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Research article

Stability analysis of boundary value problems for Caputo proportional fractional derivative of a function with respect to another function via impulsive Langevin equation

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Abstract: In this paper, we discuss existence and stability results for a new class of impulsive fractional boundary value problems with non-separated boundary conditions containing the Caputo proportional fractional derivative of a function with respect to another function. The uniqueness result is discussed via Banach's contraction mapping principle, and the existence of solutions is proved by using Schaefer's fixed point theorem. Furthermore, we utilize the theory of stability for presenting different kinds of Ulam's stability results of the proposed problem. Finally, an example is also constructed to demonstrate the application of the main results.

Keywords: existence and uniqueness; fractional Langevin equation; fixed point theorems; impulsive conditions; Ulam-Hyers stability
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1. Introduction

Fractional calculus is the generalization of the ordinary differentiation and integration to non-integer order. It has been applied in various fields such as visco-elastic materials, aerodynamics, finance, chaotic dynamics, nonlinear control, signal processing, bioengineering, chemical engineering, and applied sciences. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of many materials and processes. However, for the last few years, the fractional calculus was developed by many researchers. There are different definitions of fractional operators (derivative and integral) that have been presented such as

Riemann-Liouville, Caputo, Hadamard, Hilfer, Katugampola, and the generalized fractional operators, see [1–14] and references therein.

The impulsive differential equations have impulsive conditions at points of discontinuity. They have played an important role in discussing the dynamics process of various physical and evolutionary phenomena which have discontinuous jumps and abrupt changes in their state of systems. Such processes and phenomena appear in various applications. For some works on impulsive problems, we refer readers to [15–19] and references cited therein.

The Langevin differential equation (first introduced by Paul Langevin in 1908 to provide a complex illustration of Brownian motion [20]) is found an effective piece of equipment to explain the evolution of physical phenomena in fluctuating environments of mathematical physics. After that, the ordinary Langevin equation was replaced by the fractional Langevin equation in 1996 [21]. For some works on the fractional Langevin equation, see, for example, [22–26].

In recent years, many researchers attention studied the exclusive examination of the qualitative theory for fractional differential equations. It is existence and uniqueness theory and stability analysis. One of the most method used to examine the stability analysis of functional differential equations is the Ulam's stability such as Ulam-Hyers ($\mathbb{UH}$) stability, generalized Ulam-Hyers ($\mathbb{UH}$) stability, Ulam-Hyers-Rassias ($\mathbb{UHR}$) stability and generalized Ulam-Hyers-Rassias ($\mathbb{UHR}$) stability [27–34]. It has helpfulness in the field of numerical analysis and optimization because solving the exact solutions of the problems of fractional differential equations is very difficult. Consequently, it is imperative to develop the concepts of Ulam's stability for these problems because we need not get the exact solutions of the purpose problems when we study the properties of Ulam's stability. The qualitative theory encourages us obtain an efficient and reliable technique for approximately finding fractional differential equations because there exists a close exact solution when the purpose problem is Ulam's stable. Recently, many researchers attentively initiated and examined the existence, uniqueness, and different types of Ulam's stability of the solutions for nonlinear fractional differential equations with/without impulsive conditions; see [35–49] and references cited therein. To the best of our knowledge, there is no paper on impulsive fractional Langevin differential equations containing the Caputo proportional fractional derivative of a function concerning function.

Motivated by the papers mentioned above [13, 40, 47] and a series of papers was devoted to the investigation of existence, uniqueness, and Ulam's stability of solutions of the impulsive fractional Langevin differential equation within different kinds of fractional derivatives, this paper examines the existence results and Ulam's stability of solutions for a class of the following impulsive fractional Langevin differential equation with non-separated boundary conditions under the Caputo proportional derivative type of the form:

$$\begin{cases} {}_{t_k}^{C}\mathfrak{D}^{\beta_k,\rho,\psi_k}\left({}_{t_k}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k}+\lambda\right)x(t)=f(t,x(t),x(\mu t)), \quad t\neq t_k, \quad k=0,1,2,\ldots,m, \\ x(t_k^+)-x(t_k^-)=\varphi_k(x(t_k)), \quad k=1,2,\ldots,m, \\ {}_{t_k}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k}x(t_k^+)-{}_{t_{k-1}}^{\quad C}\mathfrak{D}^{\alpha_{k-1},\rho,\psi_{k-1}}x(t_k^-)=\varphi_k^*(x(t_k)), \quad k=1,2,\ldots,m, \\ \eta_1 x(0)+\kappa_1 x(T)=\xi_1, \qquad \eta_2{}_{t_0}^{C}\mathfrak{D}^{\alpha_0,\rho,\psi_0}x(0)+\kappa_2{}_{t_m}^{C}\mathfrak{D}^{\alpha_m,\rho,\psi_m}x(T)=\xi_2, \end{cases} \tag{1.1}$$

where ${}_{t_k}^{C}D^{\nu,\rho,\psi_k}$ denotes the Caputo proportional fractional derivative of order ν with respect to certain continuously differentiable and increasing function ψ_k with $\psi'(t)>0$ and $\nu\in\{\alpha_k,\beta_k\}$, $\alpha_k,\beta_k\in(0,1)$,

$1 < \alpha_k + \beta_k < 2$, $t \in J_k = (t_k, t_{k+1}] \subseteq J = [0, T] = \{0\} \cup (\bigcup_0^m J_k)$, $k = 0, 1, \ldots, m$. $0 = t_0 < t_1 < \cdot < t_m < t_{m+1} = T$ are impulsive points, $0 < \rho \leq 1$, $\lambda \in \mathbb{R}$, $\mu \in (0, 1)$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, φ_k, $\varphi_k^* \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \ldots, m$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$, $x(t_k^-) = x(t_k)$ and the given constants η_i, κ_i, $\xi_i \in \mathbb{R}$ for $i = 1, 2$.

The outline of the paper is as follows: Section 2 contains fundamental concepts from proportional fractional calculus and some basic lemmas needed in the sequel. An auxiliary result useful to transform problem (1.1) into an equivalent integral equation is proved in Section 2. The existence results are presented in Section 3, where the uniqueness result is proved via Banach's fixed point theorem and the existence result with the help of Schaefer's fixed point theorem. Furthermore, we study different types of Ulam's stability results for the problem (1.1). Finally, an illustrative example is constructed in Section 5 to illustrate the usefulness of the main results.

2. Preliminaries

In this section, we recall some notations, definitions, lemmas, and properties of proportional fractional derivative and fractional integral operators of a function with respect to another function that will be used throughout the remaining part of this paper. For more details, see [13, 14, 50].

Definition 2.1. (The proportional derivative of a function with respect to another function [13, 14]) *Take $\rho \in [0, 1]$ and the let the functions κ_0, $\kappa_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous such that for all $t \in \mathbb{R}$ we have*

$$\lim_{\rho \to 0^+} \kappa_1(\rho, t) = 1, \quad \lim_{\rho \to 0^+} \kappa_0(\rho, t) = 0, \quad \lim_{\rho \to 1^-} \kappa_1(\rho, t) = 0, \quad \lim_{\rho \to 1^-} \kappa_0(\rho, t) = 1,$$

and $\kappa_1(\rho, t) \neq 0$, $\rho \in [0, 1)$, $\kappa_0(\rho, t) \neq 0$, $\rho \in (0, 1]$. Let $\psi(t)$ be a continuously differentiable and increasing function. Then, the proportional differential operator of order ρ of f with respect to ψ is defined by

$$\mathfrak{D}^{\rho,\psi} f(t) = \kappa_1(\rho, t) f(t) + \kappa_0(\rho, t) \frac{f'(t)}{\psi'(t)}. \tag{2.1}$$

In particular, If $\kappa_1(\rho, t) = 1 - \rho$ and $\kappa_0(\rho, t) = \rho$, we get

$$\mathfrak{D}^{\rho,\psi} f(t) = (1 - \rho) f(t) + \rho \frac{f'(t)}{\psi'(t)}. \tag{2.2}$$

Definition 2.2. ([13, 14]*) Take $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0, 1]$, $\psi \in C^1([a, b])$, $\psi' > 0$. The proportional fractional integral of order α of the function $f \in L^1([a, b])$ with respect to another function ψ is defined by*

$${}_a\mathfrak{I}^{\alpha,\rho,\psi} f(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} (\psi(t) - \psi(s))^{\alpha-1} f(s) \psi'(s) ds, \tag{2.3}$$

where $\Gamma(\cdot)$ represents the Gamma function [4].

Definition 2.3. ([13, 14]*) Take $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0, 1]$, $\psi \in C([a, b])$, $\psi'(t) > 0$. The Riemann-Liouvill proportional fractional derivative of order α of the function $f \in C^n([a, b])$ with respect to another function ψ is defined by*

$${}_a\mathfrak{D}^{\alpha,\rho,\psi} f(t) = \mathfrak{D}^{n,\rho,\psi} {}_a\mathfrak{I}^{n-\alpha,\rho,\psi} f(t) = \frac{\mathfrak{D}_t^{n,\rho,\psi}}{\rho^{n-\alpha} \Gamma(n - \alpha)} \int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)} (\psi(t) - \psi(s))^{n-\alpha-1} f(s) \psi'(s) ds, \tag{2.4}$$

where $n = [Re(\alpha)] + 1$, $[Re(\alpha)]$ *represents the integer part of the real number* α *and* $\mathfrak{D}^{n,\rho,\psi} = \underbrace{\mathfrak{D}^{\rho,\psi}\mathfrak{D}^{\rho,\psi}\cdots\mathfrak{D}^{\rho,\psi}}_{n\ times}$.

Definition 2.4. ([13, 14]) *Take* $\alpha \in \mathbb{C}$, $Re(\alpha) > 0$, $\rho \in (0,1]$, $\psi \in C([a,b])$, $\psi'(t) > 0$. *The Caputo proportional fractional derivative of order* α *of the function* f *with respect to another function* ψ *is defined by*

$$
{}_a^C\mathfrak{D}^{\alpha,\rho,\psi}f(t) = {}_a\mathfrak{I}^{n-\alpha,\rho,\psi}\mathfrak{D}^{n,\rho,\psi}f(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)}\left(\psi(t)-\psi(s)\right)^{n-\alpha-1}\mathfrak{D}^{n,\rho,\psi}f(s)\psi'(s)ds. \tag{2.5}
$$

Lemma 2.5. ([13]) *Let* $\rho \in (0,1]$, $Re(\alpha) > 0$, $Re(\beta) > 0$. *Then, for* f *is continuous and defined for* $t \geq a$, *we have*

$$
{}_a\mathfrak{I}^{\alpha,\rho,\psi}{}_a\mathfrak{I}^{\beta,\rho,\psi}f(t) = {}_a\mathfrak{I}^{\beta,\rho,\psi}{}_a\mathfrak{I}^{\alpha,\rho,\psi}f(t) = {}_a\mathfrak{I}^{\alpha+\beta,\rho,\psi}f(t).
$$

Lemma 2.6. ([13]) *Let* $0 \leq m < [Re(\alpha)] + 1$ *and* f *be integrable in each interval* $[a,t]$, $t > a$. *Then*

$$
\mathfrak{D}^{m,\rho,\psi}{}_a\mathfrak{I}^{\alpha,\rho,\psi}f(t) = {}_a\mathfrak{I}^{\alpha-m,\rho,\psi}f(t).
$$

Corollary 2.7. ([13]) *Let* $0 < Re(\beta) < Re(\alpha)$ *and* $m-1 < Re(\beta) \leq m$. *Then, we have*

$$
{}_a\mathfrak{D}^{\beta,\rho,\psi}{}_a\mathfrak{I}^{\alpha,\rho,\psi}f(t) = {}_a\mathfrak{I}^{\alpha-\beta,\rho,\psi}f(t).
$$

Corollary 2.8. *Let* $0 < Re(\beta) < Re(\alpha)$ *and* $m-1 < Re(\beta) \leq m$. *Then, we have*

$$
{}_a^C\mathfrak{D}^{\beta,\rho,\psi}{}_a\mathfrak{I}^{\alpha,\rho,\psi}f(t) = {}_a\mathfrak{I}^{\alpha-\beta,\rho,\psi}f(t).
$$

Proof. By the help of Definition 2.4, Lemma 2.5 and Lemma 2.6, we have

$$
{}_a^C\mathfrak{D}^{\beta,\rho,\psi}{}_a\mathfrak{I}^{\alpha,\rho,\psi}f(t) = {}_a\mathfrak{I}^{m-\beta,\rho,\psi}\mathfrak{D}^{m,\rho,\psi}{}_a\mathfrak{I}^{\alpha,\rho,\psi}f(t) = {}_a\mathfrak{I}^{m-\beta,\rho,\psi}{}_a\mathfrak{I}^{\alpha-m,\rho,\psi}f(t) = {}_a\mathfrak{I}^{\alpha-\beta,\rho,\psi}f(t).
$$

The proof is completed. □

Next, the lemma presents the impact of the proportional fractional integral operator on the Caputo proportional fractional derivative operator of the same order.

Lemma 2.9. ([14]) *For* $\rho \in (0,1]$ *and* $n = [Re(\alpha)] + 1$, *we have* ${}_a^C\mathfrak{D}^{\alpha,\rho,\psi}{}_a\mathfrak{I}^{\alpha,\rho,\psi}f(t) = f(t)$, *and*

$$
{}_a\mathfrak{I}^{\alpha,\rho,\psi C}_{\ a}\mathfrak{D}^{\alpha,\rho,\psi}f(t) = f(t) - \sum_{k=0}^{n-1}\frac{\mathfrak{D}^{k,\rho,\psi}f(a)}{\rho^k k!}\left(\psi(t)-\psi(a)\right)^k e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(a)\right)}.
$$

Proposition 2.10. ([14]) *Let* $Re(\alpha) \geq 0$ *and* $Re(\beta) > 0$. *Then, for any* $\rho \in (0,1]$ *and* $n = [Re(\alpha)] + 1$, *we have*

(i) $\left({}_a\mathfrak{I}^{\alpha,\rho,\psi}e^{\frac{\rho-1}{\rho}\psi(s)}\left(\psi(s)-\psi(a)\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\rho^\alpha\Gamma(\beta+\alpha)}e^{\frac{\rho-1}{\rho}\psi(t)}\left(\psi(t)-\psi(a)\right)^{\beta+\alpha-1}$, $\quad Re(\alpha) > 0$.

(ii) $\left({}_a\mathfrak{D}^{\alpha,\rho,\psi}e^{\frac{\rho-1}{\rho}\psi(s)}\left(\psi(s)-\psi(a)\right)^{\beta-1}\right)(t) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\rho-1}{\rho}\psi(t)}\left(\psi(t)-\psi(a)\right)^{\beta-\alpha-1}$, $\quad Re(\alpha) \geq 0$.

(iii) $\left({}_a^C\mathfrak{D}^{\alpha,\rho,\psi}e^{\frac{\rho-1}{\rho}\psi(s)}(\psi(s)-\psi(a))^{\beta-1}\right)(t)=\frac{\rho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\rho-1}{\rho}\psi(t)}(\psi(t)-\psi(a))^{\beta-\alpha-1},\quad Re(\beta)>n.$
For $k=0,1,\ldots,n-1$, we have

$$\left({}_a^C\mathfrak{D}^{\alpha,\rho,\psi}e^{\frac{\rho-1}{\rho}\psi(s)}(\psi(s)-\psi(a))^{k}\right)(t)=0\quad and\quad \left({}_a^C\mathfrak{D}^{\alpha,\rho,\psi}e^{\frac{\rho-1}{\rho}\psi(s)}\right)(t)=0.$$

Throughout this paper, let $\mathbb{E}:=PC(J,\mathbb{R}):=\{x:J\to\mathbb{R}:x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)=x(t_k)$, $k=1,2,\ldots,m\}$ the space of piecewise continuous functions. Obviously, $(\mathbb{E},\|x\|)$ is a Banach space equipped with the norm $\|x\|:=\sup_{t\in J}|x(t)|$.

In the following, for the convenience for the reader, we set the functional equation $F_x(t)=f(t,x(t),x(\mu t))$, and we express the proportional fractional integral operator defined in (2.3) of a nonlinear function F_x by a subscript notation by

$$\begin{aligned}{}_a\mathfrak{I}^{\alpha,\rho,\psi}F_x(t) &= \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)}(\psi(t)-\psi(s))^{\alpha-1}F_x(s)\psi'(s)ds\\ &= \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_a^t e^{\frac{\rho-1}{\rho}\left(\psi(t)-\psi(s)\right)}(\psi(t)-\psi(s))^{\alpha-1}f(s,x(s),x(\mu s))\psi'(s)ds.\end{aligned}$$

In the sequel, for nonnegative integers $a<b$, we use the following notations:

$$\Phi^{c}(t_a,t_b) = \frac{(\psi_a(t_b)-\psi_a(t_a))^{c}}{\rho^{c}\Gamma(c+1)}, \tag{2.6}$$

$$G_i(x) = {}_{t_i}\mathfrak{I}^{\beta_i,\rho,\psi_i}F_x(t_{i+1})+\varphi_{i+1}^{*}(x(t_{i+1})), \tag{2.7}$$

$$H_i(x) = {}_{t_i}\mathfrak{I}^{\alpha_i+\beta_i,\rho,\psi_i}F_x(t_{i+1})-\lambda_{t_i}\mathfrak{I}^{\alpha_i,\rho,\psi_i}x(t_{i+1})+\varphi_{i+1}(x(t_{i+1})), \tag{2.8}$$

where $i=0,1,2,\ldots,m$.

In Lemma 2.11, we prepare an important lemma, which is used as the main results of the problem (1.1).

Lemma 2.11. *Let $0<\alpha_k,\beta_k<1$, $1<\alpha_k+\beta_k<2$, $0<\rho\le 1$, $F_x\in AC(J\times\mathbb{R}^2,\mathbb{R})$ for any $x\in C(J,\mathbb{R})$ and $\Omega_1\Omega_4\neq\Omega_2\Omega_3$. Then the following boundary value problem:*

$$\begin{cases} {}_{t_k}^{C}\mathfrak{D}^{\beta_k,\rho,\psi_k}\left({}_{t_k}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k}+\lambda\right)x(t)=F_x(t),\quad t\neq t_k,\quad k=0,1,2,\ldots,m,\\ x(t_k^+)-x(t_k^-)=\varphi_k(x(t_k)),\quad k=1,2,\ldots,m,\\ {}_{t_k}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k}x(t_k^+)-{}_{t_{k-1}}^{\;\;C}\mathfrak{D}^{\alpha_{k-1},\rho,\psi_{k-1}}x(t_k^-)=\varphi_k^{*}(x(t_k)),\quad k=1,2,\ldots,m,\\ \eta_1x(0)+\kappa_1x(T)=\xi_1,\qquad \eta_2{}_{t_0}^{C}\mathfrak{D}^{\alpha_0,\rho,\psi_0}x(0)+\kappa_2{}_{t_m}^{C}\mathfrak{D}^{\alpha_m,\rho,\psi_m}x(T)=\xi_2,\end{cases} \tag{2.9}$$

is equivalent to the following integral equation:

$$\begin{aligned}x(t) &= {}_{t_k}\mathfrak{I}^{\alpha_k+\beta_k,\rho,\psi_k}F_x(t)-\lambda_{t_k}\mathfrak{I}^{\alpha_k,\rho,\psi_k}x(t)+\left\{\sum_{i=1}^{k}H_{i-1}(x)\prod_{j=i}^{k-1}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\right.\\ &\quad+\sum_{i=1}^{k}G_{i-1}(x)\sum_{j=i}^{k-1}\left(\Phi^{\alpha_j}(t_j,t_{j+1})+\Phi^{\alpha_k}(t_k,t)\right)\prod_{j=i}^{k-1}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\end{aligned}$$

$$+\frac{\Omega_1\mathcal{R}(x,F_x)-\Omega_3\mathcal{K}(x,F_x)}{\Omega_5}\sum_{i=1}^{k}(\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+\Phi^{\alpha_k}(t_k,t))\prod_{i=1}^{k}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}$$
$$+\frac{\Omega_4\mathcal{K}(x,F_x)-\Omega_2\mathcal{R}(x,F_x)}{\Omega_5}\prod_{i=1}^{k}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\Bigg\}e^{\frac{\rho-1}{\rho}(\psi_k(t)-\psi_k(t_k))},\ t\in J_k, \tag{2.10}$$

where

$$\Omega_1 = \kappa_1\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\prod_{i=1}^{m+1}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}, \tag{2.11}$$

$$\Omega_2 = \eta_1+\kappa_1\prod_{i=1}^{m+1}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}, \tag{2.12}$$

$$\Omega_3 = \eta_2+\kappa_2\left(1-\lambda\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\right)\prod_{i=1}^{m+1}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}, \tag{2.13}$$

$$\Omega_4 = -\eta_2\lambda-\kappa_2\lambda\prod_{i=1}^{m+1}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}, \tag{2.14}$$

$$\Omega_5 = \Omega_1\Omega_4-\Omega_2\Omega_3, \tag{2.15}$$

$$\begin{aligned}\mathcal{K}(x,F_x) = {} & \xi_1-\kappa_1{}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}F_x(T)+\kappa_1\lambda{}_{t_m}\mathfrak{I}^{\alpha_m,\rho,\psi_m}x(T)\\ & -\kappa_1\sum_{i=1}^{m}G_{i-1}(x)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\\ & -\kappa_1\sum_{i=1}^{m}H_{i-1}(x)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)},\end{aligned} \tag{2.16}$$

$$\begin{aligned}\mathcal{R}(x,F_x) = {} & \xi_2-\kappa_2{}_{t_m}\mathfrak{I}^{\beta_m,\rho,\psi_m}F_x(T)+\kappa_2\lambda{}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}F_x(T)-\kappa_2\lambda^2{}_{t_m}\mathfrak{I}^{\alpha_m,\rho,\psi_m}x(T)\\ & -\kappa_2\sum_{i=1}^{m}G_{i-1}(x)\left(1-\lambda\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\right)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\\ & +\kappa_2\lambda\sum_{i=1}^{m}H_{i-1}(x)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)},\end{aligned} \tag{2.17}$$

where $\Phi^c(t_a,t_b)$*,* $G_{i-1}(x)$*,* $H_{i-1}(x)$ *are defind by* (2.6)*,* (2.7)*,* (2.8)*, respectively.*

Proof. Firstly, for $t\in J_0=[t_0,t_1]$, we transform the problem (2.9) into an integral equation by applying the proportional fractional integral of order $\beta_0\in(0,1)$ with respect to a function $\psi_0(t)$ to both sides of (2.9) and also using Lemma 2.9, we obtain

$${}^{C}_{t_0}\mathfrak{D}^{\alpha_0,\rho,\psi_0}x(t)={}_{t_0}\mathfrak{I}^{\beta_0,\rho,\psi_0}F_x(t)-\lambda x(t)+c_1e^{\frac{\rho-1}{\rho}(\psi_0(t)-\psi_0(t_0))},$$

where $c_1\in\mathbb{R}$.

In the same process, taking the proportional fractional integral of order $\alpha_0\in(0,1)$ with respect to a function $\psi_0(t)$ to both sides of (2), we get, for c_1, $c_2\in\mathbb{R}$,

$$x(t) = {}_{t_0}\mathfrak{I}^{\alpha_0+\beta_0,\rho,\psi_0}F_x(t)-\lambda{}_{t_0}\mathfrak{I}^{\alpha_0,\rho,\psi_0}x(t)$$

$$+c_1\left\{\frac{(\psi_0(t)-\psi_0(t_0))^{\alpha_0}}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}\right\}e^{\frac{\rho-1}{\rho}(\psi_0(t)-\psi_0(t_0))}+c_2e^{\frac{\rho-1}{\rho}(\psi_0(t)-\psi_0(t_0))}.$$

For $t\in J_1=(t_1,t_2]$, by applying the proportional fractional integral of order $\beta_1\in(0,1)$ with respect to a function $\psi_1(t)$ to both sides of (2.9) and again using Lemma 2.9, we have

$$ {}_{t_1}^{C}\mathfrak{D}^{\alpha_1,\rho,\psi_1}x(t)={}_{t_1}\mathfrak{J}^{\beta_1,\rho,\psi_1}F_x(t)-\lambda x(t)+d_1e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))}, \tag{2.18}$$

and the same method, it follows that

$$\begin{aligned} x(t) &= {}_{t_1}\mathfrak{J}^{\alpha_1+\beta_1,\rho,\psi_1}F_x(t)-\lambda_{t_1}\mathfrak{J}^{\alpha_1,\rho,\psi_1}x(t)\\ &\quad +d_1\frac{(\psi_1(t)-\psi_1(t_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))}+d_2e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))}, \end{aligned} \tag{2.19}$$

where $d_1,d_2\in\mathbb{R}$

By using impulsive conditions $x(t_1^+)=x(t_1^-)+\varphi_1(x(t_1))$ and ${}_{t_1}^{C}\mathfrak{D}^{\alpha_1,\rho,\psi_1}x(t_1^+)={}_{t_0}^{C}\mathfrak{D}^{\alpha_0,\rho,\psi_0}x(t_1^-)+\varphi_1^*(x(t_1))$, then

$$\begin{aligned} d_1 &= {}_{t_0}\mathfrak{J}^{\beta_0,\rho,\psi_0}F_x(t_1)+c_1e^{\frac{\rho-1}{\rho}(\psi_0(t_1)-\psi_0(t_0))}+\varphi_1^*(x(t_1)),\\ d_2 &= {}_{t_0}\mathfrak{J}^{\alpha_0+\beta_0,\rho,\psi_0}F_x(t_1)-\lambda_{t_0}\mathfrak{J}^{\alpha_0,\rho,\psi_0}x(t_1)+c_1\frac{(\psi_0(t_1)-\psi_0(t_0))^{\alpha_0}}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}e^{\frac{\rho-1}{\rho}(\psi_0(t_1)-\psi_0(t_0))}\\ &\quad +c_2e^{\frac{\rho-1}{\rho}(\psi_0(t_1)-\psi_0(t_0))}+\varphi_1(x(t_1)). \end{aligned}$$

Substituting d_1 and d_2 into (2.18) and (2.19), we obtain

$$\begin{aligned} {}_{t_1}^{C}\mathfrak{D}^{\alpha_1,\rho,\psi_1}x(t) &= {}_{t_1}\mathfrak{J}^{\beta_1,\rho,\psi_1}F_x(t)-\lambda x(t)+\left\{\left({}_{t_0}\mathfrak{J}^{\beta_0,\rho,\psi_0}F_x(t_1)+\varphi_1^*(x(t_1))\right)\right\}e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))}\\ &\quad +c_1\left\{e^{\frac{\rho-1}{\rho}(\psi_0(t_1)-\psi_0(t_0))}\right\}e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))},\quad t\in J_1,\\ x(t) &= {}_{t_1}\mathfrak{J}^{\alpha_1+\beta_1,\rho,\psi_1}F_x(t)-\lambda_{t_1}\mathfrak{J}^{\alpha_1,\rho,\psi_1}x(t)\\ &\quad +\left\{\left({}_{t_0}\mathfrak{J}^{\beta_0,\rho,\psi_0}F_x(t_1)+\varphi_1^*(x(t_1))\right)\frac{(\psi_1(t)-\psi_1(t_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\right\}e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))}\\ &\quad +\left\{\left({}_{t_0}\mathfrak{J}^{\alpha_0+\beta_0,\rho,\psi_0}F_x(t_1)-\lambda_{t_0}\mathfrak{J}^{\alpha_0,\rho,\psi_0}x(t_1)+\varphi_1(x(t_1))\right)\right\}e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))}\\ &\quad +c_1\left\{\left(\frac{(\psi_0(t_1)-\psi_0(t_0))^{\alpha_0}}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}+\frac{(\psi_1(t)-\psi_1(t_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\right)e^{\frac{\rho-1}{\rho}(\psi_0(t_1)-\psi_0(t_0))}\right\}e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))}\\ &\quad +c_2\left\{e^{\frac{\rho-1}{\rho}(\psi_0(t_1)-\psi_0(t_0))}\right\}e^{\frac{\rho-1}{\rho}(\psi_1(t)-\psi_1(t_1))},\quad t\in J_1. \end{aligned}$$

For $t\in J_2=(t_2,t_3]$, by using the proportional fractional integral of order $\beta_2\in(0,1)$ and $\alpha_2\in(0,1)$ with respect to a function $\psi_2(t)$ to both sides of (2.9), we have

$$\begin{aligned} {}_{t_2}^{C}\mathfrak{D}^{\alpha_2,\rho,\psi_2}x(t) &= {}_{t_2}\mathfrak{J}^{\beta_2,\rho,\psi_2}F_x(t)-\lambda x(t)+d_1e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}, \qquad (2.20)\\ x(t) &= {}_{t_2}\mathfrak{J}^{\alpha_2+\beta_2,\rho,\psi_2}F_x(t)-\lambda_{t_2}\mathfrak{J}^{\alpha_2,\rho,\psi_2}x(t) \end{aligned}$$

$$+d_3\frac{(\psi_2(t)-\psi_2(t_2))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}+d_4e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}. \tag{2.21}$$

where d_3, $d_4\in\mathbb{R}$. In view of the impulsive conditions $x(t_2^+)=x(t_2^-)+\varphi_2(x(t_2))$ and ${}^{C}_{t_2}\mathfrak{D}^{\alpha_2,\rho,\psi_2}x(t_2^+)={}^{C}_{t_1}\mathfrak{D}^{\alpha_1,\rho,\psi_1}x(t_2^-)+\varphi_2^*(x(t_2))$, we obtain

$$\begin{aligned}
d_3 &= \left({}_{t_0}\mathfrak{I}^{\beta_0,\rho,\psi_0}F_x(t_1)+\varphi_1^*(x(t_1))\right)e^{\frac{\rho-1}{\rho}(\psi_1(t_2)-\psi_1(t_1))}+{}_{t_1}\mathfrak{I}^{\beta_1,\rho,\psi_1}F_x(t_2)+\varphi_2^*(x(t_2))\\
&\quad +c_1e^{\frac{\rho-1}{\rho}\left[(\psi_0(t_1)-\psi_0(t_0))+(\psi_1(t_2)-\psi_1(t_1))\right]},\\
d_4 &= \frac{(\psi_1(t_2)-\psi_1(t_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\left({}_{t_0}\mathfrak{I}^{\beta_0,\rho,\psi_0}F_x(t_1)+\varphi_1^*(x(t_1))\right)e^{\frac{\rho-1}{\rho}(\psi_1(t_2)-\psi_1(t_1))}\\
&\quad +\left({}_{t_0}\mathfrak{I}^{\alpha_0+\beta_0,\rho,\psi_0}F_x(t_1)-\lambda_{t_0}\mathfrak{I}^{\alpha_0,\rho,\psi_0}x(t_1)+\varphi_1(x(t_1))\right)e^{\frac{\rho-1}{\rho}(\psi_1(t_2)-\psi_1(t_1))}\\
&\quad +\left({}_{t_1}\mathfrak{I}^{\alpha_1+\beta_1,\rho,\psi_1}F_x(t_2)-\lambda_{t_1}\mathfrak{I}^{\alpha_1,\rho,\psi_1}x(t_2)+\varphi_2(x(t_2))\right)\\
&\quad +c_1\left(\frac{(\psi_0(t_1)-\psi_0(t_0))^{\alpha_0}}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}+\frac{(\psi_1(t_2)-\psi_1(t_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}\right)e^{\frac{\rho-1}{\rho}\left[(\psi_0(t_1)-\psi_0(t_0))+(\psi_1(t_2)-\psi_1(t_1))\right]}\\
&\quad +c_2e^{\frac{\rho-1}{\rho}\left[(\psi_0(t_1)-\psi_0(t_0))+(\psi_1(t_2)-\psi_1(t_1))\right]}.
\end{aligned}$$

Substituting d_3 and d_4 into (2.20) and (2.21), we obtain

$$\begin{aligned}
{}^{C}_{t_2}\mathfrak{D}^{\alpha_2,\rho,\psi_2}x(t) &= {}_{t_2}\mathfrak{I}^{\beta_2,\rho,\psi_2}F_x(t)-\lambda x(t)\\
&\quad +\left\{\left({}_{t_0}\mathfrak{I}^{\beta_0,\rho,\psi_0}F_x(t_1)+\varphi_1^*(x(t_1))\right)e^{\frac{\rho-1}{\rho}(\psi_1(t_2)-\psi_1(t_1))}\right\}e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}\\
&\quad +\left\{\left({}_{t_1}\mathfrak{I}^{\beta_1,\rho,\psi_1}F_x(t_2)+\varphi_2^*(x(t_2))\right)\right\}e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}\\
&\quad +c_1\left\{e^{\frac{\rho-1}{\rho}\left[(\psi_0(t_1)-\psi_0(t_0))+(\psi_1(t_2)-\psi_1(t_1))\right]}\right\}e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))},\quad t\in J_2,\\
x(t) &= {}_{t_2}\mathfrak{I}^{\alpha_2+\beta_2,\rho,\psi_2}F_x(t)-\lambda_{t_2}\mathfrak{I}^{\alpha_2,\rho,\psi_2}x(t)\\
&\quad +\left\{\left({}_{t_0}\mathfrak{I}^{\beta_0,\rho,\psi_0}F_x(t_1)+\varphi_1^*(x(t_1))\right)\right.\\
&\quad \times\left(\frac{(\psi_1(t_2)-\psi_1(t_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+\frac{(\psi_2(t)-\psi_2(t_2))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)e^{\frac{\rho-1}{\rho}(\psi_1(t_2)-\psi_1(t_1))}\\
&\quad \left.+\left({}_{t_1}\mathfrak{I}^{\beta_1,\rho,\psi_1}F_x(t_2)+\varphi_2^*(x(t_2))\right)\frac{(\psi_2(t)-\psi_2(t_2))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right\}e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}\\
&\quad +\left\{\left({}_{t_0}\mathfrak{I}^{\alpha_0+\beta_0,\rho,\psi_0}F_x(t_1)-\lambda_{t_0}\mathfrak{I}^{\alpha_0,\rho,\psi_0}x(t_1)+\varphi_1(x(t_1))\right)e^{\frac{\rho-1}{\rho}(\psi_1(t_2)-\psi_1(t_1))}\right.\\
&\quad \left.+\left({}_{t_1}\mathfrak{I}^{\alpha_1+\beta_1,\rho,\psi_1}F_x(t_2)-\lambda_{t_1}\mathfrak{I}^{\alpha_1,\rho,\psi_1}x(t_2)+\varphi_2(x(t_2))\right)\right\}e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}\\
&\quad +c_1\left\{\left(\frac{(\psi_0(t_1)-\psi_0(t_0))^{\alpha_0}}{\rho^{\alpha_0}\Gamma(\alpha_0+1)}+\frac{(\psi_1(t_2)-\psi_1(t_1))^{\alpha_1}}{\rho^{\alpha_1}\Gamma(\alpha_1+1)}+\frac{(\psi_2(t)-\psi_2(t_2))^{\alpha_2}}{\rho^{\alpha_2}\Gamma(\alpha_2+1)}\right)\right.\\
&\quad \left.\times e^{\frac{\rho-1}{\rho}\left[(\psi_0(t_1)-\psi_0(t_0))+(\psi_1(t_2)-\psi_1(t_1))\right]}\right\}e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}
\end{aligned}$$

$$+c_2\left\{e^{\frac{\rho-1}{\rho}\left[(\psi_0(t_1)-\psi_0(t_0))+(\psi_1(t_2)-\psi_1(t_1))\right]}\right\}e^{\frac{\rho-1}{\rho}(\psi_2(t)-\psi_2(t_2))}, \quad t\in J_2.$$

By a similar way repeating the same process, for $t \in J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$, we have the integral equation

$$\begin{aligned} x(t) &= {}_{t_k}\mathfrak{I}^{\alpha_k+\beta_k,\rho,\psi_k}F_x(t) - \lambda_{t_k}\mathfrak{I}^{\alpha_k,\rho,\psi_k}x(t) + \left\{\sum_{i=1}^{k} H_{i-1}(x)\prod_{j=i}^{k-1} e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\right. \\ &\quad + \sum_{i=1}^{k} G_{i-1}(x)\sum_{j=i}^{k-1}\left(\Phi^{\alpha_j}(t_j,t_{j+1}) + \Phi^{\alpha_k}(t_k,t)\right)\prod_{j=i}^{k-1} e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)} \\ &\quad + c_1\sum_{i=1}^{k}(\Phi^{\alpha_{i-1}}(t_{i-1},t_i) + \Phi^{\alpha_k}(t_k,t))\prod_{i=1}^{k} e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))} \\ &\quad \left. + c_2\prod_{i=1}^{k} e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\right\} e^{\frac{\rho-1}{\rho}(\psi_k(t)-\psi_k(t_k))}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} {}_{t_k}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k}x(t) &= {}_{t_k}\mathfrak{I}^{\beta_k,\rho,\psi_k}F_x(t) - \lambda x(t) + \left\{\sum_{i=1}^{k} G_{i-1}(x)\prod_{j=i}^{k-1} e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\right. \\ &\quad \left. + c_1\prod_{i=1}^{k} e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\right\} e^{\frac{\rho-1}{\rho}(\psi_k(t)-\psi_k(t_k))}. \end{aligned} \tag{2.23}$$

From the given boundary conditions, we get the following system

$$\begin{aligned} \Omega_1 c_1 + \Omega_2 c_2 &= \mathcal{K}(x, F_x), \\ \Omega_3 c_1 + \Omega_4 c_2 &= \mathcal{R}(x, F_x). \end{aligned}$$

Solving the above system for the constants c_1 and c_2, we have

$$c_1 = \frac{\Omega_1\mathcal{R}(x,F_x) - \Omega_3\mathcal{K}(x,F_x)}{\Omega_1\Omega_4 - \Omega_2\Omega_3} \qquad \text{and} \qquad c_2 = \frac{\Omega_4\mathcal{K}(x,F_x) - \Omega_2\mathcal{R}(x,F_x)}{\Omega_1\Omega_4 - \Omega_2\Omega_3},$$

where $\Omega_1\Omega_4 \neq \Omega_2\Omega_3$ are defined by (2.11), (2.12), (2.13) and (2.14), respectively. Substituting these values of c_1 and c_2 in (2.22), yields the solution in (2.10).

Conversely, it is easily to shown by direct calculuation that the solution $x(t)$ is given by (2.10) satisfies the problem (2.9) under the given boundary conditions. This completes the proof. □

The fixed point theorems play an important role in studying the existence theory for the problem (1.1). We collect here some well-known fixed point theorems for the sake of essential in the proofs of our existence and uniqueness results.

Theorem 2.12. (Banach's fixed point theorem [50]) *Let D be a non-empty closed subset of a Banach space E. Then any contraction mapping T from D into itself has a unique fixed point.*

Theorem 2.13. (Schaefer's fixed point theorem [50]) *Let E be a Banach space and $T : E \to E$ be a completely continuous operator, and let the set $D = \{x \in E : x = \sigma Tx, 0 < \sigma \leq 1\}$ be bounded. Then T has a fixed point in E.*

3. Existence and uniqueness results

In this section, we discuss the existence and uniqueness results for the problem (1.1) via Banach's and Schaefer's fixed point theorems.

In view of Lemma 2.11 to establish existence theorems, we consider the operator equation $x = Qx$, where $Q : \mathbb{E} \to \mathbb{E}$ is defined by

$$\begin{aligned}(Qx)(t) &= {}_{t_k}\mathfrak{I}^{\alpha_k+\beta_k,\rho,\psi_k} F_x(t) - \lambda\,{}_{t_k}\mathfrak{I}^{\alpha_k,\rho,\psi_k} x(t) + \Bigg\{ \sum_{i=1}^{k} H_{i-1}(x) \prod_{j=i}^{k-1} e^{\frac{\rho-1}{\rho}(\psi_j(t_{j+1})-\psi_j(t_j))} \\ &\quad + \sum_{i=1}^{k} G_{i-1}(x) \sum_{j=i}^{k-1} \Big(\Phi^{\alpha_j}(t_j, t_{j+1}) + \Phi^{\alpha_k}(t_k, t)\Big) \prod_{j=i}^{k-1} e^{\frac{\rho-1}{\rho}(\psi_j(t_{j+1})-\psi_j(t_j))} \\ &\quad + \frac{\Omega_1 \mathcal{R}(x, F_x) - \Omega_3 \mathcal{K}(x, F_x)}{\Omega_5} \sum_{i=1}^{k} (\Phi^{\alpha_{i-1}}(t_{i-1}, t_i) + \Phi^{\alpha_k}(t_k, t)) \prod_{i=1}^{k} e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))} \\ &\quad + \frac{\Omega_4 \mathcal{K}(x, F_x) - \Omega_2 \mathcal{R}(x, F_x)}{\Omega_5} \prod_{i=1}^{k} e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))} \Bigg\} e^{\frac{\rho-1}{\rho}(\psi_k(t)-\psi_k(t_k))}, \quad t \in J_k. \qquad (3.1)\end{aligned}$$

It is clear that the problem (1.1) has a solution if and only if the operator Q has fixed points.

To simplify the computations, we use the following constants:

$$\Lambda_1 = \sum_{i=1}^{m+1} \Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1}, t_i) + \sum_{i=1}^{m} \Phi^{\beta_{i-1}}(t_{i-1}, t_i) \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}), \qquad (3.2)$$

$$\Lambda_2 = \sum_{i=1}^{m+1} \Phi^{\alpha_{i-1}}(t_{i-1}, t_i), \qquad (3.3)$$

$$\Lambda_3 = \sum_{i=1}^{m} \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}), \qquad (3.4)$$

$$\Lambda_4 = \sum_{i=1}^{m+1} \Phi^{\beta_{i-1}}(t_{i-1}, t_i), \qquad (3.5)$$

$$\Theta_1 = \Lambda_1 + \Big(|\kappa_1|\Lambda_1(\Lambda_2|\Omega_3| + |\Omega_4|) + |\kappa_2|(|\lambda|\Lambda_1 + \Lambda_4)(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}, \qquad (3.6)$$

$$\Theta_2 = 1 + \Big(|\kappa_1|(\Lambda_2|\Omega_3| + |\Omega_4|) + |\lambda||\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}, \qquad (3.7)$$

$$\Theta_3 = \Lambda_3 + \Big(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3| + |\Omega_4|) + |\kappa_2|(|\lambda|\Lambda_3 + m)(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}, \qquad (3.8)$$

$$\Theta_4 = \Big(|\xi_1|(\Lambda_2|\Omega_3| + |\Omega_4|) + |\xi_2|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}, \qquad (3.9)$$

By applying classical fixed point theorems, we prove in the next subsections, for the problems (1.1), our main existence and uniqueness results.

3.1. Uniqueness result via Banach's fixed point theorem

The first result is an existence and uniqueness result for the problem (1.1) by applying Banach's fixed point theorem.

Theorem 3.1. *Let $\psi_k \in C^2(J)$ with $\psi'_k(t) > 0$ for $t \in J$, $k = 0, 1, 2, \dots, m$. Assume that $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, φ_k, $\varphi^*_k \in C(\mathbb{R}, \mathbb{R})$, $k = 1, 2, \dots, m$ satisfy the following assumptions:*

(H_1) *There exist a constant $L_1 > 0$ such that, for every $t \in J$ and x_1, x_2, y_1, $y_2 \in \mathbb{R}$, such that*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1 \left(|x_1 - x_2| + |y_1 - y_2|\right).$$

(H_2) *There exist constants $M_1, M^*_1 > 0$, for any $x, y \in \mathbb{R}$, such that*

$$|\varphi_k(x) - \varphi_k(y)| \leq M_1|x - y|, \quad |\varphi^*_k(x) - \varphi^*_k(y)| \leq M^*_1|x - y|, \quad k = 1, 2, \dots, m.$$

Then, the problem (1.1) has a unique solution on J provided that

$$2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M^*_1\Theta_3 < 1. \tag{3.10}$$

Proof. Observe that the problem (1.1) is equivalent to a fixed point problem $x = Qx$, where the operator Q is defined by (3.1). Thus, we need to establish that the operator Q has a fixed point. This will be achieved by means of the Banach's fixed point theorem.

Let K_1, K_2 and K_3 be nonnegative constants such that $K_1 = \sup_{t\in J} |f(t, 0, 0)|$, $K_2 = \max\{\varphi_k(0) : k = 1, 2, \dots, m\}$ and $K_3 = \max\{\varphi^*_k(0) : k = 1, 2, \dots, m\}$. Next we set $B_{r_1} = \{x \in \mathbb{E} : \|x\| \leq r_1\}$ with

$$r_1 \geq \frac{K_1\Theta_1 + mK_2\Theta_2 + K_3\Theta_3 + \Theta_4}{1 - (2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M^*_1\Theta_3)}. \tag{3.11}$$

Clearly, B_{r_1} is a bounded, closed, and convex subset of $\mathbb{E}$. We complete the proof in two steps.

Step I. We show that $QB_{r_1} \subset B_{r_1}$.

For any $x \in B_{r_1}$, we have

$$\begin{aligned} |(Qx)(t)| \leq\ & {}_{t_m}\Im^{\alpha_m+\beta_m,\rho,\psi_m}|F_x(s)|(T) + |\lambda|{}_{t_m}\Im^{\alpha_m,\rho,\psi_m}|x(s)|(T) + \Bigg\{ \sum_{i=1}^{m} |H_{i-1}(x)| \prod_{j=i}^{m} e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)} \\ & + \sum_{i=1}^{m} |G_{i-1}(x)| \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \prod_{j=i}^{m} e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)} \\ & + \frac{|\Omega_1||\mathcal{R}(x, F_x)| + |\Omega_3||\mathcal{K}(x, F_x)|}{|\Omega_5|} \sum_{i=1}^{m+1} \Phi^{\alpha_{i-1}}(t_{i-1}, t_i) \prod_{i=1}^{m+1} e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))} \\ & + \frac{|\Omega_4||\mathcal{K}(x, F_x)| + |\Omega_2||\mathcal{R}(x, F_x)|}{|\Omega_5|} \prod_{i=1}^{m+1} e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))} \Bigg\} e^{\frac{\rho-1}{\rho}(\psi_m(T)-\psi_m(t_m))}. \end{aligned} \tag{3.12}$$

By using $0 < e^{\frac{\rho-1}{\rho}(\psi_a(u)-\psi_a(s))} \leq 1$ for $0 \leq s \leq u \leq T$ with (H_1) and (H_2), we have

$$|G_{i-1}(x)| \leq |G_{i-1}(x) - G_{i-1}(0)| + |G_{i-1}(0)|$$

$$
\begin{aligned}
&\leq {}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|F_x(s)-F_0(s)|(t_i)+|\varphi_i^*(x(t_i))-\varphi_i^*(0)|+{}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|F_0(s)|(t_i)+|\varphi_i^*(0)|\\
&\leq \frac{2L_1r_1}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\beta_{i-1}-1}\psi'_{i-1}(s)ds+M_1^*r_1\\
&\quad+\frac{K_1}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\beta_{i-1}-1}\psi'_{i-1}(s)ds+K_3\\
&\leq \left(2L_1\frac{(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))^{\beta_{i-1}}}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1}+1)}+M_1^*\right)r_1+K_1\frac{(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))^{\beta_{i-1}}}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1}+1)}+K_3\\
&= \left(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\right)r_1+K_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+K_3, \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
|H_{i-1}(x)| &\leq |H_{i-1}(x)-H_{i-1}(0)|+|H_{i-1}(0)|\\
&\leq {}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|F_x(s)-F_0(s)|(t_i)+|\lambda|_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1},\rho,\psi_{i-1}}|x(s)|(t_i)\\
&\quad+|\varphi_i(x(t_i))-\varphi_i(0)|+{}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|F_0(s)|(t_i)+|\varphi_i(0)|\\
&\leq \frac{2L_1r_1}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}\psi'_{i-1}(s)ds\\
&\quad+\frac{|\lambda|r_1}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}-1}\psi'_{i-1}(s)ds+M_1r_1+K_2\\
&\quad+\frac{K_1}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1},}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}\psi'_{i-1}(s)ds\\
&\leq \left(2L_1\frac{(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))^{\alpha_{i-1}+\beta_{i-1}}}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1}+1)}+|\lambda|\frac{(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))^{\alpha_{i-1}}}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+M_1\right)r_1\\
&\quad+K_1\frac{(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))^{\alpha_{i-1}+\beta_{i-1}}}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1}+1)}+K_2\\
&= \left(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\right)r_1+K_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+K_2. \qquad (3.14)
\end{aligned}
$$

From the results of the inequalities (3.13)-(3.14) with the similarly process, we obtain,

$$
\begin{aligned}
|\mathcal{K}(x,F_x)| &\leq |\mathcal{K}(x,F_x)-\mathcal{K}(0,F_0)|+|\mathcal{K}(0,F_0)|\\
&\leq |\xi_1|+|\kappa_1|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|F_x(s)-F_0(s)|(T)+|\kappa_1||\lambda|_{t_m}\mathfrak{I}^{\alpha_m,\rho,\psi_m}|x(s)|(T)\\
&\quad+|\kappa_1|\sum_{i=1}^{m}|G_{i-1}(x)-G_{i-1}(0)|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+|\kappa_1|\sum_{i=1}^{m}|H_{i-1}(x)-H_{i-1}(0)|\\
&\quad+|\kappa_1|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|F_0(s)|(T)+|\kappa_1|\sum_{i=1}^{m}|G_{i-1}(0)|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+|\kappa_1|\sum_{i=1}^{m}|H_{i-1}(0)|\\
&\leq |\xi_1|+\frac{2L_1r_1|\kappa_1|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}\psi'_m(s)ds\\
&\quad+\frac{|\kappa_1||\lambda|r_1}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m-1}\psi'_m(s)ds\\
&\quad+|\kappa_1|\sum_{i=1}^{m}\left(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\right)r_1\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})
\end{aligned}
$$

$$
\begin{aligned}
& +|\kappa_1| \sum_{i=1}^{m} \Big(2L_1 \Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1}, t_i) + |\lambda| \Phi^{\alpha_{i-1}}(t_{i-1}, t_i) + M_1\Big) r_1 \\
& + \frac{K_1 |\kappa_1|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| (\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1} \psi_m'(s) ds \\
& +|\kappa_1| \sum_{i=1}^{m} \Big(K_1 \Phi^{\beta_{i-1}}(t_{i-1}, t_i) + K_3\Big) \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) + |\kappa_1| \sum_{i=1}^{m} \Big(K_1 \Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1}, t_i) + K_2\Big) \\
\leq\ & \Bigg\{ 2L_1 \left(\sum_{i=1}^{m+1} \Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1}, t_i) + \sum_{i=1}^{m} \Phi^{\beta_{i-1}}(t_{i-1}, t_i) \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \right) \\
& + |\lambda| \sum_{i=1}^{m+1} \Phi^{\alpha_{i-1}}(t_{i-1}, t_i) + M_1^* \sum_{i=1}^{m} \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) + m M_1 \Bigg\} |\kappa_1| r_1 \\
& + \Bigg\{ K_1 \left(\sum_{i=1}^{m+1} \Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1}, t_i) + \sum_{i=1}^{m} \Phi^{\beta_{i-1}}(t_{i-1}, t_i) \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \right) \\
& + K_3 \sum_{i=1}^{m} \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) + m K_2 \Bigg\} |\kappa_1| + |\xi_1| \\
=\ & \Big(2L_1 \Lambda_1 + |\lambda| \Lambda_2 + M_1^* \Lambda_3 + m M_1\Big) |\kappa_1| r_1 + \Big(K_1 \Lambda_1 + K_3 \Lambda_3 + m K_2\Big) |\kappa_1| + |\xi_1|, \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{R}(x, F_x)| \leq\ & |\mathcal{R}(x, F_x) - \mathcal{R}(0, F_0)| + |\mathcal{R}(0, F_0)| \\
\leq\ & |\xi_2| + |\kappa_2| {}_{t_m}\mathfrak{I}^{\beta_m,\rho,\psi_m} |F_x(s) - F_0(s)|(T) \\
& + |\kappa_2||\lambda| {}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m} |F_x(s) - F_0(s)|(T) + |\kappa_2| \lambda^2 {}_{t_m}\mathfrak{I}^{\alpha_m,\rho,\psi_m} |x(s)|(T) \\
& + |\kappa_2| \sum_{i=1}^{m} |G_{i-1}(x) - G_{i-1}(0)| \left(1 + |\lambda| \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \right) \\
& + |\kappa_2||\lambda| \sum_{i=1}^{m} |H_{i-1}(x) - H_{i-1}(0)| + |\kappa_2| {}_{t_m}\mathfrak{I}^{\beta_m,\rho,\psi_m} |F_0(s)|(T) \\
& + |\kappa_2||\lambda| \sum_{i=1}^{m} |H_{i-1}(0)| + |\kappa_2||\lambda| {}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m} |F_0(s)|(T) \\
& + |\kappa_2| \sum_{i=1}^{m} |G_{i-1}(0)| \left(1 + |\lambda| \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \right) \\
\leq\ & |\xi_2| + \frac{2L_1 r_1 |\kappa_2|}{\rho^{\beta_m}\Gamma(\beta_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| (\psi_m(T)-\psi_m(s))^{\beta_m-1} \psi_m'(s) ds \\
& + \frac{2L_1 r_1 |\kappa_2||\lambda|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| (\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1} \psi_m'(s) ds \\
& + \frac{|\kappa_2| \lambda^2 r_1}{\rho^{\alpha_m}\Gamma(\alpha_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| (\psi_m(T)-\psi_m(s))^{\alpha_m-1} \psi_m'(s) ds \\
& + |\kappa_2| \sum_{i=1}^{m} \Big(2L_1 \Phi^{\beta_{i-1}}(t_{i-1}, t_i) + M_1^*\Big) r_1 \left(1 + |\lambda| \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\Big(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\Big)r_1\\
&+\frac{K_1|\kappa_2|}{\rho^{\beta_m}\Gamma(\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\beta_m-1}\psi_m'(s)ds\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\Big(K_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+K_2\Big)\\
&+\frac{K_1|\kappa_2||\lambda|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}\psi_m'(s)ds\\
&+|\kappa_2|\sum_{i=1}^{m}\Big(K_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+K_3\Big)\left(1+|\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\right)\\
\leq\;&\Bigg\{2L_1\Bigg[|\lambda|\Bigg(\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+\sum_{i=1}^{m}\Phi^{\beta_{i-1}}(t_{i-1},t_i)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)+\sum_{i=1}^{m+1}\Phi^{\beta_{i-1}}(t_{i-1},t_i)\Bigg]\\
&+M_1^*\Bigg(|\lambda|\sum_{i=1}^{m}\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+m\Bigg)+\lambda^2\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+m|\lambda|M_1\Bigg\}|\kappa_2|r_1\\
&+\Bigg\{K_1\Bigg[\sum_{i=1}^{m+1}\Phi^{\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Bigg(\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+\sum_{i=1}^{m}\Phi^{\beta_{i-1}}(t_{i-1},t_i)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)\Bigg]\\
&+K_3\Bigg[|\lambda|\sum_{i=1}^{m}\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+m\Bigg]+K_2m|\lambda|\Bigg\}|\kappa_2|+|\xi_2|\\
=\;&\Big(2L_1(|\lambda|\Lambda_1+\Lambda_4)+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\Big)|\kappa_2|r_1\\
&+\Big(K_1(\Lambda_4+|\lambda|\Lambda_1)+K_3(|\lambda|\Lambda_3+m)+K_2m|\lambda|\Big)|\kappa_2|+|\xi_2|. \qquad (3.16)
\end{aligned}
$$

Substisuting (3.13), (3.14), (3.15) and (3.16) into (3.12), we obtain

$$
\begin{aligned}
|(Qx)(t)| \leq\;& {}_{t_m}\Im^{\alpha_m+\beta_m,\rho,\psi_m}\left(|F_x(s)-F_0(s)|+|F_0(s)|\right)(T)+|\lambda|_{t_m}\Im^{\alpha_m,\rho,\psi_m}|x(s)|(T)\\
&+\sum_{i=1}^{m}\left(|H_{i-1}(x)-H_{i-1}(0)|+|H_{i-1}(0)|\right)+\sum_{i=1}^{m}\left(|G_{i-1}(x)-G_{i-1}(0)|+|G_{i-1}(0)|\right)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+\Big(|\Omega_1|(|\mathcal{R}(x,F_x)-\mathcal{R}(0,F_0)|+|\mathcal{R}(0,F_0)|)+|\Omega_3|(|\mathcal{K}(x,F_x)-\mathcal{K}(0,F_0)|+|\mathcal{K}(0,F_0)|)\Big)\\
&\times\frac{1}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+\frac{1}{|\Omega_5|}\Big(|\Omega_4|(|\mathcal{K}(x,F_x)-\mathcal{K}(0,F_0)|+|\mathcal{K}(0,F_0)|)\\
&+|\Omega_2|(|\mathcal{R}(x,F_x)-\mathcal{R}(0,F_0)|+|\mathcal{R}(0,F_0)|)\Big)\\
\leq\;&\frac{2L_1r_1+K_1}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}\psi_m'(s)ds\\
&+\frac{|\lambda|r_1}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m-1}\psi_m'(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\Bigg[\Big(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\Big)r_1+K_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+K_2\Bigg]\\
&+\sum_{i=1}^{m}\Bigg[\Big(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\Big)r_1+K_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+K_3\Bigg]\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+\frac{1}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\Bigg[|\Omega_1|\Bigg(\{2L_1[|\lambda|\Lambda_1+\Lambda_4]+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\}|\kappa_2|r_1\\
&+\{K_1(\Lambda_4+|\lambda|\Lambda_1)+K_3(|\lambda|\Lambda_3+m)+K_2m|\lambda|\}|\kappa_2|+|\xi_2|\Bigg)\\
&+|\Omega_3|\Bigg((2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1)|\kappa_1|r_1+(K_1\Lambda_1+K_3\Lambda_3+mK_2)|\kappa_1|+|\xi_1|\Bigg)\Bigg]\\
&+\frac{1}{|\Omega_5|}\Bigg[|\Omega_4|\Bigg((2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1)|\kappa_1|r_1+(K_1\Lambda_1+K_3\Lambda_3+mK_2)|\kappa_1|+|\xi_1|\Bigg)\\
&+|\Omega_2|\Bigg(\Big(2L_1(|\lambda|\Lambda_1+\Lambda_4)+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\Big)|\kappa_2|r_1\\
&+\Big(K_1(\Lambda_4+|\lambda|\Lambda_1)+K_3(|\lambda|\Lambda_3+m)+K_2m|\lambda|\Big)|\kappa_2|+|\xi_2|\Bigg)\Bigg]\\
\leq\;& 2L_1\Phi^{\alpha_m+\beta_m}(t_m,T)r_1+K_1\Phi^{\alpha_m+\beta_m}(t_m,T)+|\lambda|\Phi^{\alpha_m}(t_m,T)r_1\\
&+\sum_{i=1}^{m}\Bigg[\Big(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\Big)r_1+K_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+K_2\Bigg]\\
&+\sum_{i=1}^{m}\Bigg[\Big(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\Big)r_1+K_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+K_3\Bigg]\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+\Bigg[|\Omega_1|\Bigg(\{2L_1[|\lambda|\Lambda_1+\Lambda_4]+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\}|\kappa_2|r_1\\
&+\{K_1(\Lambda_4+|\lambda|\Lambda_1)+K_3(|\lambda|\Lambda_3+m)+K_2m|\lambda|\}|\kappa_2|+|\xi_2|\Bigg)\\
&+|\Omega_3|\Bigg((2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1)|\kappa_1|r_1+(K_1\Lambda_1+K_3\Lambda_3+mK_2)|\kappa_1|+|\xi_1|\Bigg)\Bigg]\frac{\Lambda_2}{|\Omega_5|}\\
&+\frac{1}{|\Omega_5|}\Bigg[|\Omega_4|\Bigg((2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1)|\kappa_1|r_1+(K_1\Lambda_1+K_3\Lambda_3+mK_2)|\kappa_1|+|\xi_1|\Bigg)\\
&+|\Omega_2|\Bigg(\{2L_1[|\lambda|\Lambda_1+\Lambda_4]+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\}|\kappa_2|r_1\\
&+\{K_1(\Lambda_4+|\lambda|\Lambda_1)+K_3(|\lambda|\Lambda_3+m)+K_2m|\lambda|\}|\kappa_2|+|\xi_2|\Bigg)\Bigg]\\
=\;&\Bigg(2L_1\Bigg[\Lambda_1+(|\kappa_1|\Lambda_1(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_1+\Lambda_4)(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+(mM_1+|\lambda|\Lambda_2)\left[1+(|\kappa_1|(\Lambda_2|\Omega_3|+|\Omega_4|)+|\lambda||\kappa_2|(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\right]\\
&+M_1^*\left[\Lambda_3+(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_3+m)(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\right]\Bigg)r_1\\
&+K_1\left[\Lambda_1+(|\kappa_1|\Lambda_1(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_1+\Lambda_4)(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\right]\\
&+mK_2\left[1+(|\kappa_1|(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2||\lambda|(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\right]\\
&+K_3\left[\Lambda_3+(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_3+m)(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\right]\\
&+\Big(|\xi_1|(\Lambda_2|\Omega_3|+|\Omega_4|)+|\xi_2|(\Lambda_2|\Omega_1|+|\Omega_2|)\Big)\frac{1}{|\Omega_5|}\\
=\;&\Big(2L_1\Theta_1+(mM_1+|\lambda|\Lambda_2)\Theta_2+M_1^*\Theta_3\Big)r_1+K_1\Theta_1+mK_2\Theta_2+K_3\Theta_3+\Theta_4\\
\leq\;& r_1,
\end{aligned}
$$

which implies that $QB_{r_1}\subset B_{r_1}$.

Step II. We prove that the operator Q is a contraction.

Let $x,y\in\mathbb{E}$. Then, for each $t\in J$, we have

$$
\begin{aligned}
|(Qx)(t)-(Qy)(t)| \leq\;& {}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|F_x(s)-F_y(s)|(T)+|\lambda|_{t_m}\mathfrak{I}^{\alpha_m,\rho,\psi_m}|x(s)-y(s)|(T)\\
&+\Bigg\{\sum_{i=1}^{m}|H_{i-1}(x)-H_{i-1}(y)|\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\\
&+\sum_{i=1}^{m}|G_{i-1}(x)-G_{i-1}(y)|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\\
&+\Big(|\Omega_1||\mathcal{R}(x,F_x)-\mathcal{R}(y,F_y)|+|\Omega_3||\mathcal{K}(x,F_x)-\mathcal{K}(y,F_y)|\Big)\\
&\times\frac{1}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\prod_{i=1}^{m+1}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\\
&+\Big(|\Omega_4||\mathcal{K}(x,F_x)-\mathcal{K}(y,F_y)|+|\Omega_2||\mathcal{R}(x,F_x)-\mathcal{R}(y,F_y)|\Big)\\
&\times\frac{1}{|\Omega_5|}\prod_{i=1}^{m+1}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\Bigg\}e^{\frac{\rho-1}{\rho}(\psi_m(T)-\psi_m(t_m))}.
\end{aligned}
\tag{3.17}
$$

By using $0<e^{\frac{\rho-1}{\rho}(\psi_a(u)-\psi_a(s))}\leq 1$ for $0\leq s\leq u\leq T$ and (H_1)-(H_2), we get

$$
\begin{aligned}
|G_{i-1}(x)-G_{i-1}(y)| \leq\;& {}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|F_x(s)-F_y(s)|(t_i)+|\varphi_i^*(x(t_i))-\varphi_i^*(y(t_i))|\\
\leq\;& \frac{2L_1\|x-y\|}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\beta_{i-1}-1}\psi'_{i-1}(s)ds+M_1^*\|x-y\|\\
\leq\;& \left(2L_1\frac{(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))^{\beta_{i-1}}}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1}+1)}+M_1^*\right)\|x-y\|\\
=\;& \Big(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\Big)\|x-y\|,
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
|H_{i-1}(x)-H_{i-1}(y)| &\leq {}_{t_{i-1}}\mathfrak{J}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|F_x(s)-F_y(s)|(t_i)\\
&\quad+|\lambda|_{t_{i-1}}\mathfrak{J}^{\alpha_{i-1},\rho,\psi_{i-1}}|x(s)-y(s)|(t_i)+|\varphi_i(x(t_i))-\varphi_i(y(t_i))|\\
&\leq \frac{2L_1\|x-y\|}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\\
&\quad\times(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}\psi'_{i-1}(s)ds+|\lambda|\frac{\|x-y\|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\\
&\quad\times\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}-1}\psi'_{i-1}(s)ds+M_1\|x-y\|\\
&\leq \left(2L_1\frac{(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))^{\alpha_{i-1}+\beta_{i-1}}}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1}+1)}+|\lambda|\frac{(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))^{\alpha_{i-1}}}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1}+1)}+M_1\right)\|x-y\|\\
&= \left(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\right)\|x-y\|. \qquad (3.19)
\end{aligned}
$$

By using the results of the inequalities (3.18) and (3.19), we have

$$
\begin{aligned}
|\mathcal{K}(x,F_x)-\mathcal{K}(y,F_y)| &\leq |\kappa_1|_{t_m}\mathfrak{J}^{\alpha_m+\beta_m,\rho,\psi_m}|F_x(s)-F_y(s)|(T)+|\kappa_1||\lambda|_{t_m}\mathfrak{J}^{\alpha_m,\rho,\psi_m}|x(s)-y(s)|(T)\\
&\quad+|\kappa_1|\sum_{i=1}^{m}|G_{i-1}(x)-G_{i-1}(y)|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+|\kappa_1|\sum_{i=1}^{m}|H_{i-1}(x)-H_{i-1}(y)|\\
&\leq \frac{2L_1\|x-y\||\kappa_1|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}\psi'_m(s)ds\\
&\quad+\frac{|\kappa_1||\lambda|\|x-y\|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m-1}\psi'_m(s)ds\\
&\quad+|\kappa_1|\sum_{i=1}^{m}\left(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\right)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\|x-y\|\\
&\quad+|\kappa_1|\sum_{i=1}^{m}\left(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\right)\|x-y\|\\
&\leq \Bigg[2L_1\left(\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+\sum_{i=1}^{m}\Phi^{\beta_{i-1}}(t_{i-1},t_i)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\right)\\
&\quad+|\lambda|\left(\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\right)+M_1^*\left(\sum_{i=1}^{m}\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\right)+mM_1\Bigg]|\kappa_1|\|x-y\|\\
&= \left(2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1\right)|\kappa_1|\|x-y\|, \qquad (3.20)\\
|\mathcal{R}(x,F_x)-\mathcal{R}(y,F_y)| &\leq |\kappa_2|_{t_m}\mathfrak{J}^{\beta_m,\rho,\psi_m}|F_x(s)-F_y(s)|(T)+|\kappa_2||\lambda|_{t_m}\mathfrak{J}^{\alpha_m+\beta_m,\rho,\psi_m}|F_x(s)-F_y(s)|(T)\\
&\quad+|\kappa_2|\lambda^2{}_{t_m}\mathfrak{J}^{\alpha_m,\rho,\psi_m}|x(s)-y(s)|(T)+|\kappa_2||\lambda|\sum_{i=1}^{m}|H_{i-1}(x)-H_{i-1}(y)|\\
&\quad+|\kappa_2|\sum_{i=1}^{m}|G_{i-1}(x)-G_{i-1}(y)|\left(1+|\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\right)\\
&\leq \frac{2L_1\|x-y\||\kappa_2|}{\rho^{\beta_m}\Gamma(\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\beta_m-1}\psi'_m(s)ds
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{2L_1\|x-y\||\kappa_2||\lambda|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m+\beta_m-1}\psi_m'(s)ds \\
& +\frac{|\kappa_2|\lambda^2\|x-y\|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi_m'(s)ds \\
& +|\kappa_2||\lambda|\sum_{i=1}^{m}\left(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\right)\|x-y\| \\
& +|\kappa_2|\sum_{i=1}^{m}\left(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\right)\left(1+|\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\right)\|x-y\| \\
\leq\ & \Bigg[2L_1\Bigg(|\lambda|\Bigg(\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+\sum_{i=1}^{m}\Phi^{\beta_{i-1}}(t_{i-1},t_i)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg) \\
& +\sum_{i=1}^{m+1}\Phi^{\beta_{i-1}}(t_{i-1},t_i)\Bigg)+M_1^*\Bigg(|\lambda|\Bigg(\sum_{i=1}^{m}\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)+m\Bigg) \\
& +\lambda^2\Bigg(\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\Bigg)+m|\lambda|M_1\Bigg]|\kappa_2|\|x-y\| \\
=\ & \Big(2L_1(|\lambda|\Lambda_1+\Lambda_4)+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\Big)|\kappa_2|\|x-y\|. \qquad (3.21)
\end{aligned}
$$

Substituting (3.18), (3.19), (3.20) and (3.21) into (3.17), it follows that

$$
\begin{aligned}
& |(Qx)(t)-(Qy)(t)| \\
\leq\ & \frac{2L_1\|x-y\|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m+\beta_m-1}\psi_m'(s)ds \\
& +\frac{|\lambda|\|x-y\|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi_m'(s)ds \\
& +\Bigg\{\sum_{i=1}^{m}\left(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\right)\|x-y\|\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)} \\
& +\sum_{i=1}^{m}\left(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\right)\|x-y\|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)} \\
& +\Bigg[|\Omega_1|\Bigg(2L_1(|\lambda|\Lambda_1+\Lambda_4)+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\Bigg)|\kappa_2|\|x-y\|. \\
& +|\Omega_3|\Bigg(2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1\Bigg)|\kappa_1|\|x-y\|\Bigg]\frac{1}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\prod_{i=1}^{m+1}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))} \\
& +\Bigg[|\Omega_4|\Bigg(2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1\Bigg)|\kappa_1|\|x-y\|+|\Omega_2|\Bigg(2L_1(|\lambda|\Lambda_1+\Lambda_4) \\
& +M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\Bigg)|\kappa_2|\|x-y\|\Bigg]\frac{1}{|\Omega_5|}\prod_{i=1}^{m+1}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\Bigg\}e^{\frac{\rho-1}{\rho}(\psi_m(T)-\psi_m(t_m))} \\
\leq\ & 2L_1\Phi^{\alpha_m+\beta_m}(t_m,T)\|x-y\|+|\lambda|\Phi^{\alpha_m}(t_m,T)\|x-y\|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\Big(2L_1\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+M_1\Big)\|x-y\| \\
&+\sum_{i=1}^{m}\Big(2L_1\Phi^{\beta_{i-1}}(t_{i-1},t_i)+M_1^*\Big)\|x-y\|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1}) \\
&+\frac{\Lambda_2|\Omega_1|}{|\Omega_5|}\Big(2L_1(|\lambda|\Lambda_1+\Lambda_4)+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\Big)|\kappa_2|\|x-y\| \\
&+\frac{\Lambda_2|\Omega_3|}{|\Omega_5|}\Big(2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1\Big)|\kappa_1|\|x-y\| \\
&+\frac{|\Omega_4|}{|\Omega_5|}\Big(2L_1\Lambda_1+|\lambda|\Lambda_2+M_1^*\Lambda_3+mM_1\Big)|\kappa_1|\|x-y\| \\
&+\frac{|\Omega_2|}{|\Omega_5|}\Big(2L_1(|\lambda|\Lambda_1+\Lambda_4)+M_1^*(|\lambda|\Lambda_3+m)+\lambda^2\Lambda_2+m|\lambda|M_1\Big)|\kappa_2|\|x-y\| \\
= &\Bigg(2L_1\left[\Lambda_1+(|\kappa_1|\Lambda_1(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_1+\Lambda_4)(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\right] \\
&+(mM_1+|\lambda|\Lambda_2)\left[1+(|\kappa_1|(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2||\lambda|(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\right] \\
&+M_1^*\left[\Lambda_3+(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_3+m)(\Lambda_2|\Omega_1|+|\Omega_2|))\frac{1}{|\Omega_5|}\right]\Bigg)\|x-y\| \\
= &(2L_1\Theta_1+(mM_1+|\lambda|\Lambda_2)\Theta_2+M_1^*\Theta_3)\|x-y\|,
\end{aligned}
$$

which implies that $\|Qx-Qy\|\leq(2L_1\Theta_1+(mM_1+|\lambda|\Lambda_2)\Theta_2+M_1^*\Theta_3)\|x-y\|$. Clearly $(2L_1\Theta_1+(mM_1+|\lambda|\Lambda_2)\Theta_2+M_1^*\Theta_3)<1$, thus, by the Banach's contraction principle (Theorem 2.12), the operator Q is a contraction, hence, the operator Q has a unique fixed point that is the unique solution of the problem (1.1) on J. This completes the proof. □

3.2. Existence result via Schaefer's fixed point theorem

The second existence result is based on Schaefer's fixed point theorem.

Theorem 3.2. *Let $\psi_k\in C^2(J)$ with $\psi_k'(t)>0$ for $t\in J$, $k=0,1,2,\dots,m$. Assume that $f:J\times\mathbb{R}^2\to\mathbb{R}$, $\varphi_k:\mathbb{R}\to\mathbb{R}$ and $\varphi_k^*:\mathbb{R}\to\mathbb{R}$ are continuous functions, $k=1,2,\dots,m$ satisfy the following assumptions:*

(H_3) *There exist nonnegative continuous functions h_1, h_2, $h_3\in C(J,\mathbb{R}^+)$ such that, for every $t\in J$ and x, $y\in\mathbb{R}$, such that*

$$|f(t,x,y)|\leq h_1(t)+h_2(t)(|x|+|y|),$$

with $h_1^*=\sup_{t\in J}\{h_1(t)\}$ and $h_2^*=\sup_{t\in J}\{h_2(t)\}$.

(H_4) *There exist positive constants k_1, k_1^*, for any $x\in\mathbb{R}$, such that*

$$|\varphi_k(x)|\leq k_1,\quad |\varphi_k^*(x)|\leq k_1^*,\quad k=1,2,\dots,m.$$

Then, the problem (1.1) *has at least one solution on J.*

Proof. We apply Schaefer's fixed point theorem. The proof is given in the following four steps.

Step I. We prove that the operator Q is continuous.

Let x_n be a sequence such that $x_n \to x$ in $\mathbb{E}$. Then, for any $t \in J$, we get

$$\begin{aligned}
&|(Qx_n)(t) - (Qx)(t)| \\
\leq\ & {}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|F_{x_n}(s) - F_x(s)|(t) + |\lambda|_{t_m}\mathfrak{I}^{\alpha_m,\rho,\psi_m}|x_n(s) - x(s)|(t) \\
&+ \sum_{i=1}^{m} |H_{i-1}(x_n) - H_{i-1}(x)| + \sum_{i=1}^{m} |G_{i-1}(x_n) - G_{i-1}(x)| \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \\
&+ \frac{\Lambda_2}{|\Omega_5|}\Big[|\Omega_1||\mathcal{R}(x_n, F_{x_n}) - \mathcal{R}(x, F_x)| + |\Omega_3||\mathcal{K}(x_n, F_{x_n}) - \mathcal{K}(x, F_x)|\Big] \\
&+ \frac{1}{\Omega_5}\Big[|\Omega_4||\mathcal{K}(x_n, F_{x_n}) - \mathcal{K}(x, F_x)| + |\Omega_2||\mathcal{R}(x_n, F_{x_n}) - \mathcal{R}(x, F_x)|\Big] \\
\leq\ & \frac{1}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| (\psi_m(T) - \psi_m(s))^{\alpha_m+\beta_m-1} |F_{x_n}(s) - F_x(s)| \psi'_m(s) ds \\
&+ \frac{|\lambda|}{\rho^{\alpha_m}\Gamma(\alpha_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| (\psi_m(T) - \psi_m(s))^{\alpha_m-1} |x_n(s) - x(s)| \psi'_m(s) ds \\
&+ \sum_{i=1}^{m} \Bigg(\frac{1}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| (\psi_{i-1}(t_i) - \psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1} \\
&\times |F_{x_n}(s) - F_x(s)| \psi'_{i-1}(s) ds + \frac{|\lambda|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| (\psi_{i-1}(t_i) - \psi_{i-1}(s))^{\alpha_{i-1}-1} \\
&\times |x_n(s) - x(s)| \psi'_{i-1}(s) ds + M_1 |x_n(t_i) - x(t_i)| \Bigg) + \sum_{i=1}^{m} \Bigg(\frac{1}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| \\
&\times \big(\psi_{i-1}(t_i) - \psi_{i-1}(s)\big)^{\beta_{i-1}-1} |F_{x_n}(s) - F_x(s)| \psi'_{i-1}(s) ds + M_1^* |x_n(t_{i-1}) - x(t_{i-1})| \Bigg) \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \\
&+ \frac{\Lambda_2}{|\Omega_5|} \Bigg\{ |\Omega_1| \Bigg[\frac{|\kappa_2|}{\rho^{\beta_m}\Gamma(\beta_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| \big(\psi_m(T) - \psi_m(s)\big)^{\beta_m-1} |F_{x_n}(s) - F_x(s)| \psi'_m(s) ds \\
&+ \frac{|\kappa_2||\lambda|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| \big(\psi_m(T) - \psi_m(s)\big)^{\alpha_m+\beta_m-1} |F_{x_n}(s) - F_x(s)| \psi'_m(s) ds \\
&+ \frac{|\kappa_2|\lambda^2}{\rho^{\alpha_m}\Gamma(\alpha_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| (\psi_m(T) - \psi_m(s))^{\alpha_m-1} |x_n(s) - x(s)| \psi'_m(s) ds \\
&+ |\kappa_2||\lambda| \sum_{i=1}^{m} \Bigg(\frac{1}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| (\psi_{i-1}(t_i) - \psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1} \\
&\times |F_{x_n}(s) - F_x(s)| \psi'_{i-1}(s) ds + \frac{|\lambda|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| (\psi_{i-1}(t_i) - \psi_{i-1}(s))^{\alpha_{i-1}-1} \\
&\times |x_n(s) - x(s)| \psi'_{i-1}(s) ds + M_1 |x_n(t_{i-1}) - x(t_{i-1})| \Bigg) + |\kappa_2| \sum_{i=1}^{m} \Bigg(\frac{1}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| \\
&\times \big(\psi_{i-1}(t_i) - \psi_{i-1}(s)\big)^{\beta_{i-1}-1} |F_{x_n}(s) - F_x(s)| \psi'_{i-1}(s) ds + M_1^* |x_n(t_{i-1}) - x(t_{i-1})| \Bigg) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+|\Omega_3|\Bigg[\frac{|\kappa_1|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}|F_{x_n}(s)-F_x(s)|\psi'_m(s)ds\\
&+\frac{|\kappa_1||\lambda|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m-1}|x_n(s)-x(s)|\psi'_m(s)ds\\
&+|\kappa_1|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\beta_{i-1}-1}|F_{x_n}(s)-F_x(s)|\psi'_{i-1}(s)ds\\
&+M_1^*|x_n(t_{i-1})-x(t_{i-1})|\Bigg)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+|\kappa_1|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\\
&\times(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}|F_{x_n}(s)-F_x(s)|\psi'_{i-1}(s)ds+\frac{|\lambda|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\\
&\times(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}-1}|x_n(s)-x(s)|\psi'_{i-1}(s)ds+M_1|x_n(t_{i-1})-x(t_{i-1})|\Bigg)\Bigg]\Bigg\}+\frac{1}{\Omega_5}\Bigg\{|\Omega_4|\\
&\times\Bigg(\frac{|\kappa_1|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}|F_{x_n}(s)-F_x(s)|\psi'_m(s)ds\\
&+\frac{|\kappa_1||\lambda|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m-1}|x_n(s)-x(s)|\psi'_m(s)ds\\
&+|\kappa_1|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\beta_{i-1}-1}|F_{x_n}(s)-F_x(s)|\psi'_{i-1}(s)ds\\
&+M_1^*|x_n(t_{i-1})-x(t_{i-1})|\Bigg)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+|\kappa_1|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\\
&\times(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}|F_{x_n}(s)-F_x(s)|\psi'_{i-1}(s)ds+\frac{|\lambda|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\\
&\times(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}-1}|x_n(s)-x(s)|\psi'_{i-1}(s)ds+M_1|x_n(t_{i-1})-x(t_{i-1})|\Bigg)\\
&+|\Omega_2|\Bigg[\frac{|\kappa_2|}{\rho^{\beta_m}\Gamma(\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\beta_m-1}|F_{x_n}(s)-F_x(s)|\psi'_m(s)ds\\
&+\frac{|\kappa_2||\lambda|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}|F_{x_n}(s)-F_x(s)|\psi'_m(s)ds\\
&+\frac{|\kappa_2|\lambda^2}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m-1}|x_n(s)-x(s)|\psi'_m(s)ds\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}\\
&\times|F_{x_n}(s)-F_x(s)|\psi'_{i-1}(s)ds+\frac{|\lambda|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}-1}\\
&\times|x_n(s)-x(s)|\psi'_{i-1}(s)ds+M_1|x_n(t_{i-1})-x(t_{i-1})|\Bigg)+|\kappa_2|\sum_{i=1}^{m}\Bigg(\frac{1}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|
\end{aligned}
$$

$$\times\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\beta_{i-1}-1}|F_{x_n}(s)-F_x(s)|\psi'_{i-1}(s)ds+M_1^*|x_n(t_{i-1})-x(t_{i-1})|\Bigg)\Bigg]\Bigg\}.$$

By using the fact of $0<e^{\frac{\rho-1}{\rho}(\psi_a(u)-\psi_a(s))}\leq 1$ for $0\leq s\leq u\leq T$ with the notations (2.6), (2.11)–(2.15) and (3.2)–(3.5), we obtain

$$\begin{aligned}
&|(Qx_n)(t)-(Qx)(t)|\\
\leq\;& \frac{\|F_{x_n}-F_x\|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m+\beta_m-1}\psi'_m(s)ds\\
&+\frac{|\lambda|\|x_n-x\|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m-1}\psi'_m(s)ds+\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\\
&\times\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}+\beta_{i-1}-1}\psi'_{i-1}(s)ds+\frac{|\lambda|\|x_n-x\|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}-1}\psi'_{i-1}(s)ds\\
&+M_1\|x_n-x\|\Bigg)+\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\beta_{i-1}-1}\psi'_{i-1}(s)ds+M_1^*\|x_n-x\|\Bigg)\\
&\times\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+\frac{\Lambda_2}{|\Omega_5|}\Bigg\{|\Omega_1|\Bigg[\frac{|\kappa_2|\|F_{x_n}-F_x\|}{\rho^{\beta_m}\Gamma(\beta_m)}\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\beta_m-1}\psi'_m(s)ds\\
&+\frac{|\kappa_2||\lambda|\|F_{x_n}-F_x\|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m+\beta_m-1}\psi'_m(s)ds+\frac{|\kappa_2|\lambda^2\|x_n-x\|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m-1}\\
&\times\psi'_m(s)ds+|\kappa_2||\lambda|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}+\beta_{i-1}-1}\psi'_{i-1}(s)ds\\
&+\frac{|\lambda|\|x_n-x\|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}-1}\psi'_{i-1}(s)ds+M_1\|x_n-x\|\Bigg)\\
&+|\kappa_2|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\beta_{i-1}-1}\psi'_{i-1}(s)ds+M_1^*\|x_n-x\|\Bigg)\Bigg]\\
&+|\Omega_3|\Bigg[\frac{|\kappa_1|\|F_{x_n}-F_x\|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m+\beta_m-1}\psi'_m(s)ds+\frac{|\kappa_1||\lambda|\|x_n-x\|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\\
&\times\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m-1}\psi'_m(s)ds+|\kappa_1|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\beta_{i-1}-1}\psi'_{i-1}(s)ds\\
&+M_1^*\|x_n-x\|\Bigg)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+|\kappa_1|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}+\beta_{i-1}-1}\\
&\times\psi'_{i-1}(s)ds+\frac{|\lambda|\|x_n-x\|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\alpha_{i-1}-1}\psi'_{i-1}(s)ds+M_1\|x_n-x\|\Bigg)\Bigg]\Bigg\}\\
&+\frac{1}{\Omega_5}\Bigg\{|\Omega_4|\Bigg(\frac{|\kappa_1|\|F_{x_n}-F_x\|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m+\beta_m-1}\psi'_m(s)ds+\frac{|\kappa_1||\lambda|\|x_n-x\|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\\
&\times\int_{t_m}^{T}\big(\psi_m(T)-\psi_m(s)\big)^{\alpha_m-1}\psi'_m(s)ds+|\kappa_1|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\big(\psi_{i-1}(t_i)-\psi_{i-1}(s)\big)^{\beta_{i-1}-1}\psi'_{i-1}(s)ds
\end{aligned}$$

$$
\begin{aligned}
&+M_1^*\|x_n-x\|\Bigg)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+|\kappa_1|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}\\
&\times\psi_{i-1}'(s)ds+\frac{|\lambda|\|x_n-x\|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}-1}\psi_{i-1}'(s)ds+M_1\|x_n-x\|\Bigg)\\
&+|\Omega_2|\Bigg[\frac{|\kappa_2|\|F_{x_n}-F_x\|}{\rho^{\beta_m}\Gamma(\beta_m)}\int_{t_m}^{T}(\psi_m(T)-\psi_m(s))^{\beta_m-1}\psi_m'(s)ds+\frac{|\kappa_2||\lambda|\|F_{x_n}-F_x\|}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\\
&\times\int_{t_m}^{T}(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}\psi_m'(s)ds+\frac{|\kappa_2|\lambda^2\|x_n-x\|}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}(\psi_m(T)-\psi_m(s))^{\alpha_m-1}\psi_m'(s)ds\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}\psi_{i-1}'(s)ds\\
&+\frac{|\lambda|\|x_n-x\|}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}-1}\psi_{i-1}'(s)ds+M_1\|x_n-x\|\Bigg)\\
&+|\kappa_2|\sum_{i=1}^{m}\Bigg(\frac{\|F_{x_n}-F_x\|}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\beta_{i-1}-1}\psi_{i-1}'(s)ds+M_1^*\|x_n-x\|\Bigg)\Bigg]\Bigg\}.\\
\leq\;&\Phi^{\alpha_m+\beta_m}(t_m,T)\|F_{x_n}-F_x\|+|\lambda|\Phi^{\alpha_m}(t_m,T)\|x_n-x\|\\
&+\sum_{i=1}^{m}\Big(\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\|x_n-x\|+\|\varphi_k(x_n)-\varphi_k(x)\|\Big)\\
&+\sum_{i=1}^{m}\Big(\Phi^{\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|+\|\varphi_k^*(x_n)-\varphi_k^*(x)\|\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+\frac{\Lambda_2}{|\Omega_5|}\Bigg[|\Omega_1|\Bigg(|\kappa_2|\Phi^{\beta_m}(t_m,T)\|F_{x_n}-F_x\|+|\kappa_2||\lambda|\Phi^{\alpha_m+\beta_m}(t_m,T)\|F_{x_n}-F_x\|\\
&+|\kappa_2|\lambda^2\Phi^{\alpha_m}(t_m,T)\|x_n-x\|+|\kappa_2||\lambda|\sum_{i=1}^{m}\Big(\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|\\
&+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\|x_n-x\|+\|\varphi_k(x_n)-\varphi_k(x)\|\Big)+|\kappa_2|\sum_{i=1}^{m}\Big(\Phi^{\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|\\
&+\|\varphi_k^*(x_n)-\varphi_k^*(x)\|\Big)\Bigg(1+|\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)\Bigg)+|\Omega_3|\Bigg(|\kappa_1|\Phi^{\alpha_m+\beta_m}(t_m,T)\|F_{x_n}-F_x\|\\
&+|\kappa_1||\lambda|\Phi^{\alpha_m}(t_m,T)\|x_n-x\|+|\kappa_1|\sum_{i=1}^{m}\Big(\Phi^{\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|+\|\varphi_k^*(x_n)-\varphi_k^*(x)\|\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+|\kappa_1|\sum_{i=1}^{m}\Big(\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\|x_n-x\|+\|\varphi_k(x_n)-\varphi_k(x)\|\Big)\Bigg)\Bigg]\\
&+\frac{1}{\Omega_5}\Bigg[|\Omega_4|\Bigg(|\kappa_1|\Phi^{\alpha_m+\beta_m}(t_m,T)\|F_{x_n}-F_x\|+|\kappa_1||\lambda|\Phi^{\alpha_m}(t_m,T)\|x_n-x\|\\
&+|\kappa_1|\sum_{i=1}^{m}\Big(\Phi^{\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|+\|\varphi_k^*(x_n)-\varphi_k^*(x)\|\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})
\end{aligned}
$$

$$
\begin{aligned}
&+|\kappa_1|\sum_{i=1}^{m}\Big(\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\|x_n-x\|+\|\varphi_k(x_n)-\varphi_k(x)\|\Big)\Big)\\
&+|\Omega_2|\Bigg(|\kappa_2|\Phi^{\beta_m}(t_m,T)\|F_{x_n}-F_x\|+|\kappa_2||\lambda|\Phi^{\alpha_m+\beta_m}(t_m,T)\|F_{x_n}-F_x\|+|\kappa_2|\lambda^2\Phi^{\alpha_m}(t_m,T)\|x_n-x\|\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\Big(\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\|x_n-x\|+\|\varphi_k(x_n)-\varphi_k(x)\|\Big)\\
&+|\kappa_2|\sum_{i=1}^{m}\Big(\Phi^{\beta_{i-1}}(t_{i-1},t_i)\|F_{x_n}-F_x\|+\|\varphi_k^*(x_n)-\varphi_k^*(x)\|\Big)\Bigg(1+|\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)\Bigg)\Bigg]\\
=\;&\left[\Lambda_1+\Big(|\kappa_1|\Lambda_1(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_1+\Lambda_4)(\Lambda_2|\Omega_1|+|\Omega_2|)\Big)\frac{1}{|\Omega_5|}\right]\|F_{x_n}-F_x\|\\
&+|\lambda|\Lambda_2\left[1+\Big(|\kappa_1|(\Lambda_2|\Omega_3|+|\Omega_4|)+|\lambda||\kappa_2|(\Lambda_2|\Omega_1|+|\Omega_2|)\Big)\frac{1}{\Omega_5}\right]\|x_n-x\|\\
&+\left[\Lambda_3+(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_3+m)(\Lambda_2|\Omega_1|+|\Omega_2|))\,\frac{1}{|\Omega_5|}\right]\|\varphi_k^*(x_n)-\varphi_k^*(x)\|\\
=\;&\Theta_1\|F_{x_n}-F_x\|+\Theta_2\|x_n-x\|+\Theta_3\|\varphi_k^*(x_n)-\varphi_k^*(x)\|.
\end{aligned}
$$

Since f, λ, φ_k and φ_k^* are continuous, this implies that Q is also continuous. Then, $\|F_{x_n}-F_x\|\to 0$, and $\|x_n-x\|\to 0$, as $n\to\infty$, and$\|\varphi_k(x_n)-\varphi_k(x)\|\to 0$, and $\|\varphi_k^*(x_n)-\varphi_k^*(x)\|\to 0$ as $n\to\infty$.

Step II. We prove that the operator Q maps a bounded set into a bounded set in $\mathbb{E}$.

For $r_2>0$, there exists a constant $N>0$ such that, for each $x\in B_{r_2}=\{x\in\mathbb{E}:\|x\|\le r_2\}$, then $\|Qx\|\le N$. Then, for any $t\in J$ and $x\in B_{r_2}$, we have

$$
\begin{aligned}
|(Qx)(t)| \;\le\;& {}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|F_x(s)|(T)+|\lambda|_{t_m}\mathfrak{I}^{\alpha_m,\rho,\psi_m}|x(s)|(T)+\sum_{i=1}^{m}|H_{i-1}(x)|+\sum_{i=1}^{m}|G_{i-1}(x)|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+\frac{|\Omega_1||\mathcal{R}(x,F_x)|+|\Omega_3||\mathcal{K}(x,F_x)|}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+\frac{|\Omega_4||\mathcal{K}(x,F_x)|+|\Omega_2||\mathcal{R}(x,F_x)|}{|\Omega_5|}. \quad (3.22)
\end{aligned}
$$

It follows from (H_3) and (H_4), that

$$
|F_x(t)|\le h_1^*+2h_2^*r_2,\quad |\varphi_k(x)|\le k_1,\quad |\varphi_k^*(x)|\le k_1^*,\quad k=1,2,\ldots,m. \quad (3.23)
$$

Then by substituting (3.23) into (3.22) with the notations (2.6), (2.11)–(2.15) and (3.2)–(3.5), we have

$$
\begin{aligned}
|(Qx)(t)| \;\le\;& \frac{h_1^*+2h_2^*r_2}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m+\beta_m-1}\psi_m'(s)ds\\
&+\frac{|\lambda|r_2}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|(\psi_m(T)-\psi_m(s))^{\alpha_m-1}\psi_m'(s)ds\\
&+\sum_{i=1}^{m}\Bigg[\frac{h_1^*+2h_2^*r_2}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|(\psi_{i-1}(t_i)-\psi_{i-1}(s))^{\alpha_{i-1}+\beta_{i-1}-1}\psi_{i-1}'(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&\frac{|\lambda|r_2}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\psi'_{i-1}(s)ds+k_1\Bigg]\\
&+\sum_{i=1}^{m}\Bigg[\frac{h_1^*+2h_2^*r_2}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\beta_{i-1}-1}\psi'_{i-1}(s)ds\\
&+\frac{h_1^*+2h_2^*r_2}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\beta_{i-1}-1}\psi'_{i-1}(s)ds+k_1^*\Bigg]\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+\Bigg\{|\Omega_1|\Bigg(|\xi_2|+\frac{|\kappa_2|(h_1^*+2h_2^*r_2)}{\rho^{\beta_m}\Gamma(\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\beta_m-1}\psi'_m(s)ds\\
&+\frac{|\kappa_2||\lambda|(h_1^*+2h_2^*r_2)}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m+\beta_m-1}\psi'_m(s)ds\\
&+\frac{|\kappa_2|\lambda^2r_2}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi'_m(s)ds+|\kappa_2|\sum_{i=1}^{m}\Bigg[\frac{h_1^*+2h_2^*r_2}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\\
&\times\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\beta_{i-1}-1}\psi'_{i-1}(s)ds+k_1^*\Bigg]\Bigg[1+|\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg]\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\Bigg[\frac{h_1^*+2h_2^*r_2}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}+\beta_{i-1}-1}\\
&\times\psi'_{i-1}(s)ds+\frac{|\lambda|r_2}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\psi'_{i-1}(s)ds+k_1\Bigg]\Bigg)\\
&+|\Omega_3|\Bigg(|\xi_1|+\frac{|\kappa_1||\lambda|r_2}{\rho^{\alpha_m}\Gamma(\alpha_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi'_m(s)ds\\
&\frac{|\kappa_1|(h_1^*+2h_2^*r_2)}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m+\beta_m-1}\psi'_m(s)ds\\
&+|\kappa_1|\sum_{i=1}^{m}\Bigg[\frac{h_1^*+2h_2^*r_2}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\beta_{i-1}-1}\psi'_{i-1}(s)ds+k_1^*\Bigg]\\
&\times\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+|\kappa_1|\sum_{i=1}^{m}\Bigg[\frac{h_1^*+2h_2^*r_2}{\rho^{\alpha_{i-1}+\beta_{i-1}}\Gamma(\alpha_{i-1}+\beta_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\\
&\times\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}+\beta_{i-1}-1}\psi'_{i-1}(s)ds+\frac{|\lambda|r_2}{\rho^{\alpha_{i-1}}\Gamma(\alpha_{i-1})}\int_{t_{i-1}}^{t_i}\left|e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)}\right|\\
&\times\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1}\psi'_{i-1}(s)ds+k_1\Bigg]\Bigg)\Bigg\}\frac{\Lambda_2}{|\Omega_5|}+\frac{1}{|\Omega_5|}\Bigg\{|\Omega_4|\Bigg(|\xi_1|+\frac{|\kappa_1||\lambda|r_2}{\rho^{\alpha_m}\Gamma(\alpha_m)}\\
&\times\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1}\psi'_m(s)ds+\frac{|\kappa_1|(h_1^*+2h_2^*r_2)}{\rho^{\alpha_m+\beta_m}\Gamma(\alpha_m+\beta_m)}\\
&\times\int_{t_m}^{T}\left|e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)}\right|\left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m+\beta_m-1}\psi'_m(s)ds+|\kappa_1|\sum_{i=1}^{m}\Bigg[\frac{h_1^*+2h_2^*r_2}{\rho^{\beta_{i-1}}\Gamma(\beta_{i-1})}
\end{aligned}
$$

$$
\begin{aligned}
& \times \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| \left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\beta_{i-1}-1} \psi'_{i-1}(s)ds + k_1^* \Bigg] \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \\
& + |\kappa_1| \sum_{i=1}^{m} \left[\frac{h_1^* + 2h_2^* r_2}{\rho^{\alpha_{i-1}+\beta_{i-1}} \Gamma(\alpha_{i-1}+\beta_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| \left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}+\beta_{i-1}-1} \right. \\
& \left. \times \psi'_{i-1}(s)ds + \frac{|\lambda| r_2}{\rho^{\alpha_{i-1}} \Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| \left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1} \psi'_{i-1}(s)ds + k_1 \right] \Bigg) \\
& + |\Omega_2| \Bigg(|\xi_2| + \frac{|\kappa_2|(h_1^* + 2h_2^* r_2)}{\rho^{\beta_m} \Gamma(\beta_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| \left(\psi_m(T)-\psi_m(s)\right)^{\beta_m-1} \psi'_m(s)ds \\
& + \frac{|\kappa_2||\lambda|(h_1^* + 2h_2^* r_2)}{\rho^{\alpha_m+\beta_m} \Gamma(\alpha_m+\beta_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| \left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m+\beta_m-1} \psi'_m(s)ds \\
& + \frac{|\kappa_2|\lambda^2 r_2)}{\rho^{\alpha_m} \Gamma(\alpha_m)} \int_{t_m}^{T} \left| e^{\frac{\rho-1}{\rho}\left(\psi_m(T)-\psi_m(s)\right)} \right| \left(\psi_m(T)-\psi_m(s)\right)^{\alpha_m-1} \psi'_m(s)ds + |\kappa_2| \sum_{i=1}^{m} \Bigg[\frac{h_1^* + 2h_2^* r_2}{\rho^{\beta_{i-1}} \Gamma(\beta_{i-1})} \\
& \times \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| \left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\beta_{i-1}-1} \psi'_{i-1}(s)ds + k_1^* \Bigg] \left[1 + |\lambda| \sum_{j=i}^{m} \Phi^{\alpha_j}(t_j, t_{j+1}) \right] \\
& + |\kappa_2||\lambda| \sum_{i=1}^{m} \left[\frac{h_1^* + 2h_2^* r_2}{\rho^{\alpha_{i-1}+\beta_{i-1}} \Gamma(\alpha_{i-1}+\beta_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| \left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}+\beta_{i-1}-1} \right. \\
& \left. \times \psi'_{i-1}(s)ds + \frac{|\lambda| r_2}{\rho^{\alpha_{i-1}} \Gamma(\alpha_{i-1})} \int_{t_{i-1}}^{t_i} \left| e^{\frac{\rho-1}{\rho}\left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)} \right| \left(\psi_{i-1}(t_i)-\psi_{i-1}(s)\right)^{\alpha_{i-1}-1} \psi'_{i-1}(s)ds + k_1 \right] \Bigg) \Bigg\} \\
\leq & \left[\Lambda_1 + \Big(|\kappa_1| \Lambda_1 (|\Omega_3| \Lambda_2 + |\Omega_4|) + |\kappa_2| (|\lambda| \Lambda_1 + \Lambda_4)(|\Omega_1| \Lambda_2 + |\Omega_2|) \Big) \frac{1}{|\Omega_5|} \right] (h_1^* + 2h_2^* r_2) \\
& + |\lambda| \Lambda_2 \left[1 + \Big(|\kappa_1| (\Lambda_2 |\Omega_3| + \Omega_4|) + |\lambda||\kappa_2| (\Lambda_2 |\Omega_1| + |\Omega_2|) \Big) \frac{1}{|\Omega_5|} \right] r_2 \\
& + m \left[1 + \Big(|\kappa_1| (\Lambda_2 |\Omega_3| + |\Omega_4|) + |\lambda||\kappa_2| (\Lambda_2 |\Omega_1| + |\Omega_2|) \Big) \frac{1}{|\Omega_5|} \right] k_1 \\
& + \left[\Lambda_3 + \Big(|\kappa_1| \Lambda_3 (\Lambda_2 |\Omega_3| + |\Omega_4|) + |\kappa_2| (|\lambda| \Lambda_3 + m)(\Lambda_2 |\Omega_1| + |\Omega_2|) \Big) \frac{1}{|\Omega_5|} \right] k_1^* \\
& + \Big(|\xi_1| (\Lambda_2 |\Omega_3| + |\Omega_4|) + |\xi_2| (\Lambda_2 |\Omega_1| + |\Omega_2|) \Big) \frac{1}{|\Omega_5|},
\end{aligned}
$$

we estimate $\|Qx\| \leq \Theta_1 \left(h_1^* + 2h_2^* r_2\right) + (|\lambda|\Lambda_2 r_2 + mk_1)\Theta_2 + \Theta_3 k_1^* + \Theta_4 := N$, which implies that $\|Qx\| \leq N$. Hence, the set QB_{r_2} is uniformly bounded.

Step III. We prove that Q maps a bounded set into an equicontinuous set of $\mathbb{E}$.

Let $\tau_1, \tau_2 \in J_k$ for some $k \in \{0, 1, 2, \ldots, m\}$ with $\tau_1 < \tau_2$. Then, for any $x \in B_{r_2}$, where B_{r_2} is as defined in Step II, by using the property of f is bounded on the compact set $J \times B_{r_2}$, we have

$$
\begin{aligned}
& |(Qx)(\tau_2) - (Qx)(\tau_1)| \\
\leq & \left\{ \sum_{i=1}^{m} \Big(\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1}, t_i) \left(h_1^* + 2h_2^* r_2\right) + |\lambda| \Phi^{\alpha_{i-1}}(t_{i-1}, t_i) r_2 + k_1 \Big) \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\Big(\Phi^{\beta_{i-1}}(t_{i-1},t_i)\,(h_1^*+2h_2^*r_2)+k_1^*\Big)\sum_{j=i}^{k-1}\bigg(\Phi^{\alpha_j}(t_j,t_{j+1})+\Big|\Phi^{\alpha_k}(t_k,\tau_2)-\Phi^{\alpha_k}(t_k,\tau_1)\Big|\bigg)\\
&+\frac{|\Omega_1||\mathcal{R}(x,F_x)|+|\Omega_3||\mathcal{K}(x,F_x)|}{|\Omega_5|}\sum_{i=1}^{k}\bigg(\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+\Big|\Phi^{\alpha_k}(t_k,\tau_2)-\Phi^{\alpha_k}(t_k,\tau_1)\Big|\bigg)\\
&+\frac{|\Omega_4||\mathcal{K}(x,F_x)|+|\Omega_2||\mathcal{R}(x,F_x)|}{|\Omega_5|}\Bigg\}\bigg|e^{\frac{\rho-1}{\rho}(\psi_k(\tau_2)-\psi_k(t_k))}-e^{\frac{\rho-1}{\rho}(\psi_k(\tau_1)-\psi_k(t_k))}\bigg|\\
&+\frac{h_1^*+2h_2^*r_2}{\rho^{\alpha_k+\beta_k}\Gamma(\alpha_k+\beta_k)}\bigg(\int_{\tau_1}^{\tau_2}e^{\frac{\rho-1}{\rho}\left(\psi_k(\tau_2)-\psi_k(s)\right)}(\psi_k(\tau_2)-\psi_k(s))^{\alpha_k+\beta_k-1}\psi_k'(s)ds\\
&+\int_{t_k}^{\tau_1}\bigg|e^{\frac{\rho-1}{\rho}\left(\psi_k(\tau_2)-\psi_k(s)\right)}(\psi_k(\tau_2)-\psi_k(s))^{\alpha_k+\beta_k-1}-e^{\frac{\rho-1}{\rho}\left(\psi_k(\tau_1)-\psi_k(s)\right)}(\psi_k(\tau_1)-\psi_k(s))^{\alpha_k+\beta_k-1}\bigg|\psi_k'(s)ds\bigg)\\
&+\frac{|\lambda|r_2}{\Gamma(\alpha_k)}\bigg(\int_{\tau_1}^{\tau_2}e^{\frac{\rho-1}{\rho}\left(\psi_k(\tau_2)-\psi_k(s)\right)}(\psi_k(\tau_2)-\psi_k(s))^{\alpha_k-1}\psi_k'(s)ds\\
&+\int_{t_k}^{\tau_1}\bigg|e^{\frac{\rho-1}{\rho}\left(\psi_k(\tau_2)-\psi_k(s)\right)}(\psi_k(\tau_2)-\psi_k(s))^{\alpha_k-1}-e^{\frac{\rho-1}{\rho}\left(\psi_k(\tau_1)-\psi_k(s)\right)}(\psi_k(\tau_1)-\psi_k(s))^{\alpha_k-1}\bigg|\psi_k'(s)ds\bigg).
\end{aligned}
$$

By using the notations (2.6), (2.11)–(2.15) and (3.2)–(3.5), we obtain that

$$
\begin{aligned}
&|(Qx)(\tau_2)-(Qx)(\tau_1)|\\
\le\;&\Bigg\{\sum_{i=1}^{m}\Big(\Phi^{\alpha_{i-1}+\beta_{i-1}}(t_{i-1},t_i)\,(h_1^*+2h_2^*r_2)+|\lambda|\Phi^{\alpha_{i-1}}(t_{i-1},t_i)r_2+k_1\Big)\\
&+\sum_{i=1}^{m}\Big(\Phi^{\beta_{i-1}}(t_{i-1},t_i)\,(h_1^*+2h_2^*r_2)+k_1^*\Big)\sum_{j=i}^{k-1}\bigg(\Phi^{\alpha_j}(t_j,t_{j+1})+\Big|\Phi^{\alpha_k}(t_k,\tau_2)-\Phi^{\alpha_k}(t_k,\tau_1)\Big|\bigg)\\
&+\bigg[\Big(|\kappa_1|\Lambda_1|\Omega_3|+|\kappa_2||\Omega_1|(|\lambda|\Lambda_1+\Lambda_4)\Big)(h_1^*+2h_2^*r_2)+|\lambda|\Lambda_2\Big(|\kappa_1||\Omega_3|+|\kappa_2||\lambda||\Omega_1|\Big)r_2\\
&+m\Big(|\kappa_1||\Omega_3|+|\kappa_2||\lambda||\Omega_1|\Big)k_1+\Big(|\kappa_1|\Lambda_3|\Omega_3|+|\kappa_2||\Omega_1|(|\lambda|\Lambda_3+m)\Big)k_1^*+|\xi_1||\Omega_3|+|\xi_2||\Omega_1|\bigg]\\
&\times\frac{1}{|\Omega_5|}\sum_{i=1}^{m}\bigg(\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+\Big|\Phi^{\alpha_k}(t_k,\tau_2)-\Phi^{\alpha_k}(t_k,\tau_1)\Big|\bigg)+\bigg[|\lambda|\Lambda_2\Big(|\kappa_1||\Omega_4|+|\kappa_2||\lambda||\Omega_2|\Big)r_2\\
&+\Big(|\kappa_1|\Lambda_1|\Omega_4|+|\kappa_2||\Omega_2|(|\lambda|\Lambda_1+\Lambda_4)\Big)(h_1^*+2h_2^*r_2)+m\Big(|\kappa_1||\Omega_4|+|\kappa_2||\lambda||\Omega_2|\Big)k_1\\
&+\Big(|\kappa_1|\Lambda_3|\Omega_4|+|\kappa_2||\Omega_2|(|\lambda|\Lambda_3+m)\Big)k_1^*+|\xi_1||\Omega_4|+|\xi_2||\Omega_2|\bigg]\frac{1}{|\Omega_5|}\Bigg\}\bigg|e^{\frac{\rho-1}{\rho}(\psi_k(\tau_2)-\psi_k(t_k))}-e^{\frac{\rho-1}{\rho}(\psi_k(\tau_1)-\psi_k(t_k))}\bigg|\\
&+\frac{h_1^*+2h_2^*r_2}{\rho^{\alpha_k+\beta_k}\Gamma(\alpha_k+\beta_k+1)}\bigg(2\Big|\psi_k(\tau_2)-\psi_k(\tau_1)\Big|^{\alpha_k+\beta_k}+\Big|(\psi_k(\tau_2)-\psi_k(t_k))^{\alpha_k+\beta_k}-(\psi_k(\tau_1)-\psi_k(t_k))^{\alpha_k+\beta_k}\Big|\bigg)\\
&+\frac{|\lambda|r_2}{\Gamma(\alpha_k+1)}\bigg(2\Big|\psi_k(\tau_2)-\psi_k(\tau_1)\Big|^{\alpha_k}+\Big|(\psi_k(\tau_2)-\psi_k(t_k))^{\alpha_k}-(\psi_k(\tau_1)-\psi_k(t_k))^{\alpha_k}\Big|\bigg).
\end{aligned}
$$

From the above inequality, we get that $\Big|e^{\frac{\rho-1}{\rho}(\psi_k(\tau_2)-\psi_k(t_k))}-e^{\frac{\rho-1}{\rho}(\psi_k(\tau_1)-\psi_k(t_k))}\Big|\to 0$, $\Big|\psi_k(\tau_2)-\psi_k(\tau_1)\Big|^u\to 0$ and $\Big|(\psi_k(\tau_2)-\psi_k(t_k))^u-(\psi_k(\tau_1)-\psi_k(t_k))^u\Big|\to 0$ as $\tau_2\to\tau_1$, where $u=\{\alpha_k,\alpha_k+\beta_k\}$. This inequality is independent of unknown variable $x \in B_{r_2}$ and tends to zero as $\tau_2 \to \tau_1$, which implies that $\|(Qx)(\tau_2) - (Qx)(\tau_1)\| \to 0$ as $\tau_2 \to \tau_1$. Therefore by the Arzelá-Ascoli theorem, we can conclude that the operator $Q : \mathbb{E} \to \mathbb{E}$ is completely continuous.

Step IV. The set $\mathbb{D} = \{x \in \mathbb{E} : x = \sigma Qx, \}$ is bounded (a priori bounds).

Let $x \in \mathbb{D}$, then $x = \sigma Qx$ for some $0 < \sigma < 1$. From (H_3) and (H_4), for each $t \in J$, we get the result by using the same process in Step II,

$$\begin{aligned}
|x(t)| &= |\sigma(Qx)(t)| \\
&\leq \Bigg([\Lambda_1 + \Big(|\kappa_1|\Lambda_1(|\Omega_3|\Lambda_2 + |\Omega_4|) + |\kappa_2|(|\lambda|\Lambda_1 + \Lambda_4)(|\Omega_1|\Lambda_2 + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg] (h_1^* + 2h_2^* r_2) \\
&\quad + |\lambda|\Lambda_2 \left[1 + \Big(|\kappa_1|(\Lambda_2|\Omega_3| + \Omega_4|) + |\lambda||\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\right] r_2 \\
&\quad + m \left[1 + \Big(|\kappa_1|(\Lambda_2|\Omega_3| + |\Omega_4|) + |\lambda||\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\right] k_1 \\
&\quad + \left[\Lambda_3 + \Big(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3| + |\Omega_4|) + |\kappa_2|(|\lambda|\Lambda_3 + m)(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\right] k_1^* \\
&\quad + \Big(|\xi_1|(\Lambda_2|\Omega_3| + |\Omega_4|) + |\xi_2|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}.
\end{aligned}$$

Then, $\|x\| \leq \Theta_1 \left(h_1^* + 2h_2^* r_2\right) + (|\lambda|\Lambda_2 r_2 + mk_1)\Theta_2 + \Theta_3 k_1^* + \Theta_4 := N < \infty$. This implies that the set $\mathbb{D}$ is bounded. By all the assumptions of Theorem 3.2, we conclude that there exists a positive constant N such that $\|x\| \leq N < \infty$. By applying Schaefer's fixed point theorem (Theorem 2.13), the operator Q has at least one fixed point which is a solution of problem (1.1). The proof is completed. □

4. Ulam stability results

This section is discussed the different type of Ulam's stability such as $\mathbb{UH}$ stable, generalized $\mathbb{UH}$ stable, $\mathbb{UHR}$ stable and generalized $\mathbb{UHR}$ stable of the problem (1.1).

Now, we introduce Ulam's stability concepts for the problem (1.1). Let $\phi \in C(J, \mathbb{R}^+)$ be a nondecreasing function, $\epsilon > 0$, $\upsilon \geq 0$, $z \in \mathbb{E}$ such that, for $t \in J_k$, $k = 1, 2, \ldots, m$, the following sets of inequalities are satisfied:

$$\begin{cases}
\left|{}^C_{t_k}\mathfrak{D}^{\beta_k,\rho,\psi_k}\left({}^C_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k} + \lambda\right) z(t) - f(t, z(t), z(\mu t))\right| \leq \epsilon, \\
\left|z(t_k^+) - z(t_k^-) - \varphi_k(z(t_k))\right| \leq \epsilon, \\
\left|{}^C_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k} x(t_k^+) - {}_{t_{k-1}}{}^C\mathfrak{D}^{\alpha_k,\rho,\psi_k} x(t_k^-) - \varphi_k^*(x(t_k))\right| \leq \epsilon.
\end{cases} \tag{4.1}$$

$$\begin{cases}
\left|{}^C_{t_k}\mathfrak{D}^{\beta_k,\rho,\psi_k}\left({}^C_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k} + \lambda\right) z(t) - f(t, z(t), z(\mu t))\right| \leq \phi(t), \\
\left|z(t_k^+) - z(t_k^-) - \varphi_k(z(t_k))\right| \leq \upsilon, \\
\left|{}^C_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k} x(t_k^+) - {}_{t_{k-1}}{}^C\mathfrak{D}^{\alpha_k,\rho,\psi_k} x(t_k^-) - \varphi_k^*(x(t_k))\right| \leq \upsilon.
\end{cases} \tag{4.2}$$

$$\begin{cases} \left| {}^{C}_{t_k}\mathfrak{D}^{\beta_k,\rho,\psi_k} \left({}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k} + \lambda \right) z(t) - f(t, z(t), z(\mu t)) \right| \leq \epsilon\phi(t), \\ \left| z(t_k^+) - z(t_k^-) - \varphi_k(z(t_k)) \right| \leq \epsilon\upsilon, \\ \left| {}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k} x(t_k^+) - {}_{t_{k-1}}{}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k} x(t_k^-) - \varphi_k^*(x(t_k)) \right| \leq \epsilon\upsilon. \end{cases} \tag{4.3}$$

Definition 4.1. *If for $\epsilon > 0$ there exists a constant $C_f > 0$ such that, for any solution $z \in \mathbb{E}$ of inequality* (4.1)*, there is a unique solution $x \in \mathbb{E}$ of system* (1.1) *that satisfies*

$$|z(t) - x(t)| \leq C_f\epsilon, \quad t \in J,$$

then system (1.1) *is* $\mathbb{UH}$ *stable.*

Definition 4.2. *If for $\epsilon > 0$ and set of positive real numbers $\mathbb{R}^+$ there exists $\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$, with $\phi(0) = 0$ such that, for any solution $z \in \mathbb{E}$ of inequality* (4.2)*, there exist $\epsilon > 0$ and a unique solution $x \in \mathbb{E}$ of system* (1.1) *that satisfies*

$$|z(t) - x(t)| \leq \phi(\epsilon), \quad t \in J,$$

then system (1.1) *is generalized* $\mathbb{UH}$ *stable.*

Definition 4.3. *If for $\epsilon > 0$ there exists a real number $C_f > 0$ such that, for any solution $z \in \mathbb{E}$ of inequality* (4.3)*, there is a unique solution $x \in \mathbb{E}$ of system* (1.1) *that satisfies*

$$|z(t) - x(t)| \leq C_f\epsilon(\upsilon + \phi(t)), \quad t \in J,$$

then system (1.1) *is* $\mathbb{UHR}$ *stable with respect to* (υ, ϕ).

Definition 4.4. *If there exists a real number $C_f > 0$ such that, for any solution $z \in \mathbb{E}$ of inequality* (4.2)*, there is a unique solution $x \in \mathbb{E}$ of system* (1.1) *that satisfies*

$$|z(t) - x(t)| \leq C_f(\upsilon + \phi(t)), \quad t \in J,$$

then system (1.1) *is generalized* $\mathbb{UHR}$ *stable with respect to* (υ, ϕ).

Remark 4.5. *It is clear that:* (*i*) *Definition* 4.1 $\Longrightarrow$ *Definition* 4.2*;* (*ii*) *Definition* 4.3 $\Longrightarrow$ *Definition* 4.4*;* (*iii*) *Definition* 4.3 *for $\upsilon + \phi(t) = 1$* $\Longrightarrow$ *Definition* 4.1.

Remark 4.6. *The function $z \in \mathbb{E}$ is called a solution for inequality* (4.1) *if there exists a function $w \in \mathbb{E}$ together with a sequence w_k, $k = 1, 2, \ldots, m$ (which depends on z) such that*

(A_1) $|w(t)| \leq \epsilon$, $|w_k| \leq \epsilon$, $t \in J$,
(A_2) ${}^{C}_{t_k}\mathfrak{D}^{\beta_k,\rho,\psi_k} \left({}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k} + \lambda \right) z(t) = f(t, z(t), z(\mu t)) + w(t)$, $t \in J$,
(A_3) $z(t_k^+) - z(t_k^-) = \varphi_k(z(t_k)) + w_k$, $t \in J$,
(A_4) ${}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k} z(t_k^+) - {}_{t_{k-1}}{}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k} z(t_k^-) = \varphi_k^*(z(t_k)) + w_k$, $t \in J$.

Remark 4.7. *The function $z \in \mathbb{E}$ is called a solution for inequality* (4.2) *if there exists a function $w \in \mathbb{E}$ together with a sequence w_k, $k = 1, 2, \ldots, m$ (which depends on z) such that*

(B_1) $|w(t)| \leq \phi(t)$, $|w_k| \leq \upsilon$, $t \in J$,

(B_2) ${}^{C}_{t_k}\mathfrak{D}^{\beta_k,\rho,\psi_k}\left({}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k}+\lambda\right)z(t)=f(t,z(t),z(\mu t))+w(t)$, $t\in J$,
(B_3) $z(t_k^+)-z(t_k^-)=\varphi_k(z(t_k))+w_k$, $t\in J$,
(B_4) ${}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k}z(t_k^+)-{}_{t_{k-1}}{}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k}z(t_k^-)=\varphi_k^*(z(t_k))+w_k$, $t\in J$.

Remark 4.8. *The function $z\in\mathbb{E}$ is called a solution for inequality* (4.3) *if there exists a function $w\in\mathbb{E}$ together with a sequence w_k, $k=1,2,\ldots,m$ (which depends on z) such that*

(C_1) $|w(t)|\le\epsilon\phi(t)$, $|w_k|\le\epsilon\upsilon$, $t\in J$,
(C_2) ${}^{C}_{t_k}\mathfrak{D}^{\beta_k,\rho,\psi_k}\left({}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k}+\lambda\right)z(t)=f(t,z(t),z(\mu t))+w(t)$, $t\in J$,
(C_3) $z(t_k^+)-z(t_k^-)=\varphi_k(z(t_k))+w_k$, $t\in J$,
(C_4) ${}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k}z(t_k^+)-{}_{t_{k-1}}{}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k}z(t_k^-)=\varphi_k^*(z(t_k))+w_k$, $t\in J$.

4.1. Ulam–Hyers stability

In this subsection, we establish the results related to $\mathbb{UH}$ stability of system (1.1).

Theorem 4.9. *Assume that $f:J\times\mathbb{R}^2\to\mathbb{R}$, $\varphi_k:\mathbb{R}\to\mathbb{R}$ is continuous functions. If assumptions (H_1), (H_2) and the inequality*

$$2L_1\Theta_1+(mM_1+|\lambda|\Lambda_2)\Theta_2+M_1^*\Theta_3<1 \tag{4.4}$$

are satiafied, then system (1.1) *is $\mathbb{UH}$ stable.*

Proof. Let z be any solution of inequality (4.1). Then, by Remark 4.6 (A_2)–(A_4), we have

$$\begin{cases} {}^{C}_{t_k}\mathfrak{D}^{\beta_k,\rho,\psi_k}\left({}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k}+\lambda\right)z(t)=f(t,z(t),z(\mu t))+w(t),\\ z(t_k^+)-z(t_k^-)=\varphi_k(z(t_k))+w_k,\\ {}^{C}_{t_k}\mathfrak{D}^{\alpha_k,\rho,\psi_k}z(t_k^+)-{}_{t_{k-1}}{}^{C}\mathfrak{D}^{\alpha_k,\rho,\psi_k}z(t_k^-)=\varphi_k^*(z(t_k))+w_k,\\ \eta_1 z(0)+\kappa_1 z(T)=\xi_1,\qquad \eta_2{}^{C}_{t_0}\mathfrak{D}^{\alpha_0,\rho,\psi_0}z(0)+\kappa_2{}^{C}_{t_m}\mathfrak{D}^{\alpha_m,\rho,\psi_m}z(T)=\xi_2,\end{cases} \tag{4.5}$$

By Lemma 2.11, the solution of (4.5) is given by

$$\begin{aligned} z(t) ={}& {}_{t_k}\mathfrak{I}^{\alpha_k+\beta_k,\rho,\psi_k}F_z(t)-\lambda_{t_k}\mathfrak{I}^{\alpha_k,\rho,\psi_k}z(t)+\Bigg\{\sum_{i=1}^{k}H_{i-1}(z)\prod_{j=i}^{k-1}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\\ &+\sum_{i=1}^{k}G_{i-1}(z)\sum_{j=i}^{k-1}\left(\Phi^{\alpha_j}(t_j,t_{j+1})+\Phi^{\alpha_k}(t_k,t)\right)\prod_{j=i}^{k-1}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\\ &+\frac{\Omega_1\mathcal{R}(z,F_z)-\Omega_3\mathcal{K}(z,F_z)}{\Omega_5}\sum_{i=1}^{k}(\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+\Phi^{\alpha_k}(t_k,t))\prod_{i=1}^{k}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\\ &+\frac{\Omega_4\mathcal{K}(z,F_z)-\Omega_2\mathcal{R}(z,F_z)}{\Omega_5}\prod_{i=1}^{k}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\Bigg\}e^{\frac{\rho-1}{\rho}(\psi_k(t)-\psi_k(t_k))}\\ &+{}_{t_k}\mathfrak{I}^{\alpha_k+\beta_k,\rho,\psi_k}w(t)+\Bigg\{\sum_{i=1}^{k}\left({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\right)\prod_{j=i}^{k-1}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)} \end{aligned}$$

$$+\sum_{i=1}^{k}\Big({}_{t_{i-1}}\mathfrak{J}^{\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\sum_{j=i}^{k-1}\Big(\Phi^{\alpha_j}(t_j,t_{j+1})+\Phi^{\alpha_k}(t_k,t)\Big)\prod_{j=i}^{k-1}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}$$

$$+\Bigg[\Omega_1\Bigg(-\kappa_2{}_{t_m}\mathfrak{J}^{\beta_m,\rho,\psi_m}w(T)+\kappa_2\lambda_{t_m}\mathfrak{J}^{\alpha_m+\beta_m,\rho,\psi_m}w(T)$$

$$-\kappa_2\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{J}^{\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\Bigg(1-\lambda\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}$$

$$+\kappa_2\lambda\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{J}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\Bigg)$$

$$-\Omega_3\Bigg(-\kappa_1{}_{t_m}\mathfrak{J}^{\alpha_m+\beta_m,\rho,\psi_m}w(T)-\kappa_1\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{J}^{\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})$$

$$\times\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}-\kappa_1\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{J}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\Bigg)\Bigg]$$

$$\times\frac{1}{\Omega_5}\sum_{i=1}^{k}(\Phi^{\alpha_{i-1}}(t_{i-1},t_i)+\Phi^{\alpha_k}(t_k,t))\prod_{i=1}^{k}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}$$

$$+\Bigg[\Omega_4\Bigg(-\kappa_1{}_{t_m}\mathfrak{J}^{\alpha_m+\beta_m,\rho,\psi_m}w(T)-\kappa_1\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{J}^{\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})$$

$$\times\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}-\kappa_1\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{J}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\Bigg)$$

$$-\Omega_2\Bigg(-\kappa_2{}_{t_m}\mathfrak{J}^{\beta_m,\rho,\psi_m}w(T)+\kappa_2\lambda_{t_m}\mathfrak{J}^{\alpha_m+\beta_m,\rho,\psi_m}w(T)$$

$$-\kappa_2\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{J}^{\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\Bigg(1-\lambda\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}$$

$$+\kappa_2\lambda\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{J}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}w(t_i)+w_k\Big)\prod_{j=i}^{m}e^{\frac{\rho-1}{\rho}\left(\psi_j(t_{j+1})-\psi_j(t_j)\right)}\Bigg)\Bigg]$$

$$\times\frac{1}{\Omega_5}\prod_{i=1}^{k}e^{\frac{\rho-1}{\rho}(\psi_{i-1}(t_i)-\psi_{i-1}(t_{i-1}))}\Bigg\}e^{\frac{\rho-1}{\rho}(\psi_k(t)-\psi_k(t_k))},\quad t\in J_k,\ k=0,1,2,\ldots,m.$$

From Remark 4.6 (A_1) with (H_1), (H_2) and the fact of $0<e^{\frac{\rho-1}{\rho}(\psi_a(u)-\psi_a(s))}\leq 1$ for $0\leq s\leq u\leq T$, it follows that

$$\begin{aligned}|z(t)-x(t)| \leq\ & {}_{t_m}\mathfrak{J}^{\alpha_m+\beta_m,\rho,\psi_m}|F_z(s)-F_x(s)|(T)+|\lambda|_{t_m}\mathfrak{J}^{\alpha_m,\rho,\psi_m}|z(s)-x(s)|(T)\\ &+\sum_{i=1}^{m}|H_{i-1}(z)-H_{i-1}(x)|+\sum_{i=1}^{m}|G_{i-1}(z)-G_{i-1}(x)|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\ &+\Big(|\Omega_1||\mathcal{R}(z,F_z)-\mathcal{R}(x,F_x)|+|\Omega_3||\mathcal{K}(z,F_z)-\mathcal{K}(x,F_x)|\Big)\frac{1}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\end{aligned}$$

$$
\begin{aligned}
&+\Big(|\Omega_4||\mathcal{K}(z,F_z)-\mathcal{K}(x,F_x)|+|\Omega_2||\mathcal{R}(z,F_z)-\mathcal{R}(x,F_x)|\Big)\frac{1}{|\Omega_5|}\\
&+{}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(t)|+\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\\
&+\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})+\Bigg[|\Omega_1|\Bigg(|\kappa_2|_{t_m}\mathfrak{I}^{\beta_m,\rho,\psi_m}|w(T)|\\
&+|\kappa_2||\lambda|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(T)|+|\kappa_2|\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\Bigg(1+|\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\Bigg)+|\Omega_3|\Bigg(|\kappa_1|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(T)|\\
&+|\kappa_1|\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+|\kappa_1|\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\Bigg)\Bigg]\frac{1}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\\
&+\Bigg[|\Omega_4|\Bigg(|\kappa_1|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(T)|+|\kappa_1|\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+|\kappa_1|\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\Bigg)+|\Omega_2|\Bigg(|\kappa_2|_{t_m}\mathfrak{I}^{\beta_m,\rho,\psi_m}|w(T)|\\
&+|\kappa_2||\lambda|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(T)|+|\kappa_2|\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\Bigg(1+|\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\Bigg)\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)|+|w_k|\Big)\Bigg)\Bigg]\frac{1}{|\Omega_5|}\\
\leq\;&\Bigg\{2L_1\Bigg[\Lambda_1+\Big(|\kappa_1|\Lambda_1(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_1+\Lambda_4)(\Lambda_2|\Omega_1|+|\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+(mM_1+|\lambda|\Lambda_2)\Bigg[1+\Big(|\kappa_1|(\Lambda_2|\Omega_3|+|\Omega_4|)+|\lambda||\kappa_2|(\Lambda_2|\Omega_1|+|\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+M_1^*\Bigg[\Lambda_3+\Big(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(\Lambda_2|\Omega_1|+|\Omega_2|)(|\lambda|\Lambda_3+m)\Big)\frac{1}{|\Omega_5|}\Bigg]\Bigg\}|z(t)-x(t)|\\
&+\Bigg\{\Bigg[\Lambda_1+\Big(|\kappa_1|\Lambda_1(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(|\lambda|\Lambda_1+\Lambda_4)(\Lambda_2|\Omega_1|+|\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+m\Bigg[1+\Big(|\kappa_1|(\Lambda_2|\Omega_3|+|\Omega_4|)+|\lambda||\kappa_2|(\Lambda_2|\Omega_1|+|\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+\Bigg[\Lambda_3+\Big(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3|+|\Omega_4|)+|\kappa_2|(\Lambda_2|\Omega_1|+|\Omega_2|)(|\lambda|\Lambda_3+m)\Big)\frac{1}{|\Omega_5|}\Bigg]\Bigg\}\epsilon\\
=\;&\Big(2L_1\Theta_1+(mM_1+|\lambda|\Lambda_2)\Theta_2+M_1^*\Theta_3\Big)|z(t)-x(t)|+\Big(\Theta_1+m\Theta_2+\Theta_3\Big)\epsilon.
\end{aligned}
$$

This implies that

$$|z(t) - x(t)| \leq \frac{(\Theta_1 + m\Theta_2 + \Theta_3)\,\epsilon}{1 - \Big(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3\Big)},$$

with $(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3) < 1$. By setting

$$C_f = \frac{\Theta_1 + m\Theta_2 + \Theta_3}{1 - \Big(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3\Big)},$$

we end up with $|z(t) - x(t)| \leq C_f\epsilon$. Hence, the system (1.1) is $\mathbb{UH}$ stable. The proof is completed. □

Corollary 4.10. *In Theorem* 4.9*, if we set* $\phi(\epsilon) = C_f(\epsilon)$ *such that* $\phi(0) = 0$*, then the system* (1.1) *is generalized* $\mathbb{UH}$ *stable.*

4.2. Ulam–Hyers–Rassias stability

For the proof of our next result, we assume the following assumption

(H_5) There eixsts a nondecreasing function $\phi \in C(J, \mathbb{R})$ and constants $\omega_\phi > 0$, $\epsilon > 0$ such that the following inequality holds:

$$_a\mathfrak{I}^{\alpha,\rho,\psi}\phi(t) \leq \omega_\phi\phi(t).$$

Theorem 4.11. *Assume that* $f : J \times \mathbb{R}^2 \to \mathbb{R}$, $\varphi_k : \mathbb{R} \to \mathbb{R}$ *is continuous functions. If assumptions* (H_1)*,* (H_2)*,* (H_5) *and the inequality*

$$2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3 < 1 \tag{4.6}$$

are satiafied, then system (1.1) *is* $\mathbb{UHR}$ *stable with respect to* (υ, ϕ)*. where* ϕ *is a nondecreasing function and* $\upsilon \geq 0$*.*

Proof. Let z be any solution of the inequality (4.3) and x be the unique solution of the system (1.1). Then, for $t \in J_k$, we have

$$\begin{aligned}
|z(t) - x(t)| \leq\ & {}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|F_z(s) - F_x(s)|(T) + |\lambda|_{t_m}\mathfrak{I}^{\alpha_m,\rho,\psi_m}|z(s) - x(s)|(T) \\
& + \sum_{i=1}^{m}|H_{i-1}(z) - H_{i-1}(x)| + \sum_{i=1}^{m}|G_{i-1}(z) - G_{i-1}(x)|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j, t_{j+1}) \\
& + \Big(|\Omega_1||\mathcal{R}(z, F_z) - \mathcal{R}(x, F_x)| + |\Omega_3||\mathcal{K}(z, F_z) - \mathcal{K}(x, F_x)|\Big)\frac{1}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1}, t_i) \\
& + \Big(|\Omega_4||\mathcal{K}(z, F_z) - \mathcal{K}(x, F_x)| + |\Omega_2||\mathcal{R}(z, F_z) - \mathcal{R}(x, F_x)|\Big)\frac{1}{|\Omega_5|} \\
& + {}_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(t)| + \sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\Big) \\
& + \sum_{i=1}^{m}\Big({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\Big)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j, t_{j+1}) + \Big[|\Omega_1|\Big(|\kappa_2|_{t_m}\mathfrak{I}^{\beta_m,\rho,\psi_m}|w(T)|
\end{aligned}$$

$$
\begin{aligned}
&+|\kappa_2||\lambda|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(T)| + |\kappa_2|\sum_{i=1}^{m}\left({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\right)\left(1 + |\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\right)\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\left({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\right)\Bigg) + |\Omega_3|\Bigg(|\kappa_1|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(T)|\\
&+|\kappa_1|\sum_{i=1}^{m}\left({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\right)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+|\kappa_1|\sum_{i=1}^{m}\left({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\right)\Bigg)\Bigg]\frac{1}{|\Omega_5|}\sum_{i=1}^{m+1}\Phi^{\alpha_{i-1}}(t_{i-1},t_i)\\
&+\Bigg[|\Omega_4|\Bigg(|\kappa_1|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(T)| + |\kappa_1|\sum_{i=1}^{m}\left({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\right)\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\\
&+|\kappa_1|\sum_{i=1}^{m}\left({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\right)\Bigg) + |\Omega_2|\Bigg(|\kappa_2|_{t_m}\mathfrak{I}^{\beta_m,\rho,\psi_m}|w(T)|\\
&+|\kappa_2||\lambda|_{t_m}\mathfrak{I}^{\alpha_m+\beta_m,\rho,\psi_m}|w(T)| + |\kappa_2|\sum_{i=1}^{m}\left({}_{t_{i-1}}\mathfrak{I}^{\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\right)\left(1 + |\lambda|\sum_{j=i}^{m}\Phi^{\alpha_j}(t_j,t_{j+1})\right)\\
&+|\kappa_2||\lambda|\sum_{i=1}^{m}\left({}_{t_{i-1}}\mathfrak{I}^{\alpha_{i-1}+\beta_{i-1},\rho,\psi_{i-1}}|w(t_i)| + |w_k|\right)\Bigg)\Bigg]\frac{1}{|\Omega_5|}
\end{aligned}
$$

By using Remark 4.8 (C_1) with (H_1), (H_2), (H_5) and the fact of $0 < e^{\frac{\rho-1}{\rho}(\psi_a(u)-\psi_a(s))} \le 1$ for $0 \le s \le u \le T$, we obtain the following inequality

$$
\begin{aligned}
|z(t)-x(t)| \le\ & \Bigg\{2L_1\Bigg[\Lambda_1 + \Big(|\kappa_1|\Lambda_1(\Lambda_2|\Omega_3| + |\Omega_4|) + |\kappa_2|(|\lambda|\Lambda_1 + \Lambda_4)(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+(mM_1 + |\lambda|\Lambda_2)\Bigg[1 + \Big(|\kappa_1|(\Lambda_2|\Omega_3| + |\Omega_4|) + |\lambda||\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+M_1^*\Bigg[\Lambda_3 + \Big(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3| + |\Omega_4|) + |\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)(|\lambda|\Lambda_3 + m)\Big)\frac{1}{|\Omega_5|}\Bigg]\Bigg\}|z(t)-x(t)|\\
&+\Bigg\{(1+m)\Bigg[1 + \Big(|\kappa_1|(\Lambda_2|\Omega_3| + |\Omega_4|) + |\lambda||\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+\Bigg[\Lambda_3 + \Big(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3| + |\Omega_4|) + |\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)(|\lambda|\Lambda_3 + m)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+\Big(|\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg\}\epsilon\omega_\phi\phi(t)\\
&+\Bigg\{m\Bigg[1 + \Big(|\kappa_1|(\Lambda_2|\Omega_3| + |\Omega_4|) + |\kappa_2||\lambda|(\Lambda_2|\Omega_1| + |\Omega_2|)\Big)\frac{1}{|\Omega_5|}\Bigg]\\
&+\Bigg[\Lambda_3 + \Big(|\kappa_1|\Lambda_3(\Lambda_2|\Omega_3| + |\Omega_4|) + |\kappa_2|(\Lambda_2|\Omega_1| + |\Omega_2|)(|\lambda|\Lambda_3 + m)\Big)\frac{1}{|\Omega_5|}\Bigg]\Bigg\}\epsilon\upsilon\\
\le\ & \Big(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3\Big)|z(t)-x(t)|
\end{aligned}
$$

$$+\Big(\Theta_1 + (1+m)\Theta_2 + \Theta_3\Big)(1+\omega_\phi)\epsilon(\upsilon + \phi(t))$$

which implies that

$$|z(t) - x(t)| \le \frac{(\Theta_1 + (1+m)\Theta_2 + \Theta_3)(1+\omega_\phi)\epsilon(\upsilon + \phi(t))}{1 - \Big(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3\Big)},$$

with $(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3) < 1$. By setting

$$C_f = \frac{(\Theta_1 + (1+m)\Theta_2 + \Theta_3)(1+\omega_\phi)}{1 - \Big(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3\Big)},$$

we end up with $|z(t) - x(t)| \le C_f\epsilon(\upsilon + \phi(t))$. Therefore, the system (1.1) is $\mathbb{UHR}$ stable. This completes the proof. □

Corollary 4.12. *In Theorem* 4.11*, if we set* $\epsilon = 1$ *then the system* (1.1) *is generalized* $\mathbb{UHR}$ *stable.*

5. An example

This section give an example which illustrate the validity and applicability of main results.

Example 5.1. *Consider the following an impulsive boundary value problem is given by:*

$$\begin{cases} {}^{C}_{\frac{k}{4}}\mathfrak{D}^{\frac{k+2}{k+3},\frac{1}{2},e^{\frac{t}{2}}+\frac{k}{8}}\left({}^{C}_{\frac{k}{4}}\mathfrak{D}^{\frac{k+1}{k+2},\frac{1}{2},e^{\frac{t}{2}}+\frac{k}{8}} + \frac{1}{49}\right)x(t) = f(t, x(t), x(3t/4)),\ t \neq \frac{k}{4},\ k = 0,1,2, \\ x(t_k^+) - x(t_k^-) = \varphi_k(x(t_k)), \quad k = 1,2, \\ {}^{C}_{\frac{k}{4}}\mathfrak{D}^{\frac{k+1}{k+2},\frac{1}{2},e^{\frac{t}{2}}+\frac{k}{8}}x(t_k^+) - {}^{C}_{\frac{k-1}{4}}\mathfrak{D}^{\frac{k}{k+1},\frac{1}{2},e^{\frac{t}{2}}+\frac{k-1}{8}}x(t_k^-) = \varphi_k^*(x(t_k)), \quad k = 1,2, \\ \frac{1}{5}x(0) - \frac{\sqrt{2}}{3}x\left(\frac{3}{2}\right) = \frac{1}{2}, \qquad \sqrt{7}\,{}^{C}_{0}\mathfrak{D}^{\frac{1}{2},\frac{1}{2},e^{\frac{t}{2}}}x(0) - {}^{C}_{1}\mathfrak{D}^{\frac{5}{6},\frac{1}{2},e^{\frac{t}{2}}+\frac{1}{2}}x\left(\frac{3}{2}\right) = \sqrt{3}, \end{cases} \tag{5.1}$$

Here $\alpha_k = (k+1)/(k+2)$, $\beta_k = (k+2)/(k+3)$, $\psi_k(t) = \exp(t/2)+k/8$, $t_k = k/4$, $k = 0,1,2$, $\rho = 1/2$, $\lambda = 1/49$, $\mu = 3/4$, $m = 4$, $T = 3/2$, $\eta_1 = 1/5$, $\eta_2 = \sqrt{7}$, $\kappa_1 = -\sqrt{2}/3$, $\kappa_2 = -1$, $\xi_1 = 1/2$, $\xi_2 = \sqrt{3}$. *Using the all datas, we find that* $\Omega_1 \approx -0.4687161284$, $\Omega_2 \approx -0.0990842078$, $\Omega_3 \approx 2.031589673$, $\Omega_4 \approx -0.04104689584$, $\Omega_5 \approx 0.2205377954$, $\Lambda_1 \approx 1.147842297$, $\Lambda_2 \approx 1.567171105$, $\Lambda_3 \approx 1.471412869$, $\Lambda_4 \approx 1.356509541$, $\Theta_1 \approx 14.27646343$, $\Theta_2 \approx 7.970430815$, *and* $\Theta_3 \approx 19.28788257$. *Let* $f : J \times \mathbb{R}^2 \to \mathbb{R}$, φ, $\varphi_k : \mathbb{R} \to \mathbb{R}$ *be the functions defined by*

$$\begin{aligned} f(t, x(t), x(3t/4)) &= \frac{1}{2} + t + \frac{4t^2+1}{9^{t+1}(9 + \sin^2 \pi t)}\left(\frac{|x(t)|}{10 + |x(t)|} + \frac{|x(3t/4)|}{10 + |x(3t/4)|}\right), \\ \varphi_k(x(t_k)) &= \frac{1}{(10+k)^2}\sin|x(t_k)| + \frac{1}{11}, \qquad k = 1,2, \\ \varphi_k^*(x(t_k)) &= \frac{1}{(8+k)^2}\tan^{-1}|x(t_k)| + \frac{1}{9}, \qquad k = 1,2. \end{aligned}$$

By (H_1)–(H_2), *for any* x_i, $y_i \in \mathbb{R}$, $i = 1,2$, *and* $t \in J$, *we have* $|f(t,x_1(t),x_2(3t/4)) - f(t,y_1(t),y_2(3t/4))| \leq (1/81)(|x_1(t) - y_1(t)| + |x_2(3t/4) - y_2(3t/4)|)$, $|\varphi_k(x_1) - \varphi_k(y_1)| \leq (1/121)|x_1(t_k) - y_1(t_k)|$, *and* $|\varphi_k^*(x_1) - \varphi_k^*(y_1)| \leq (1/81)|x_1(t_k) - y_1(t_k)|$, *for* $k = 1,2$. *The* (H_1)–(H_2) *are satisfied with* $L_1 = 1/81$, $M_1 = 1/121$ *and* $M_1^* = 1/81$. *Therefore, we get that*

$$2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3 \approx 0.9772888914 < 1.$$

Thus, all the assumptions of Theorem 3.1 *are fulfilled, which implies that the problem* (5.1) *has a unique solution on* $[0,3/2]$. *Also* (H_3)–(H_4) *holds with* $h_1(t) = (1/2)+t$, $h_2(t) = (4t^2+1)/((10)(9^{t+1}(9+\sin^2 \pi t))$, *where* $h_1^* = 2$, $h_2^* = 1/81$ *and* $k_1 = 12/121$, $k_1^* = 10/81$. *So, all the assumptions of Theorem* 3.2 *are satisfied, then the problem* (5.1) *has at least one solution on* $[0,3/2]$.

Moreover, we also calculate that

$$C_f = \frac{\Theta_1 + m\Theta_2 + \Theta_3}{1 - \left(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3\right)} \approx 2,179.779442 > 0.$$

Hence, by Theorem 4.9 *is both* $\mathbb{UH}$ *stable and also generalized* $\mathbb{UH}$ *stable. Further, by setting* $\phi(t) = e^{\frac{\rho-1}{\rho}\psi_k(t)}(\psi_k(t) - \psi_k(0))$ *and* $\upsilon = 1$, *for any* $t \in [0,3/2]$, *then*

$${}_{t_k}\mathfrak{I}^{\alpha_k+\beta_k,\rho,\psi_k}\phi(t) \leq \frac{(2)^{\frac{31}{20}}}{\Gamma(\frac{71}{20})}\left(e^{\frac{3}{4}} - 1\right)^{\frac{51}{20}}\phi(t).$$

From the inequality in (H_5) *is satisfy with* $\omega_\phi = \frac{(2)^{\frac{31}{20}}}{\Gamma(\frac{71}{20})}\left(e^{\frac{3}{4}} - 1\right)^{\frac{51}{20}} > 0$, *we have*

$$C_f = \frac{(\Theta_1 + (1+m)\Theta_2 + \Theta_3)(1 + \omega_\phi)}{1 - \left(2L_1\Theta_1 + (mM_1 + |\lambda|\Lambda_2)\Theta_2 + M_1^*\Theta_3\right)} \approx 5,327.572054 > 0.$$

Consequently, by all the assumptions in Theorem 4.11, *the problem* (5.1) *is* $\mathbb{UHR}$ *stable and generalized* $\mathbb{UHR}$ *stable with respect to* (υ,ϕ).

6. Conclusions

In this paper, we have studied the existence, uniqueness, and stability of solutions for a new class of impulsive fractional differential equation augmented by non-separated boundary conditions involving Caputo proportional derivative of a function with respect to another function. The uniqueness of solutions is obtained by using Banach's contraction mapping principle, whereas the existence result is established via Schaefer's fixed point theorem. Moreover, by the application of qualitative theory and nonlinear functional analysis, we investigated results concerning to different kinds of Ulam-Hyers stability such as, Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. The concerned results have been examined by a suitable example to illustrate the main results.

Further, our results are interesting special cases for different values of the parameters involved in the considered problem. For instance, our results correspond to a considered problem with

(i) periodic boundary conditions:

$$x(0) = x(T), \qquad {}^{C}_{t_0}\mathfrak{D}^{\alpha_0,\rho,\psi_0}x(0) = {}^{C}_{t_m}\mathfrak{D}^{\alpha_m,\rho,\psi_m}x(T),$$

for $\eta_1 = \eta_2 = 1$, $\kappa_1 = \kappa_2 = -1$ and $\xi_1 = \xi_2 = 0$

(ii) anti-periodic boundary conditions:

$$x(0) = -x(T), \qquad {}^{C}_{t_0}\mathfrak{D}^{\alpha_0,\rho,\psi_0}x(0) = -{}^{C}_{t_m}\mathfrak{D}^{\alpha_m,\rho,\psi_m}x(T),$$

for $\eta_1 = \eta_2 = \kappa_1 = \kappa_2 = 1$ and $\xi_1 = \xi_2 = 0$.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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