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Research article

On hypergeometric Cauchy numbers of higher grade

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Abstract: In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli, Cauchy, and Euler numbers. Cauchy numbers can be generalized to the hypergeometric Cauchy numbers. Recently, Barman et al. study more general numbers in terms of determinants, which involve Bernoulli, Euler and Lehmer's generalized Euler numbers. However, Cauchy numbers and their generalizations are not involved in these generalized numbers. In this paper, we study more general numbers in terms of determinants, which involve Cauchy numbers. The motivations and backgrounds of the definition are in an operator related to graph theory. We also give several expressions and identities by Trudi's and inversion formulae.

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1. Introduction

In 1935, D. H. Lehmer [20] introduced and investigated generalized Euler numbers W_n, defined by the generating function

$$\frac{3}{e^t + e^{\omega t} + e^{\omega^2 t}} = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!}\,, \tag{1.1}$$

where $\omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \bar{\omega} = \frac{-1-\sqrt{-3}}{2}$ are the cube roots of unity. Notice that $W_n = 0$ unless $n \equiv 0 \pmod 3$. The sequence of these numbers is given by

$$\{W_{3n}\}_{n\ge 0} = 1, -1, 19, -1513, 315523, -136085041, 105261234643, \\ -132705221399353, 254604707462013571, \cdots$$

and the sequence of these absolute values is recorded in [22, A002115]. In [15], the complementary numbers $W_n^{(j)}$ $(j = 0, 1, 2)$ to Lehmer's Euler numbers are defined by the generating function

$$\sum_{n=0}^{\infty} W_n^{(j)} \frac{t^n}{n!} = \left(1 + \sum_{l=1}^{\infty} \frac{t^{3l}}{(3l+j)!}\right)^{-1} . \tag{1.2}$$

Notice that $W_n^{(j)} = 0$ unless $n \equiv 0 \pmod 3$. When $j = 0$, $W_n = W_n^{(0)}$ are the original Lehmer's Euler numbers. When $j = 1$, we also have

$$\sum_{n=0}^{\infty} W_n^{(1)} \frac{t^n}{n!} = \frac{3t}{e^t + \omega^2 e^{\omega t} + \omega e^{\omega^2 t}} . \tag{1.3}$$

Lehmer's Euler numbers and their complementary numbers $W_n^{(j)}$ can be considered analogous of the classical Euler numbers E_n and their complementary Euler numbers $\widehat{E_n}$ ([11, 19]). For, their generating functions are given by

$$\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \left(\sum_{l=0}^{\infty} \frac{t^{2l}}{(2l)!}\right)^{-1} \tag{1.4}$$

and

$$\sum_{n=0}^{\infty} \widehat{E}_n \frac{t^n}{n!} = \frac{t}{\sinh t} = \frac{2t}{e^t - e^{-t}} = \left(\sum_{l=0}^{\infty} \frac{t^{2l}}{(2l+1)!}\right)^{-1} , \tag{1.5}$$

respectively. Still similar numbers are the well-known classical Bernoulli numbers defined by

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} = \left(\sum_{l=0}^{\infty} \frac{t^l}{(l+1)!}\right)^{-1} . \tag{1.6}$$

Recently, Barman et al. ([3]) introduce more general numbers, so-called *hypergeometric Lehmer-Euler numbers* $W_{N,n,r}^{(j)}$ $(j = 0, 1)$ of grade r , defined by

$$\begin{aligned}
&\sum_{n=0}^{\infty} W_{N,n,r}^{(j)} \frac{t^n}{n!} \\
&= \left({}_1F_r\left(1; \frac{rN+j+1}{r}, \frac{rN+j+2}{r}, \dots, \frac{rN+j+r}{r}; \left(\frac{t}{r}\right)^r\right)\right)^{-1} \\
&= \left(1 + \sum_{n=1}^{\infty} \frac{(rN+j)!}{(rN+rn+j)!} t^{rn}\right)^{-1} \quad (N \ge 0) ,
\end{aligned}$$

where ${}_1F_r(a; b_1, \dots, b_r; z)$ is the hypergeometric function, defined by

$${}_1F_r(a; b_1, \dots, b_r; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b_1)^{(n)} \cdots (b_r)^{(n)}} \frac{z^n}{n!} .$$

and $(x)^{(n)} = x(x+1)\cdots(x+n-1)$ $(n \ge 1)$ is the rising factorial with $(x)^{(0)} = 1$. A determinant expression is given by

$$W_{N,n,r}^{(j)} = (-1)^n (rn)! \begin{vmatrix} \frac{(rN+j)!}{(rN+j+r)!} & 1 & 0 & & \\ \frac{(rN+j)!}{(rN+j+2r)!} & \frac{(rN+j)!}{(rN+j+r)!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{(rN+j)!}{(rN+rn+j-r)!} & \frac{(rN+j)!}{(rN+rn+j-2r)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!} & 1 \\ \frac{(rN+j)!}{(rN+rn+j)!} & \frac{(rN+j)!}{(rN+rn+j-r)!} & \cdots & \frac{(rN+j)!}{(rN+j+2r)!} & \frac{(rN+j)!}{(rN+j+r)!} \end{vmatrix}. \tag{1.7}$$

When $N = 0$ and $r = 3$, $W_n^{(j)} = W_{0,n,3}^{(j)}$ are the Lehmer's generalized Euler numbers ($j = 0$) in (1.1) and their complementary numbers ($j = 1$) in (1.3). When $N = 0$ and $r = 2$, $E_n = W_{0,n,2}^{(j)}$ are the classical Euler numbers ($j = 0$) in (1.4) and their complementary numbers ($j = 1$) in (1.5). A famous determinant expression of Euler numbers discovered by Glaisher in 1875 ([6, p.52])

$$E_{2n} = (-1)^n (2n)! \begin{vmatrix} \frac{1}{2} & 1 & 0 & & \\ \frac{1}{4!} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2} \end{vmatrix} \tag{1.8}$$

and an expression of the complementary numbers ([11, 19])

$$\widehat{E}_{2n} = (-1)^n (2n)! \begin{vmatrix} \frac{1}{3!} & 1 & 0 & & \\ \frac{1}{5!} & \frac{1}{3!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{(2n-1)!} & \frac{1}{(2n-3)!} & \cdots & \frac{1}{3!} & 1 \\ \frac{1}{(2n+1)!} & \frac{1}{(2n-1)!} & \cdots & \frac{1}{5!} & \frac{1}{3!} \end{vmatrix}. \tag{1.9}$$

When $r = 1$ and $j = 0$, $B_{N,n} = W_{N,n,1}^{(0)}$ are the hypergeometric Bernoulli numbers. When $N = r = 1$ and $j = 0$ in (1.7), $B_n = W_{1,n,1}^{(0)}$ are the classical Bernoulli numbers in (1.6). The determinant expression for the classical Bernoulli numbers was discovered by Glaisher ([6, p.53]).

$$B_n = (-1)^n n! \begin{vmatrix} \frac{1}{2} & 1 & 0 & & \\ \frac{1}{3!} & \frac{1}{2} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2} \end{vmatrix}. \tag{1.10}$$

However, the classical Cauchy numbers and their generalized numbers are not involved in the numbers $W_{N,n,r}^{(j)}$. Hypergeometric Cauchy numbers $c_{N,n}$ ([9]) are defined by

$$\frac{1}{{}_2F_1(1, N; N+1; -t)} = \frac{(-1)^{N-1} t^N / N}{\log(1+t) - \sum_{n=1}^{N-1} (-1)^{n-1} t^n / n}$$

$$= \sum_{n=0}^{\infty} c_{N,n} \frac{t^n}{n!}\,, \tag{1.11}$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(a)^{(b)}}{(c)^{(n)}} \frac{z^n}{n!}\,.$$

When $N = 1$, $c_n = c_{1,n}$ are the classical Cauchy numbers defined by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!}\,. \tag{1.12}$$

The determinant expression of hypergeometric Cauchy numbers is given by

$$c_{N,n} = n! \begin{vmatrix} \frac{N}{N+1} & 1 & 0 & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix} \tag{1.13}$$

([2, 18]). The determinant expression for the classical Cauchy numbers was discovered by Glaisher ([6, p.50]). Other generalized Cauchy numbers, having similar properties, are Leaping Cauchy numbers [13] and Shifted Cauchy numbers [16].

A generalized version for Bernoulli and Euler numbers has been established in [17], where the elements contain factorials, as seen in (1.8), (1.9), (1.10) and (1.7). However, expressions for Cauchy and their generalized numbers cannot be included because they do not contain the factorial elements, as seen in (1.13). Universal Bernoulli numbers were studied in [1] and [8], and particularly, some universal Kummer congruences were established in [1] and [8].

In this paper, we introduce the hypergeometric Cauchy numbers of higher grade that are introduced as generalizations of both hypergeometric Cauchy numbers and the classical Cauchy numbers. We give several expressions and identities.

2. Hypergeometric Cauchy numbers of higher grade

For $N \ge 1$ and $n \ge 0$, define *hypergeometric Cauchy numbers* $V_{N,n,r}^{(j)}$ $(j = 0, 1)$ of grade r by

$$\sum_{n=0}^{\infty} V_{N,n,r}^{(j)} \frac{t^n}{n!} = \left({}_2F_1\left(1, N + \frac{j}{r}; N + 1 + \frac{j}{r}; (-t)^r\right)\right)^{-1}\,, \tag{2.1}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function, defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}} \frac{z^n}{n!}\,.$$

From the definition, $V_{N,n,r}^{(j)} \equiv 0 \pmod r$ unless $n \equiv 0 \pmod r$. When $r = 1$ and $j = 0$ in (2.1), $c_{N,n} = V_{N,n,1}^{(0)}$ are the hypergeometric Cauchy numbers in (1.11). When $N = 1$, $r = 1$ and $j = 0$ in (2.1), $c_n = V_{1,n,1}^{(0)}$ are the classical Cauchy numbers in (1.12).

We can write (2.1) as

$$
\begin{aligned}
&{}_2F_1\left(1, N + \frac{j}{r}; N + 1 + \frac{j}{r}; (-t)^r\right)\\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (rN + j)}{rN + rn + j} t^{rn} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (rN + j)}{rN + rn + j} t^{rn}\,. \qquad (2.2)
\end{aligned}
$$

The definition (2.1) with (2.2) may be obvious or artificial for the readers with different backgrounds. However, our initial motivations were from Combinatorics, in particular, graph theory. In 1989, Cameron [5] considered the operator A defined on the set of sequences of non-negative integers as follows: for $x = \{x_n\}_{n\ge 1}$ and $z = \{z_n\}_{n\ge 1}$, set $Ax = z$, where

$$
1 + \sum_{n=1}^{\infty} z_n t^n = \left(1 - \sum_{n=1}^{\infty} x_n t^n\right)^{-1}\,. \qquad (2.3)
$$

Cameron's operators deal with only nonnegative integers, but the operators can be used extensively for rational numbers. In the sense of Cameron's operator A, we have the following relation.

$$
A\left\{\frac{(-1)^{n-1}(rN + j)}{rN + rn + j}\right\} = \left\{\frac{V_{N,rn,r}^{(j)}}{(rn)!}\right\}
$$

This relation is interchangeable in the sense of determinants too. See Section 5 about Trudi's formula.

We have the following recurrence relation.

Proposition 1. *For $N \ge 0$ and $j = 0, 1$, we have*

$$
V_{N,rn,r}^{(j)} = \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}(rn)!(rN + j)}{(rN + rn - rk + j)(rk)!} V_{N,rk,r}^{(j)} \quad (n \ge 1)
$$

with $V_{N,0,r}^{(j)} = 1$.

Proof. By (2.1), we get

$$
\begin{aligned}
1 &= \left(1 + \sum_{l=1}^{\infty} \frac{(-1)^l (rN + j)}{rN + rl + j} t^{rl}\right)\left(\sum_{n=0}^{\infty} V_{N,rn,r}^{(j)} \frac{t^{rn}}{(rn)!}\right)\\
&= \sum_{n=0}^{\infty} V_{N,rn,r}^{(j)} \frac{t^{rn}}{(rn)!} + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \frac{(-1)^{n-k}(rN + j) V_{N,rk,r}^{(j)}}{(rN + rn - rk + j)(rk)!} t^{rn}\,.
\end{aligned}
$$

Comparing the coefficient on both sides, we obtain

$$
\frac{V_{N,rn,r}^{(j)}}{(rn)!} + \sum_{k=0}^{n-1} \frac{(-1)^{n-k}(rN + j) V_{N,rk,r}^{(j)}}{(rN + rn - rk + j)(rk)!} = 0 \quad (n \ge 1)\,.
$$

□

We have an explicit expression of $V_{N,n,r}^{(j)}$.

Theorem 1. *Let* $j=0,1$. *For* $n\ge 1$*,*

$$V_{N,rn,r}^{(j)}=(rn)!\sum_{k=1}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\,.$$

Proof. The proof is done by induction on n. From Proposition 1 with $n=1$,

$$V_{N,r,r}^{(j)}=\frac{r!(rN+j)}{rN+j+r}V_{N,0,r}^{(j)}=\frac{r!(rN+j)}{rN+j+r}\,.$$

This matches the result when $n=1$. Assume that the result is valid up to $n-1$. Then by Proposition 1

$$\begin{aligned}
\frac{V_{N,rn,r}^{(j)}}{(rn)!}&=\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}(rN+j)}{rN+rn-rl+j}\frac{V_{N,rl,r}^{(j)}}{(rl)!}\\
&=\sum_{l=1}^{n-1}\frac{(-1)^{n-l-1}(rN+j)}{rN+rn-rl+j}\sum_{k=1}^{l}(-1)^{l-k}\sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\dots,i_k\ge 1}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\
&\quad+\frac{(-1)^{n-1}(rN+j)}{rN+rn+j}\\
&=\sum_{k=1}^{n-1}(-1)^{n-k-1}\sum_{l=k}^{n-1}\frac{(rN+j)}{rN+rn-rl+j}\sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\dots,i_k\ge 1}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\
&\quad+\frac{(-1)^{n-1}(rN+j)}{rN+rn+j}\\
&=\sum_{k=2}^{n}(-1)^{n-k}\sum_{l=k-1}^{n-1}\frac{rN+j}{rN+rn-rl+j}\sum_{\substack{i_1+\cdots+i_{k-1}=l\\ i_1,\dots,i_{k-1}\ge 1}}\frac{(rN+j)^{k-1}}{(rN+ri_1+j)\cdots(rN+ri_{k-1}+j)}\\
&\quad+\frac{(-1)^{n-1}(rN+j)}{rN+rn+j}\\
&=\sum_{k=2}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\
&\quad+\frac{(-1)^{n-1}(rN+j)}{(rN+rn+j)}\quad (n-l=i_k)\\
&=\sum_{k=1}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\,.
\end{aligned}$$

□

There is an alternative form of $V_{N,n,r}^{(j)}$ by using binomial coefficients. The proof is similar to that of Theorem 1 and is omitted.

Theorem 2. *For $n \ge 1$,*

$$V_{N,rn,r}^{(j)} = (rn)! \sum_{k=1}^{n} (-1)^{n-k} \binom{n+1}{k+1} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}} \frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\,.$$

3. Determinantal expressions

In this section, we shall show an expression of hypergeometric Cauchy numbers of higher grade in terms of determinants. This result is a generalization of those of the hypergeometric and the classical Cauchy numbers. For simplification of determinant expressions, we use the Jordan matrix

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

J^0 is the identity matrix and J^T is the transpose matrix of J.

Theorem 3. *For $n \ge 1$,*

$$V_{N,rn,r}^{(j)} = (rn)! \left| J^T + \sum_{k=1}^{n} \frac{rN+j}{r(N+k)+j} J^{k-1} \right|.$$

Proof. For simplicity, put $\widetilde{V}_{N,n} = V_{N,n,r}^{(j)}/n!$. Then, we shall prove that for any $n \ge 1$

$$\widetilde{V}_{N,rn} = \left| J^T + \sum_{k=1}^{n} \frac{rN+j}{r(N+k)+j} J^{k-1} \right|. \tag{3.1}$$

When $n = 1$, (3.1) is valid because by Theorem 1 we get

$$\widetilde{V}_{N,r} = \frac{rN+j}{rN+j+r}\,.$$

Assume that (3.1) is valid up to $n-1$. Notice that by Proposition 1, we have

$$\widetilde{V}_{N,rn} = \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}(rN+j)}{rN+rn-rk+j} \widetilde{V}_{N,rk}\,.$$

Thus, by expanding the right-hand side of (3.1) along the first row, it is equal to

$$\frac{rN+j}{rN+j+r}\widetilde{V}_{N,rn-r} - \begin{vmatrix} \frac{rN+j}{rN+j+2r} & 1 & 0 & & \\ \frac{rN+j}{rN+j+3r} & \frac{rN+j}{rN+j+r} & & & \\ \vdots & \vdots & \ddots & 1 & 0\\ \frac{rN+j}{rN+rn+j-r} & \frac{rN+j}{rN+rn+j-3r} & \cdots & \frac{rN+j}{rN+j+r} & 1\\ \frac{rN+j}{rN+rn+j} & \frac{rN+j}{rN+rn+j-2r} & \cdots & \frac{rN+j}{rN+j+2r} & \frac{rN+j}{rN+j+r} \end{vmatrix}$$

$$= \frac{rN+j}{rN+j+r}\widetilde{V}_{N,rn-r} - \frac{rN+j}{rN+j+2r}\widetilde{V}_{N,rn-2r} + \cdots + (-1)^n \begin{vmatrix} \frac{rN+j}{rN+rn+j-r} & 1 \\ \frac{rN+j}{rN+rn+j} & \frac{rN+j}{rN+j+r} \end{vmatrix}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}(rN+j)}{rN+rn-rk+j}\widetilde{V}_{N,rk} = \widetilde{V}_{N,rn}\,.$$

□

Remark. When $r = 1$ and $j = 0$, the determinant expression in Theorem 3 is reduced to that in (1.13) for the hypergeometric Cauchy numbers $c_{N,n} = V_{N,n,1}^{(0)}$. When $N = 1$, $r = 1$ and $j = 0$, we have a determinant expression of the Cauchy numbers $c_n = V_{1,n,1}^{(0)}$ ([6, p.50]).

4. Incomplete hypergeometric Cauchy numbers of higher grade

As applications or variations to generalize the hypergeometric numbers $V_{N,n,r}^{(j)}$ of higher grade, we shall introduce two kinds of incomplete hypergeometric Cauchy numbers of higher grade. Similar but slightly different kinds of incomplete numbers are considered in [10, 12, 14, 17]. In addition, similar techniques can be found in [24] and later cited in [7]. For $j = 0, 1$ and $n \ge m \ge 1$, define the *restricted* hypergeometric Cauchy numbers $V_{N,n,r,\le m}^{(j)}$ of grade r by

$$\sum_{n=0}^{\infty} V_{N,n,r,\le m}^{(j)} \frac{t^n}{n!} = \left(1 + \sum_{l=1}^{m} \frac{(-1)^l (rN+j)}{rN+rl+j} t^{rl}\right)^{-1} \tag{4.1}$$

and the *associated* hypergeometric Cauchy numbers $V_{N,n,r,\ge m}^{(j)}$ of grade r by

$$\sum_{n=0}^{\infty} V_{N,n,r,\ge m}^{(j)} \frac{t^n}{n!} = \left(1 + \sum_{l=m}^{\infty} \frac{(-1)^l (rN+j)}{rN+rl+j} t^{rl}\right)^{-1}. \tag{4.2}$$

When $m \to \infty$ in (4.1) and $m = 1$ in (4.2), $V_{N,n,r}^{(j)} = V_{N,n,r,\le\infty}^{(j)} = V_{N,n,r,\ge 1}^{(j)}$ are the original hypergeometric Cauchy numbers of grade r, defined in (2.1) with (2.2). Hence, both incomplete numbers are reduced to the hypergeometric Cauchy numbers too.

Notice that $V_{N,n,r,\le m}^{(j)} = V_{N,n,r,\ge m}^{(j)} = 0$ unless $n \equiv 0 \pmod r$.

The restricted and associated hypergeometric Cauchy numbers satisfy the following recurrence relations.

Proposition 2. *For $j = 0, 1$, we have*

$$V_{N,rn,r,\le m}^{(j)} = \sum_{k=\max\{n-m,0\}}^{n-1} \frac{(-1)^{n-k-1}(rn)!(rN+j)}{(rN+rn-rk+j)(rk)!} V_{N,rk,r,\le m}^{(j)} \quad (n \ge 1)$$

with $V_{N,0,r,\le m}^{(j)} = 1$, and

$$V_{N,rn,r,\ge m}^{(j)} = \sum_{k=0}^{n-m} \frac{(-1)^{n-k-1}(rn)!(rN+j)}{(rN+rn-rk+j)(rk)!} V_{N,rk,r,\ge m}^{(j)} \quad (n \ge m)$$

with $V_{N,0,r,\ge m}^{(j)} = 1$ and $V_{N,r,r,\ge m}^{(j)} = \cdots = V_{N,r(m-1),r,\ge m}^{(j)} = 0$.

Proof. First, we shall prove the relation for the restricted hypergeometric Cauchy numbers. By the definition (4.1), we get

$$\begin{aligned}1 &= \left(1+\sum_{l=1}^{m}\frac{(-1)^l(rN+j)t^{rl}}{rN+rl+j}\right)\left(\sum_{n=0}^{\infty}V_{N,rn,r,\le m}^{(j)}\frac{t^{rn}}{(rn)!}\right)\\ &= \sum_{n=0}^{\infty}V_{N,rn,r,\le m}^{(j)}\frac{t^{rn}}{(rn)!}+\sum_{n=1}^{\infty}\sum_{k=\max\{n-m,0\}}^{n-1}\frac{(-1)^{n-k}(rN+j)V_{N,rk,r,\le m}^{(j)}}{(rN+rn-rk+j)(rk)!}t^{rn}\,.\end{aligned}$$

Comparing the coefficient on both sides, we obtain the first identity.

Next, we prove the relation for the associated hypergeometric Cauchy numbers. By the definition (4.2), we get

$$\begin{aligned}1 &= \left(1+\sum_{l=m}^{\infty}\frac{(-1)^l(rN+j)!t^{rl}}{rN+rl+j}\right)\left(\sum_{n=0}^{\infty}V_{N,rn,r,\ge m}^{(j)}\frac{t^{rn}}{(rn)!}\right)\\ &= \sum_{n=0}^{\infty}V_{N,rn,r,\ge m}^{(j)}\frac{t^{rn}}{(rn)!}+\sum_{n=m}^{\infty}\sum_{k=0}^{n-m}\frac{(-1)^{n-k}(rN+j)V_{N,rk,r,\ge m}^{(j)}}{(rN+rn-rk+j)(rk)!}t^{rn}\,.\end{aligned}$$

Comparing the coefficient on both sides, we obtain the desired result. □

The restricted and associated hypergeometric Cauchy numbers have the following expressions in terms of determinants. From the expression of Theorem 3, all the elements change to 0 in more diagonal directed bands.

Theorem 4. *For integers n and m with $n \ge m \ge 1$, we have*

$$V_{N,rn,r,\le m}^{(j)} = (rn)!\left|J^T+\sum_{k=1}^{m}\frac{rN+j}{r(N+k)+j}J^{k-1}\right|$$

and

$$V_{N,rn,r,\ge m}^{(j)} = (rn)!\left|J^T+\sum_{k=m}^{n}\frac{rN+j}{r(N+k)+j}J^{k-1}\right|\,.$$

Proof. First, we shall prove the first expression for the restricted hypergeometric Cauchy numbers. For simplicity, put $\widetilde{V}_{N,rn,\le m} = V_{N,rn,r,\le m}^{(j)}/(rn)!$ and prove that for $n \ge m \ge 1$

$$\widetilde{V}_{N,rn,\le m} = \left|J^T+\sum_{k=1}^{m}\frac{rN+j}{r(N+k)+j}J^{k-1}\right|\,. \tag{4.3}$$

When $n = m$, we have $\widetilde{V}_{N,rm,\le m} = \widetilde{V}_{N,rm}$, and the result reduces to Theorem 3. Assume that (4.3) is valid up to $n-1$. If $n \ge 2m$, then the determinant on the right-hand side of (4.3) is equal to

$$\frac{\widetilde{V}_{N,rn-r,\le m}(rN+j)}{rN+j+r}-\frac{\widetilde{V}_{N,rn-2r,\le m}(rN+j)}{rN+j+2r}+\cdots$$

$$
+(-1)^{m-1}\begin{vmatrix}
\frac{rN+j}{rN+rm+j} & 1 & 0 & & & \\
0 & \frac{rN+j}{rN+r+j} & 1 & & & \\
 & \vdots & & & & \\
 & \frac{rN+j}{rN+rm+j} & & & & \\
 & & \ddots & & & 1\\
 & & & \frac{rN+j}{rN+rm+j} & \cdots & \frac{rN+j}{rN+r+j}
\end{vmatrix}
$$

$$
=\frac{\widetilde{V}_{N,rn-r,\le m}(rN+j)}{rN+r+j}-\frac{\widetilde{V}_{N,rn-2r,\le m}(rN+j)}{rN+2r+j}+\cdots+(-1)^{m-1}\frac{\widetilde{V}_{N,rn-rm,\le m}(rN+j)}{rN+rm+j}
$$

$$
=\widetilde{V}_{N,rn,\le m}\,.
$$

If $m<n\le 2m$, then the determinant on the right-hand side of (4.3) is equal to

$$
\frac{\widetilde{V}^{(j)}_{N,rn-r,\le m}(rN+j)}{rN+r+j}-\frac{\widetilde{V}^{(j)}_{N,rn-2r,\le m}(rN+j)}{rN+2r+j}+\cdots
$$

$$
+(-1)^{m-n-1}\begin{vmatrix}
\frac{rN+j}{rN+rn-rm+j} & 1 & 0 & & & \\
\vdots & \vdots & & & & \\
\frac{rN+j}{rN+rm+j} & \frac{rN+j}{rN+2rm-rn+j} & & & & \\
0 & \vdots & & & & \\
\vdots & & & & & 1\\
0 & \frac{rN+j}{rN+rm+j} & \cdots & & \cdots & \frac{rN+j}{rN+r+j}
\end{vmatrix}
$$

$$
=\frac{\widetilde{V}_{N,rn-r,\le m}(rN+j)}{rN+r+j}-\frac{\widetilde{V}_{N,rn-2r,\le m}(rN+j)}{rN+2r+j}+\cdots+(-1)^{n-m-1}\frac{\widetilde{V}_{N,rm,\le m}(rN+j)}{rN+rn-rm+j}
$$

$$
=\frac{\widetilde{V}_{N,rn-r,\le m}(rN+j)}{rN+r+j}-\frac{\widetilde{V}_{N,rn-2r,\le m}(rN+j)}{rN+2r+j}+\cdots+(-1)^{m-1}\frac{\widetilde{V}_{N,rn-rm,\le m}(rN+j)}{rN+rm+j}
$$

$$
=\widetilde{V}_{N,rn,\le m}\,.
$$

Next, we prove the second expression for the associated hypergeometric Cauchy numbers. For simplicity, put $\widetilde{V}_{N,rn,\ge m}=V_{N,rn,r,\ge m}/(rn)!$ and we prove that

$$
\widetilde{V}_{N,rn,\ge m}=\left|J^T+\sum_{k=m}^{n}\frac{rN+j}{r(N+k)+j}J^{k-1}\right|\,. \tag{4.4}
$$

If $m\le n\le 2m$, the determinant on the right-hand side of (4.4) is equal to

$$
(-1)^{n-m}\begin{vmatrix}
0 & 1 & 0 & & \\
\vdots & 0 & \ddots & & \\
0 & \vdots & & & \\
\frac{rN+j}{rN+rm+j} & & & \ddots & 0\\
\vdots & \vdots & & & 1\\
\frac{rN+j}{rN+rn+j} & \underbrace{0 \quad \cdots}_{} & & \underbrace{\cdots \quad 0}_{m-1} &
\end{vmatrix}
$$

$$= (-1)^{n-m}(-1)^{m+1}\frac{rN+j}{rN+rn+j}\begin{vmatrix} 1 & 0 & & \\ 0 & \ddots & & \\ & & \ddots & 0 \\ 0 & & 0 & 1 \end{vmatrix} = (-1)^{n+1}\frac{rN+j}{rN+rn+j}\,.$$

Since only the term for $k = 0$ does not vanish in the second relation of Proposition 2, we have

$$\widetilde{V}_{N,rn,\ge m} = (-1)^{n+1}\frac{rN+j}{rN+rn+j}\,.$$

If $n \ge 2m$, the determinant on the right-hand side of (4.4) is equal to

$$\begin{aligned}
&(-1)^{m-1}\begin{vmatrix}
\frac{rN+j}{rN+rm+j} & 1 & 0 & & & & \\
\vdots & 0 & \ddots & & & & \\
 & \vdots & \ddots & & & & \\
 & 0 & & & & & \\
 & \frac{rN+j}{rN+rm+j} & & & & \ddots & 0 \\
\vdots & \vdots & \ddots & & & \ddots & 1 \\
\frac{rN+j}{rN+rn+j} & \underbrace{\frac{rN+j}{r(N+n-m)+j} & \cdots & \frac{rN+j}{r(N+m)+j}}_{n-2m+1} & & \underbrace{0 \ \cdots & 0}_{m-1}
\end{vmatrix}\\
&= (-1)^{m-1}\frac{\widetilde{V}^{(j)}_{N,rn-rm,\ge m}(rN+j)}{rN+rm+j}\\
&\quad + (-1)^{m}\begin{vmatrix}
\frac{rN+j}{rN+rm+r+j} & 1 & 0 & & & & \\
\vdots & 0 & \ddots & & & & \\
 & \vdots & \ddots & & & & \\
 & 0 & & & & & \\
 & \frac{rN+j}{rN+rm+j} & & & & \ddots & 0 \\
\vdots & \vdots & \ddots & & & \ddots & 1 \\
\frac{rN+j}{rN+rn+j} & \underbrace{\frac{rN+j}{r(N+n-m-1)+j} & \cdots & \frac{rN+j}{r(N+m)+j}}_{n-2m} & & \underbrace{0 \ \cdots & 0}_{m-1}
\end{vmatrix}\\
&= \cdots\\
&= (-1)^{m-1}\frac{\widetilde{V}^{(j)}_{N,rn-rm,\ge m}(rN+j)}{rN+rm+j} + (-1)^{m}\frac{\widetilde{V}^{(j)}_{N,r(n-m-1),\ge m}(rN+j)}{rN+r(m+1)+j}\\
&\quad + \cdots + (-1)^{n-m+1}\frac{\widetilde{V}^{(j)}_{N,rm,\ge m}(rN+j)}{rN+r(n-m)+j} + (-1)^{n-m+1}(-1)^{m}\frac{rN+j}{rN+rn+j}\\
&= -\sum_{k=m}^{n-m}\frac{(-1)^{n-k}\widetilde{V}_{N,rk,\ge m}(rN+j)}{r(N+n-k)+j} = \widetilde{V}_{N,rn,\ge m}\,.
\end{aligned}$$

Here, we used the second relation of Proposition 2 again. □

There exist explicit expressions for both incomplete Cauchy numbers.

Theorem 5. *For $n, m \ge 1$,*

$$V_{N,rn,r,\le m}^{(j)} = (rn)! \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n\\ 1\le i_1,\dots,i_k\le m}} \frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\,.$$

For $n, m \ge 1$,

$$V_{N,rn,\ge m}^{(j)} = (rn)! \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge m}} \frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\,.$$

Proof. First, we shall prove the first expression for the restricted hypergeometric Cauchy numbers. When $n \le m$, the proof is similar to that of Proposition 1. Note that in the proof of Proposition 1,

$$1 \le n-l = i_k \le n-k+1 \le n\,.$$

Let $n \ge m+1$. By the first relation of Proposition 2

$$\begin{aligned}
\frac{V_{N,rn,r,\le m}^{(j)}}{(rn)!} &= \sum_{l=n-m}^{n-1} \frac{(-1)^{n-l-1}(rN+j)V_{N,rl,r,\le m}^{(j)}}{(rN+rn-rl+j)(rl)!}\\
&= \sum_{l=n-m}^{n-1} \frac{(-1)^{n-l-1}(rN+j)}{rN+rn-rl+j} \sum_{k=1}^{l} (-1)^{l-k} \sum_{\substack{i_1+\cdots+i_k=l\\ 1\le i_1,\dots,i_k\le m}} \frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\
&= \sum_{l=1}^{n-1} \frac{(-1)^{n-k-1}(rN+j)}{rN+rn-rl+j} \sum_{k=1}^{l} \sum_{\substack{i_1+\cdots+i_k=l\\ 1\le i_1,\dots,i_k\le m}} \frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\
&\quad + \sum_{l=1}^{n-m-1} \frac{(-1)^{n-k}(rN+j)}{rN+rn-rl+j} \sum_{k=1}^{l} (-1)^{k} \sum_{\substack{i_1+\cdots+i_k=l\\ 1\le i_1,\dots,i_k\le m}} \frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\
&= \sum_{k=1}^{n-1} (-1)^{n-k-1} \sum_{l=k}^{n-1} \frac{rN+j}{rN+rn-rl+j} \sum_{\substack{i_1+\cdots+i_k=l\\ 1\le i_1,\dots,i_k\le m}} \frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\
&\quad + \sum_{k=1}^{n-m-1} (-1)^{n-k} \sum_{l=k}^{n-m-1} \frac{rN+j}{rN+rn-rl+j} \sum_{\substack{i_1+\cdots+i_k=l\\ 1\le i_1,\dots,i_k\le m}} \frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\
&= \sum_{k=2}^{n} (-1)^{n-k} \sum_{l=k-1}^{n-1} \frac{rN+j}{rN+rn-rl+j} \sum_{\substack{i_1+\cdots+i_{k-1}=l\\ 1\le i_1,\dots,i_{k-1}\le m}} \frac{(rN+j)^{k-1}}{(rN+ri_1+j)\cdots(rN+ri_{k-1}+j)}\\
&\quad + \sum_{k=2}^{n-m} (-1)^{n-k-1} \sum_{l=k-1}^{n-m-1} \frac{rN+j}{rN+rn-rl+j} \sum_{\substack{i_1+\cdots+i_{k-1}=l\\ 1\le i_1,\dots,i_{k-1}\le m}} \frac{(rN+j)^{k-1}}{(rN+ri_1+j)\cdots(rN+ri_{k-1}+j)}\\
&= \sum_{k=n-m+1}^{n} (-1)^{n-k} \sum_{l=k-1}^{n-1} \frac{rN+j}{rN+rn-rl+j} \sum_{\substack{i_1+\cdots+i_{k-1}=l\\ 1\le i_1,\dots,i_{k-1}\le m}} \frac{(rN+j)^{k-1}}{(rN+ri_1+j)\cdots(rN+ri_{k-1}+j)}
\end{aligned}$$

$$+\sum_{k=2}^{n-m}(-1)^{n-k}\sum_{l=n-m}^{n-1}\frac{rN+j}{rN+rn-rl+j}\sum_{\substack{i_1+\cdots+i_{k-1}=l\\ 1\le i_1,\dots,i_{k-1}\le m}}\frac{(rN+j)^{k-1}}{(rN+ri_1+j)\cdots(rN+ri_{k-1}+j)}\,.$$

By putting $i_k=n-l$, in the first term by $n-1\ge l\ge k-1\ge n-m$, in the second term by $n-1\ge l\ge n-m$, we have

$$1\le n-l=i_k\le m\,.$$

Therefore,

$$\begin{aligned}\frac{V^{(j)}_{N,rn,r,\le m}}{(rn)!}&=\sum_{k=n-m+1}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ 1\le i_1,\dots,i_k\le m}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\ &\quad+\sum_{k=2}^{n-m}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ 1\le i_1,\dots,i_k\le m}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\\ &=\sum_{k=1}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ 1\le i_1,\dots,i_k\le m}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\,.\end{aligned}$$

Note that the expression vanishes for $k=1$ as $n>m$.

Next, we prove the second expression for the associated hypergeometric Cauchy numbers. Since the set

$$\{(i_1,\dots,i_k)|i_1+\cdots+i_k=n,\ i_1,\dots,i_k\ge m\}$$

is empty for $n=1,\dots,m-1$, we have $V^{(j)}_{N,r,r,\ge m}=\cdots=V^{(j)}_{N,rm-r,r,\ge m}=0$. For $n=m$, by the second expression of Theorem 4

$$\begin{aligned}V^{(j)}_{N,rm,r,\ge m}&=(rm)!\begin{vmatrix}0&1&&\\ \vdots&&&\\ 0&&&1\\ \frac{rN+j}{rN+rm+j}&0&\cdots&0\end{vmatrix}\\ &=(rm)!(-1)^{m-1}\frac{rN+j}{rN+rm+j}=\frac{(-1)^{m-1}(rN+j)}{rN+rm+j}\,,\end{aligned}$$

which matches the result for $n=m$. Assume that the result is valid up to $n-1(\ge m)$. Then by the second relation of Proposition 2

$$\begin{aligned}\frac{V_{N,rn,r,\ge m}}{(rn)!}&=\sum_{l=0}^{n-m}\frac{(-1)^{n-l-1}(rN+j)}{(rN+rn-rl+j)(rl)!}V^{(j)}_{N,rl,r,\ge m}\\ &=\frac{(-1)^{n-1}(rN+j)}{rN+rn+j}\\ &\quad+\sum_{l=1}^{n-m}\frac{(-1)^{n-l-1}(rN+j)}{rN+rn-rl+j}\sum_{k=1}^{l}(-1)^{l-k}\sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\dots,i_k\ge m}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^{n-1}(rN+j)}{rN+rn+j} \\
&\quad + \sum_{k=1}^{n-m}(-1)^{n-k-1}\sum_{l=k}^{n-m}\frac{rN+j}{rN+rn-rl+j}\sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\dots,i_k\ge m}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)} \\
&= \frac{(-1)^{n-1}(rN+j)}{rN+rn+j} \\
&\quad + \sum_{k=2}^{n-m+1}(-1)^{n-k}\sum_{l=k-1}^{n-m}\frac{rN+j}{rN+rn-rl+j}\sum_{\substack{i_1+\cdots+i_{k-1}=l\\ i_1,\dots,i_{k-1}\ge m}}\frac{(rN+j)^{k-1}}{(rN+ri_1+j)\cdots(rN+ri_{k-1}+j)} \\
&= \frac{(-1)^{n-1}(rN+j)}{rN+rn+j} \\
&\quad + \sum_{k=2}^{n-m+1}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge m}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)} \\
&\qquad\qquad (i_k=n-l) \\
&= \sum_{k=1}^{n-m+1}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge m}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)} \\
&= \sum_{k=1}^{n}(-1)^{n-k}\sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge m}}\frac{(rN+j)^k}{(rN+ri_1+j)\cdots(rN+ri_k+j)}\,.
\end{aligned}
$$

Note that $i_k = n-l \ge m$ as $l \le n-m$. As $1 \le m \le n-1$, we have $m(n-m+2) > n$, so the set

$$\{(i_1,\dots,i_k) | i_1+\cdots+i_k = n,\ i_1,\dots,i_k \ge m\}$$

is empty for $n-m+2 \le k \le n$. □

5. Applications by Trudi's formula

We shall use Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $V_{N,n}^{(j)}$. Denote the multinomial coefficients by $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$.

Lemma 1. *For a positive integer n, we have*

$$
\begin{vmatrix}
a_1 & a_0 & 0 & \cdots & \\
a_2 & a_1 & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
a_{n-1} & & \cdots & a_1 & a_0 \\
a_n & a_{n-1} & \cdots & a_2 & a_1
\end{vmatrix}
= \sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-a_0)^{n-t_1-\cdots-t_n}a_1^{t_1}a_2^{t_2}\cdots a_n^{t_n}\,.
$$

This relation is known as Trudi's formula [21, Vol.3, p.214],[23] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [4],[21, Vol.3, pp.208–209].

In addition, there exists the following inversion formula (see, e.g., [17]), which is based upon the relation

$$\sum_{k=0}^{n}(-1)^{n-k}\alpha_k D(n-k) = 0 \quad (n \geq 1)$$

or Cameron's operator in (2.3).

Lemma 2. *For the sequence* $\{\alpha_n\}_{n\geq 0}$ *defined by* $\alpha_0 = 1$ *and*

$$\alpha_n = \begin{vmatrix} D(1) & 1 & & \\ D(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ D(n) & \cdots & D(2) & D(1) \end{vmatrix}, \text{ we have } D(n) = \begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}D(1)^{t_1}D(2)^{t_2}\cdots D(n)^{t_n}.$$

By applying these lemmas to Theorem 4, we obtain explicit expressions for the incomplete hypergeometric Cauchy numbers of higher grade defined in (4.1) and (4.2).

Theorem 6. *For* $n \geq m \geq 1$, *we have*

$$V_{N,rn,r,\leq m}^{(j)} = (rn)!\sum_{t_1+2t_2+\cdots+mt_m=n}\binom{t_1+\cdots+t_m}{t_1,\dots,t_m}(-1)^{n-t_1-\cdots-t_m}\left(\frac{rN+j}{rN+j+r}\right)^{t_1}\cdots\left(\frac{rN+j}{rN+rm+j}\right)^{t_m}$$

and

$$V_{N,rn,r,\geq m}^{(j)} = (rn)!\sum_{mt_m+(m+1)t_{m+1}+\cdots+nt_n=n}\binom{t_m+t_{m+1}+\cdots+t_n}{t_m,t_{m+1},\dots,t_n}$$
$$\times(-1)^{n-t_m-t_{m+1}-\cdots-t_n}\left(\frac{rN+j}{rN+rm+j}\right)^{t_m}\left(\frac{rN+j}{rN+rm+j+r}\right)^{t_{m+1}}\left(\frac{rN+j}{rN+rn+j}\right)^{t_n}.$$

As a special case of Theorem 6, we can obtain the expressions for the original hypergeometric Cauchy numbers.

Corollary 1. *For* $n \geq 1$, *we have*

$$V_{N,rn,r}^{(j)} = (rn)!\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}(-1)^{n-t_1-\cdots-t_n}\left(\frac{rN+j}{rN+j+r}\right)^{t_1}\cdots\left(\frac{rN+j}{rN+rn+j}\right)^{t_n}.$$

By applying the inversion relation in Lemma 2 to Theorem 3, we have the following.

Theorem 7. *Let $j = 0, 1$. For $n \ge 1$, we have*

$$\frac{rN+j}{rN+rn+j} = \left| J^T + \sum_{k=1}^{n} \frac{V_{N,kr,r}^{(j)}}{(kr)!} J^{k-1} \right| .$$

In this sense, we have the inversion relation of Corollary 1 too.

Corollary 2. *For $n \ge 1$, we have*

$$\frac{rN+j}{rN+rn+j} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} \left(\frac{V_{N,r,r}^{(j)}}{r!} \right)^{t_1} \cdots \left(\frac{V_{N,rn,r}^{(j)}}{(rn)!} \right)^{t_n} .$$

6. Conclusions

In this paper, we proposed one type of generalizations of the classical Cauchy numbers and hypergeometric Cauchy numbers. Many other generalizations are known, but the focus of this paper is on the determinant, which originated in Glaisher and others. Similar determinants have been dealt with by Brioshi, Trudi and others, but have long been forgotten. A similar generalization attempt, made by the first author of this paper with Barman in 2019, has proposed generalized numbers including the classical Bernoulli numbers, hypergeometric Bernoulli numbers, Euler numbers, hypergeometric Euler numbers, and so on. However, classical Cauchy numbers and hypergeometric Cauchy numbers cannot be included in the generalization by Barman et al., and this is achieved in this paper. The background and motivation for generalization is Cameron's operator, which is related to graph theory. There, only integers were targeted, but in this paper, we extended this to rational numbers and applied it.

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