcation are that Eq (2.10) has a pair of purely imaginary roots and a root with the negative real and the equilibrium point E_+ is asymptotically stable for $k^2 = 0$. These conditions need also the following results simultaneously hold

$$\begin{cases} m_1(k^2)m_2(k^2) - m_3(k^2) = 0, \\ m_1(0) > 0, \\ m_3(0) > 0, \\ m_1(0)m_2(0) - m_3(0) > 0. \end{cases} \tag{2.17}$$

In order to get the critical value of the bifurcation parameter τ about the wave bifurcation, we firstly let $z = k^2$ and $F(z) = m_1(z)m_2(z) - m_3(z)$. By the detailed calculations, we get the expression of function $F(z)$ about the variable z as follows

$$F(z) = 8d^3 z^3 + 8d^2(2\tau - b_{11})z^2 + 2d(\tau - b_{11})(5\tau - b_{11})z + \tau(2\tau - b_{11})(\tau - 2b_{11}) + \tau^2(b_{11} + b_{12}).$$

The derivative function of $F(z)$ about the variable z is

$$\frac{dF(z)}{dz} = 24d^3z^2 + 16d^2(2\tau - b_{11})z + 2d(\tau - b_{11})(5\tau - b_{11}). \tag{2.18}$$

Letting $\frac{dF(z)}{dz} = 0$, we obtain that the zero points of the derivative function $\frac{dF(z)}{dz}$ are $z_{min} = \frac{b_{11}-5\tau}{6d}$ and $z_{max} = \frac{b_{11}-\tau}{2d}$ which correspond the local extreme maximum value and local minimum value of the function $F(z)$, respectively. Let the local extreme minimum $F(z_{max})$ of the function $F(z)$ be equal to zero, we get the critical value of the bifurcation parameter

$$\tau_{swc} = \frac{b_{11}}{b_{11} + b_{12}}, \tag{2.19}$$

and the corresponding critical wave number $k_{wc}^2 = z_{max} = \frac{b_{11}-\tau_{swc}}{2d}$. In order to obtain the transversality condition, we need to differentiate about the parameter τ on the two sides of Eq (2.10) and get

$$\frac{d\lambda}{d\tau} = \frac{-\lambda^2 \frac{dm_1(k^2)}{d\tau} - \lambda \frac{dm_2(k^2)}{d\tau} - \frac{dm_3(k^2)}{d\tau}}{3\lambda^2 + 2m_1(k^2)\lambda + m_2(k^2)}. \tag{2.20}$$

Connecting

$$\frac{dm_1(k^2)}{d\tau} = 2, \frac{dm_2(k^2)}{d\tau} = 2(dk^2 + \tau) + 2(dk^2 - b_{11}) \text{ and } \frac{dm_3(k^2)}{d\tau} = 2(dk^2 - b_{11})(dk^2 + \tau) - 2\tau b_{12},$$

we have

$$\frac{d\lambda}{d\tau} = \frac{p_1}{3\lambda^2 + 2m_1(k^2)\lambda + m_2(k^2)},$$

where

$$p_1 = -2\lambda^2 - 2\lambda((dk^2 + \tau) + (dk^2 - b_{11})) - 2((dk^2 - b_{11})(dk^2 + \tau) - \tau b_{12}).$$

So when $\tau = \tau_{swc}$ and $k^2 = k_{wc}^2$,

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_{swc}} = \frac{p_2}{q},$$

where

$$p_2 = \omega_0^2 - ((dk_{wc}^2 - b_{11})(dk_{wc}^2 + \tau_{swc}) - \tau_{swc}b_{12}) + \omega_0((dk_{wc}^2 + \tau_{swc}) + (dk_{wc}^2 - b_{11}))i$$

and

$$q = -\omega_0^2 + m_1(k_{wc}^2)\omega_0 i.$$

The transversality condition of the wave bifurcation

$$\left.\frac{d\text{Re}\lambda}{d\tau}\right|_{\tau=\tau_{swc}} = \frac{\tau_{shc}r(1 + \alpha - \beta)}{\omega_0^2 + m_1^2(k_{wc}^2)} > 0$$

is also satisfied. The proof is completed.

3. Numerical simulations of spatiotemporal patterns

In this section we will give some numerical results when this single-species reaction-diffusion model with the strong kernel function. According to the results in Theorem 2.2 of subsection 2.2, we fix $d = 1, r = 2, \beta = 0.2$ and take the parameters τ and α as variables. The pictures of Hopf bifurcation curve and wave bifurcation curve are given in the parameter plane spanned by τ and α, see Figure 1.
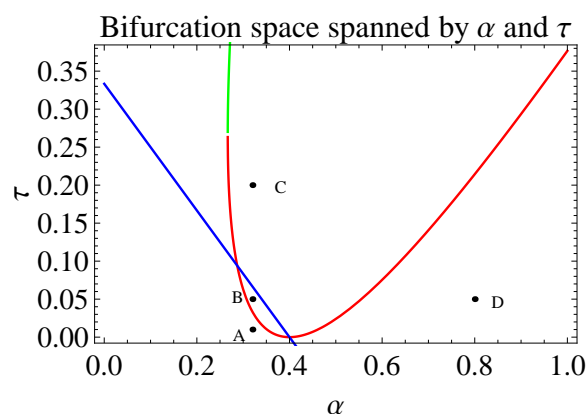

Figure 1. Parameter space of bifurcation spanned by α and τ according to Theorem 2.2. Red line and green line are Hopf bifurcation lines which is determined by τ_{shc-} and τ_{shc+}, respectively, and blue line is wave bifurcation line.

Next, we fix the parameters $\alpha = 0.32$ and obtain the critical value of $\tau_{shc-} = 0.333939$ about Hopf bifurcation and the critical value of $\tau_{swc} = 0.066667$ about wave bifurcation, respectively. If we take $\tau = 0.01$, $\tau = 0.05$ and $\tau = 0.2$, respectively corresponding to the points A, B and C, we would get the spatiotemporal evolution pictures of u with the initial conditions $u(0, x) = 1 - 0.1\cos(1.3x)$, $v(0, x) = 1 - 0.1\cos(1.3x)$ and $w(0, x) = 1 - 0.1\cos(1.3x)$, see Figure 2 (A–C).

Because of $\tau = 0.01 < 0.333939 = \tau_{shc-}$, Figure 2 (A) presents that the state of u will tend to the stable state $E_+ = (1, 1, 1)$ as the time varies.

Because of $\tau = 0.05$ and $\tau_{shc-} = 0.333939 < 0.05 < 0.066667 = \tau_{swc}$, Figure 2 (B) describes that there exists the spatially homogeneous and temporally periodic state induced by Hopf bifurcation.

Because of $\tau = 0.2 > 0.066667 = \tau_{swc}$, Figure 2 (C) shows that there exists the temporally oscillation pattern induced by the interactions of Hopf bifurcation and wave bifurcation. Because the Hopf bifurcation is much more prominent than the wave bifurcation, we would obtain the seriously oscillatory pattern in the temporal direction.

Besides, if we take the parameters $\alpha = 0.8$ and $\tau = 0.05$ corresponding to the point D, we get the spatiotemporal pictures about wave bifurcation which leads to the spatially heterogeneous pattern to occur, see Figure 2 (D).
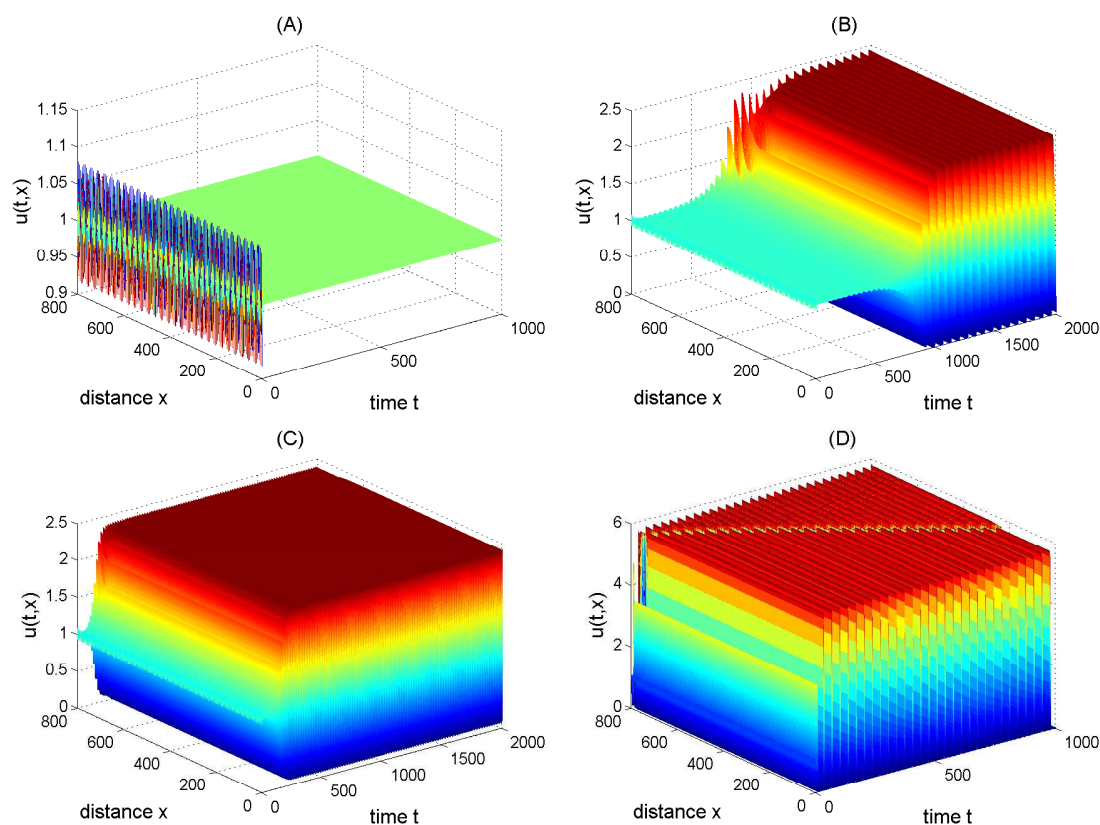
Figure 2. Spatiotemporal evolution pictures of species u at the different position of parameter space spanned by α and τ. (A) $\alpha = 0.32, \tau = 0.01$, (B) $\alpha = 0.32, \tau = 0.05$, (C) $\alpha = 0.32, \tau = 0.2$ and (D) $\alpha = 0.8, \tau = 0.05$.

4. Conclusions

The bifurcation behaviours resulting in the occurrence of all kinds of patterns in the reaction-diffusion equations have been focused in the mathematical biology. It is well known that these bifurcation behaviors including Hopf bifurcation, Turing bifurcation, wave bifurcation and interactions of them are reported by many researchers. Recently, the spatiotemporal patterns in the reaction-diffusion equations due to the presence of the nonlocal reaction have been a fascinating subject on the study of the pattern formation [5, 8]. What's more, the effect of the kernel functions on the spatial pattern formation are also discussed by [19]. In this paper, we mainly pay our attention to influences of the different kernel function, which are respectively called weak kernel and strong kernel, on pattern formation in a single-species reaction-diffusion equation. We have found that when the weak kernel function is introduced there only exists Hopf bifurcation which induces the spatially homogeneous pattern to occur, but when the strong kernel is introduced there are Hopf bifurcation and wave bifurcation which can induce the spatially homogeneous pattern and the spatially heterogeneous pattern to occur, respectively. It is shown that the rich spatiotemporal patterns occur when the different kernel functions are introduced.

Acknowledgments

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. N. F. Britton, Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model, *SIAM J. Appl. Math.,* **50** (1990), 1663–1688.

2. N. F. Britton, Aggregation and the competitive exclusion principle, *J. Theor. Biol.,* **136** (1989), 57–66.

3. S. A. Gourley, M. A. Chaplain, F. A. Davidson, Spatio-temporal pattern formation in a nonlocal reaction-diffusion equation, *Dyn. Syst.,* **16** (2001), 173–192.

4. J. Billingham, Dynamics of a strongly nonlocal reaction-diffusion population model, *Nonlinearity,* **17** (2004), 313–346.

5. S. Merchant, W. Nagata, Instabilities and spatiotemporal patterns behind predator invasions with nonlocal prey competition, *Theor. Popul. Biol.,* **80** (2011), 289–297.

6. W. Zuo, Y. Song, Stability and bifurcation analysis of a reaction-diffusion equation with spatio-temporal delay, *J. Math. Anal. Appl.,* **430** (2015), 243–261.

7. B. Han, Z. Wang, Turing patterns of a Lotka-Volterra competitive system with nonlocal delay, *Int. J. Bifurcat. Chaos,* **28** (2018), 1830021.

8. M. Banerjee, V. Volpert, Spatio-temporal pattern formation in Rosenzweig-MacArthur model: Effect of nonlocal interactions, *Ecol. Complex.,* **30** (2017), 2–10.

9. S. Wu, Y. Song, Stability and spatiotemporal dynamics in a diffusive predator-prey model with nonlocal prey competition, *Nonlinear Anal.-Real World Appl.,* **48** (2019), 12–39.

10. H. Jiang, Turing bifurcation in a diffusive predator-prey model with schooling behavior, *Appl. Math. Lett.,* **96** (2019), 230–235.

11. M. Baurmann, T. Gross, U. Feudel, Instabilities in spatially extended predator-prey systems: Spatio-temporal patterns in the neighborhood of Turing-Hopf bifurcations, *J. Theor. Biol.,* **245** (2007), 220–229.

12. Y. Song, H. Jiang, Y. Yuan, Turing-Hopf bifurcation in the reaction-diffusion system with delay and application to a diffusive predator-prey model, *J. Appl. Anal. Comput.,* **9** (2019), 1132–1146.

13. Y. Song, T. Zhang, Y. Peng, Turing-Hopf bifurcation in the reaction-diffusion equations and its applications, *Commun. Nonlinear Sci.,* **33** (2016), 229–258.

14. Y. Song, S. Wu, H. Wang, Spatiotemporal dynamics in the single population model with memory-based diffusion and nonlocal effect, *J. Differ. Equations,* **267** (2019), 6316–6351.

15. L. Yang, M. Dolnik, A. M. Zhabotinsky, I. R. Epstein, Pattern formation arising from interactions between Turing and wave instabilities, *J. Chem. Phys.,* **117** (2002), 7259–7265.

16. B. D. Hassard, N. D. Kazarinoff, Y. H. Wan, *Theory and applications of Hopf bifurcation*, Cambridge University Press, 1981.

17. J. D. Murray, *Mathematical biology I: An introduction (3 Eds.)*, Springer-Verlag, 2002.

18. S. A. Gourley, J. W. H. So, Dynamics of a food-limited population model incorporating nonlocal delays on a finite domain, *J. Math. Biol.,* **77** (2002), 49–78.

19. S. Pal, S. Ghorai, M. Banerjee, Effect of kernels on spatio-temporal patterns of a non-local prey-predator model, *Math. Biosci.,* **310** (2019), 96–107.