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Research Article

A generalized conic domain and its applications to certain subclasses of multivalent functions associated with the basic (or q-) calculus

H. M. Srivastava[1,2,3,4,*], **T. M. Seoudy**[5,6] **and M. K. Aouf**[7]

[1] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada

[2] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[3] Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan

[4] Section of Mathematics, International Telematic University Uninettuno, I-00186 Rome, Italy

[5] Department of Mathematics, Faculty of Science, Fayoum University, Fayoum 63514, Egypt

[6] Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah, Kingdom of Saudi Arabia

[7] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

* **Correspondence:** Email: harimsri@math.uviv.ca; Tel: +12504725313, +12504776960; Fax: +12507218962.

Abstract: In this paper, by using the concept of the basic (or q-) calculus and a generalized conic domain, we define two subclasses of normalized multivalent functions which map the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

onto this generalized conic domain. We investigate a number of useful properties including (for example) the coefficient estimates and the Fekete-Szegö inequalities for each of these multivalent function classes. Our results are connected with those in several earlier works which are related to this field of Geometric Function Theory of Complex Analysis.

Keywords: analytic functions; multivalent functions; conic domain; basic (or q-) calculus; generalized conic domain; principle of subordination; coefficient estimates; Fekete-Szegö inequalities
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1. Introduction and definitions

The theory of the basic and the fractional quantum calculus, that is, the basic (or q-) calculus and the fractional basic (or q-) calculus, play important roles in many diverse areas of the mathematical, physical and engineering sciences (see, for example, [10, 15, 33, 45]). Our main objective in this paper is to introduce and study some subclasses of the class of the normalized p-valently analytic functions in the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\}$$

by applying the q-derivative operator in conjunction with the principle of subordination between analytic functions (see, for details, [8, 30]).

We begin by denoting by $\mathcal{A}(p)$ the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathbb{U}$. In particular, we write $\mathcal{A}(1) =: \mathcal{A}$.

A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}_p^*(\alpha)$ of p-valently starlike functions of order α in $\mathbb{U}$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (0 \leqq \alpha < p;\ z \in \mathbb{U}). \tag{1.2}$$

Moreover, a function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{C}_p(\alpha)$ of p-valently convex functions of order α in $\mathbb{U}$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (0 \leqq \alpha < p;\ z \in \mathbb{U}). \tag{1.3}$$

The p-valent function classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{C}_p(\alpha)$ were studied by Owa [32], Aouf [2, 3] and Aouf *et al.* [4, 5]. From (1.2) and (1.3), it follows that

$$f(z) \in \mathcal{C}_p(\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha). \tag{1.4}$$

Let $\mathcal{P}$ denote the Carathéodory class of functions $\mathfrak{p}(z)$, analytic in $\mathbb{U}$, which are normalized by

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{1.5}$$

such that $\Re(p(z)) > 0$.

Recently, Kanas and Wiśniowska [18, 19] (see also [17, 31]) introduced the conic domain Ω_k $(k \geqq 0)$, which we recall here as follows:

$$\Omega_k = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2}\right\}$$

or, equivalently,

$$\Omega_k = \{w : w \in \mathbb{C} \qquad \text{and} \qquad \Re(w) > k\,|w - 1|\}.$$

By using the conic domain Ω_k, Kanas and Wiśniowska [18,19] also introduced and studied the class k-$\mathcal{UCV}$ of k-uniformly convex functions in $\mathbb{U}$ as well as the corresponding class k-$\mathcal{ST}$ of k-starlike functions in $\mathbb{U}$. For fixed k, Ω_k represents the conic region bounded successively by the imaginary axis when $k = 0$. For $k = 1$, the domain Ω_k represents a parabola. For $1 < k < \infty$, the domain Ω_k represents the right branch of a hyperbola. And, for $k > 1$, the domain Ω_k represents an ellipse. For these conic regions, the following function plays the role of the extremal function:

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z} & (k=0) \\[2ex] 1 + \dfrac{2}{\pi^2}\left[\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2 & (k=1) \\[2ex] 1 + \dfrac{1}{1-k^2}\cos\left(\dfrac{2i}{\pi}(\arccos k)\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right) & (0<k<1) \\[2ex] 1 + \dfrac{1}{k^2-1}\sin\left(\dfrac{\pi}{2\mathrm{K}(\kappa)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{\kappa}}} \dfrac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right) + \dfrac{k^2}{k^2-1} & (1<k<\infty) \end{cases} \tag{1.6}$$

with

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa z}} \qquad (0 < \kappa < 1;\ z \in \mathbb{U}),$$

where κ is so chosen that

$$k = \cosh\left(\frac{\pi \mathrm{K}'(\kappa)}{4\mathrm{K}(\kappa)}\right).$$

Here $\mathrm{K}(\kappa)$ is Legendre's complete elliptic integral of the first kind and

$$\mathrm{K}'(\kappa) = \mathrm{K}\left(\sqrt{1-\kappa^2}\right),$$

that is, $\mathrm{K}'(\kappa)$ is the complementary integral of $\mathrm{K}(\kappa)$ (see, for example, [48, p. 326, Eq 9.4 (209)]).

We now recall the definitions and concept details of the basic (or q-) calculus, which are used in this paper (see, for details, [13, 14, 45]; see also [1, 6, 7, 11, 34, 38, 39, 42, 54, 59]). Throughout the paper, unless otherwise mentioned, we suppose that $0 < q < 1$ and

$$\mathbb{N} = \{1, 2, 3 \cdots\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 := \{0, 1, 2, \cdots\}).$$

Definition 1. The q-number $[\lambda]_q$ is defined by

$$[\lambda]_q = \begin{cases} \dfrac{1-q^\lambda}{1-q} & (\lambda \in \mathbb{C}) \\[2ex] \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 \cdots + q^{n-1} & (\lambda = n \in \mathbb{N}), \end{cases} \tag{1.7}$$

so that

$$\lim_{q \to 1-} [\lambda]_q = \frac{1-q^\lambda}{1-q} = \lambda.$$

.

Definition 2. For functions given by (1.1), the q-derivative (or the q-difference) operator D_q of a function f is defined by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{1.8}$$

provided that $f'(0)$ exists.

We note from Definition 2 that

$$\lim_{q \to 1-} D_q f(z) = \lim_{q \to 1-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. It is readily deduced from (1.1) and (1.8) that

$$D_q f(z) = [p]_q z^{p-1} + \sum_{n=p+1}^{\infty} [n]_q a_n z^{n-1}. \tag{1.9}$$

We remark in passing that, in the above-cited recently-published survey-cum-expository review article, the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [42, p. 340]).

Making use of the q-derivative operator D_q given by (1.6), we introduce the subclass $\mathcal{S}^*_{q,p}(\alpha)$ of p-valently q-starlike functions of order α in $\mathbb{U}$ and the subclass $C_{q,p}(\alpha)$ of p-valently q-convex functions of order α in $\mathbb{U}$ as follows (see [54]):

$$f(z) \in \mathcal{S}^*_{q,p}(\alpha) \Longleftrightarrow \Re\left(\frac{1}{[p]_q}\frac{zD_q f(z)}{f(z)}\right) > \alpha \tag{1.10}$$

$$(0 < q < 1;\ 0 \leqq \alpha < 1;\ z \in \mathbb{U})$$

and

$$f(z) \in C_{q,p}(\alpha) \Longleftrightarrow \Re\left(\frac{1}{[p]_q}\frac{D_{p,q}\left(zD_q f(z)\right)}{D_q f(z)}\right) > \alpha \tag{1.11}$$

$$(0 < q < 1;\ 0 \leqq \alpha < 1;\ z \in \mathbb{U}),$$

respectively. From (1.10) and (1.11), it follows that

$$f(z) \in C_{q,p}(\alpha) \Longleftrightarrow \frac{zD_q f(z)}{[p]_q} \in \mathcal{S}^*_{q,p}(\alpha). \tag{1.12}$$

For the simpler classes $\mathcal{S}^*_{q,p}$ and $C^*_{q,p}$ of p-valently q-starlike functions in $\mathbb{U}$ and p-valently q-convex functions in $\mathbb{U}$, respectively, we have write

$$\mathcal{S}^*_{q,p}(0) =: \mathcal{S}^*_{q,p} \qquad \text{and} \qquad C_{q,p}(0) =: C_{q,p}.$$

Obviously, in the limit when $q \to 1-$, the function classes $\mathcal{S}^*_{q,p}(\alpha)$ and $C_{q,p}(\alpha)$ reduce to the familiar function classes $\mathcal{S}^*_p(\alpha)$ and $C_p(\alpha)$, respectively.

Definition 3. A function $f \in \mathcal{A}(p)$ is said to belong to the class $\mathcal{S}^*_{q,p}$ of p-valently q-starlike functions in $\mathbb{U}$ if

$$\left|\frac{zD_q f(z)}{[p]_q f(z)} - \frac{1}{1-q}\right| \leq \frac{1}{1-q} \qquad (z \in \mathbb{U}). \tag{1.13}$$

In the limit when $q \to 1-$, the closed disk

$$\left|w - \frac{1}{1-q}\right| \leqq \frac{1}{1-q} \qquad (0 < q < 1)$$

becomes the right-half plane and the class $\mathcal{S}^*_{q,p}$ of p-valently q-starlike functions in $\mathbb{U}$ reduces to the familiar class $\mathcal{S}^*_p$ of p-valently starlike functions with respect to the origin ($z = 0$). Equivalently, by using the principle of subordination between analytic functions, we can rewrite the condition (1.13) as follows (see [58]):

$$\frac{zD_q f(z)}{[p]_q f(z)} \prec \hat{p}(z) \quad \left(\hat{p}(z) = \frac{1+z}{1-qz}\right). \tag{1.14}$$

We note that $\mathcal{S}^*_{q,1} = \mathcal{S}^*_q$ (see [12,41]).

Definition 4. (see [50]) A function $\mathfrak{p}(z)$ given by (1.5) is said to be in the class k-$\mathcal{P}_q$ if and only if

$$\mathfrak{p}(z) \prec \frac{2p_k(z)}{(1+q) + (1-q)\,p_k(z)},$$

where $p_k(z)$ is given by (1.6).

Geometrically, the function $p \in k$-$\mathcal{P}_q$ takes on all values from the domain $\Omega_{k,q}$ ($k \geqq 0$) which is defined as follows:

$$\Omega_{k,q} = \left\{w : \Re\left(\frac{(1+q)w}{(q-1)w+2}\right) > k\left|\frac{(1+q)w}{(q-1)w+2} - 1\right|\right\}. \tag{1.15}$$

The domain $\Omega_{k,q}$ represents a generalized conic region which was introduced and studied earlier by Srivastava *et al.* (see, for example, [43,50]). It reduces, in the limit when $q \to 1-$, to the conic domain Ω_k studied by Kanas and Wiśniowska [18]. We note the following simpler cases.

(1) k-$\mathcal{P}_q \subseteq \mathcal{P}\left(\frac{2k}{2k+1+q}\right)$, where $\mathcal{P}\left(\frac{2k}{2k+1+q}\right)$ is the familiar class of functions with real part greater than $\frac{2k}{2k+1+q}$;
(2) $\lim_{q\to 1-}\{k\text{-}\mathcal{P}_q\} = \mathcal{P}(p_k(z))$, where $\mathcal{P}(p_k(z))$ is the known class introduced by Kanas and Wiśniowska [18];
(3) $\lim_{q\to 1-}\{0\text{-}\mathcal{P}_q\} = \mathcal{P}$, where $\mathcal{P}$ is Carathéodory class of analytic functions with positive real part.

Definition 5. A function $f \in \mathcal{A}(p)$ is said to be in the class k-$\mathcal{ST}_{q,p}$ if and only if

$$\Re\left(\frac{(1+q)\dfrac{zD_q f(z)}{[p]_q f(z)}}{(q-1)\dfrac{zD_q f(z)}{[p]_q f(z)} + 2}\right) > k\left|\frac{(1+q)\dfrac{zD_q f(z)}{[p]_q f(z)}}{(q-1)\dfrac{zD_q f(z)}{[p]_q f(z)} + 2} - 1\right| \quad (z \in \mathbb{U}) \tag{1.16}$$

or, equivalently,

$$\frac{zD_q f(z)}{[p]_q f(z)} \in k\text{–}\mathcal{P}_q. \tag{1.17}$$

The folowing special cases are worth mentioning here.

(1) $k\text{-}\mathcal{ST}_{q,1} = k\text{-}\mathcal{ST}_q$, where $k\text{-}\mathcal{ST}_q$ is the function class introduced and studied by Srivastava *et al.* [50] and Zhang *et al.* [60] with $\gamma = 1$;
(2) $0\text{-}\mathcal{ST}_{q,p} = \mathcal{S}_{q,p}$;
(3) $\lim_{q\to 1}\{k\text{-}\mathcal{ST}_{q,p}\} = k\text{-}\mathcal{ST}_p$, where $k\text{-}\mathcal{ST}_p$ is the class of p-valently uniformly starlike functions;
(4) $\lim_{q\to 1}\{0\text{-}\mathcal{ST}_{q,p}\} = \mathcal{S}_p$, where $\mathcal{S}_p^*$ is the class of p-valently starlike functions;
(5) $0\text{-}\mathcal{ST}_{q,1} = \mathcal{S}_q^*$, where $\mathcal{S}_q^*$ (see [12, 41]);
(6) $\lim_{q\to 1}\{k\text{-}\mathcal{ST}_{q,1}\} = k\text{-}\mathcal{ST}$, where $k\text{-}\mathcal{ST}$ is a function class introduced and studied by Kanas and Wiśniowska [19];
(7) $\lim_{q\to 1}\{0\text{-}\mathcal{ST}_{q,1}\} = \mathcal{S}^*$, where $\mathcal{S}^*$ is the familiar class of starlike functions in $\mathbb{U}$.

Definition 6. By using the idea of Alexander's theorem [9], the class $k\text{-}\mathcal{UCV}_{q,p}$ can be defined in the following way:

$$f(z) \in k-\mathcal{UCV}_{q,p} \Longleftrightarrow \frac{zD_q f(z)}{[p]_q} \in k-\mathcal{ST}_{q,p}. \tag{1.18}$$

In this paper, we investigate a number of useful properties including coefficient estimates and the Fekete-Szegö inequalities for the function classes $k\text{-}\mathcal{ST}_{q,p}$ and $k\text{-}\mathcal{UCV}_{q,p}$, which are introduced above. Various corollaries and consequences of most of our results are connected with earlier works related to the field of investigation here.

2. A set of Lemmas

In order to establish our main results, we need the following lemmas.

Lemma 1. (see [16]) *Let* $0 \leqq k < \infty$ *be fixed and let* p_k *be defined by* (1.6). *If*

$$p_k(z) = 1 + Q_1 z + Q_2 z^2 + \cdots,$$

then

$$Q_1 = \begin{cases} \dfrac{2A^2}{1-k^2} & (0 \leqq k < 1) \\[2ex] \dfrac{8}{\pi^2} & (k = 1) \\[2ex] \dfrac{\pi^2}{4\sqrt{t}\,(k^2-1)\,[\mathrm{K}(t)]^2\,(1+t)} & (1 < k < \infty) \end{cases} \tag{2.1}$$

and

$$Q_2 = \begin{cases} \frac{A^2+2}{3} Q_1 & (0 \leqq k < 1) \\[2ex] \dfrac{2}{3} Q_1 & (k = 1) \\[2ex] \dfrac{4[\mathrm{K}(t)]^2\left(t^2+6t+1\right) - \pi^2}{24\sqrt{t}[\mathrm{K}(t)]^2\,(1+t)} Q_1 & (1 < k < \infty), \end{cases} \tag{2.2}$$

where

$$A = \frac{2\arccos k}{\pi},$$

and $t \in (0,1)$ *is so chosen that*

$$k = \cosh\left(\frac{\pi \mathrm{K}'(t)}{\mathrm{K}(t)}\right),$$

$\mathrm{K}(t)$ *being Legendre's complete elliptic integral of the first kind.*

Lemma 2. *Let* $0 \leqq k < \infty$ *be fixed and suppose that*

$$p_{k,q}(z) = \frac{2p_k(z)}{(1+q)+(1-q)\,p_k(z)}, \tag{2.3}$$

where $p_k(z)$ *be defined by* (1.6). *Then*

$$p_{k,q}(z) = 1 + \frac{1}{2}(1+q)\,Q_1 z + \frac{1}{2}(1+q)\left[Q_2 - \frac{1}{2}(1-q)\,Q_1^2\right]z^2 + \cdots, \tag{2.4}$$

where Q_1 *and* Q_2 *are given by* (2.1) *and* (2.2), *respectively.*

Proof. By using (1.6) in (2.3), we can easily derive (2.4). □

Lemma 3. (see [26]) *Let the function h given by*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P}$$

be analytic in $\mathbb{U}$ *and satisfy* $\Re(h(z)) > 0$ *for* z *in* $\mathbb{U}$. *Then the following sharp estimate holds true*:

$$\left|c_2 - v c_1^2\right| \leqq 2\max\{1, |2v-1|\} \qquad (\forall\ v \in \mathbb{C}).$$

The result is sharp for the functions given by

$$g(z) = \frac{1+z^2}{1-z^2} \qquad \text{or} \qquad g(z) = \frac{1+z}{1-z}. \tag{2.5}$$

Lemma 4. (see [26]) *If the function h is given by*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{P},$$

then

$$\left|c_2 - v c_1^2\right| \leqq \begin{cases} -4v + 2 & (v \leqq 0) \\ 2 & (0 \leqq v \leqq 1) \\ 4v - 2 & (v \geqq 1). \end{cases} \tag{2.6}$$

When $\nu > 1$, the equality holds true if and only if

$$h(z) = \frac{1+z}{1-z}$$

or one of its rotations. If $0 < \nu < 1$, then the equality holds true if and only if

$$h(z) = \frac{1+z^2}{1-z^2}$$

or one of its rotations. If $\nu = 0$, the equality holds true if and only if

$$h(z) = \left(\frac{1+\lambda}{2}\right)\left(\frac{1+z}{1-z}\right) + \left(\frac{1-\lambda}{2}\right)\left(\frac{1-z}{1+z}\right) \qquad (0 \leqq \lambda \leqq 1)$$

or one of its rotations. If $\nu = 1$, the equality holds true if and only if the function h is the reciprocal of one of the functions such that equality holds true in the case when $\nu = 0$.

The above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \leqq 2 \qquad \left(0 \leqq \nu \leqq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) \left|c_1\right|^2 \leqq 2 \qquad \left(\frac{1}{2} \leq \nu \leqq 1\right).$$

3. Main results

We assume throughout our discussion that, unless otherwise stated, $0 \leqq k < \infty$, $0 < q < 1$, $p \in \mathbb{N}$, Q_1 is given by (2.1), Q_2 is given by (2.2) and $z \in \mathbb{U}$.

Theorem 1. *If a function $f \in \mathcal{A}(p)$ is of the form* (1.1) *and satisfies the following condition*:

$$\sum_{n=p+1}^{\infty} \left\{2(k+1)\left([n]_q - [p]_q\right) + q^n + 2[p]_q - 1\right\} |a_n| < (1+q)[p]_q, \tag{3.1}$$

then the function $f \in k\text{-}\mathcal{ST}_{q,p}$.

Proof. Suppose that the inequality (3.1) holds true. Then it suffices to show that

$$k\left|\frac{(1+q)\frac{zD_qf(z)}{[p]_qf(z)}}{(q-1)\frac{zD_qf(z)}{[p]_qf(z)}+2} - 1\right| - \Re\left(\frac{(1+q)\frac{zD_qf(z)}{[p]_qf(z)}}{(q-1)\frac{zD_qf(z)}{[p]_qf(z)}+2} - 1\right) < 1.$$

In fact, we have

$$k\left|\frac{(1+q)\frac{zD_qf(z)}{[p]_qf(z)}}{(q-1)\frac{zD_qf(z)}{[p]_qf(z)}+2} - 1\right| - \Re\left(\frac{(1+q)\frac{zD_qf(z)}{[p]_qf(z)}}{(q-1)\frac{zD_qf(z)}{[p]_qf(z)}+2} - 1\right)$$

$$
\begin{aligned}
&\leqq (k+1)\left|\frac{(1+q)\,\frac{zD_qf(z)}{[p]_qf(z)}}{(q-1)\,\frac{zD_qf(z)}{[p]_qf(z)}+2}-1\right| \\
&= 2\,(k+1)\left|\frac{zD_qf\,(z)-[p]_q\,f\,(z)}{(q-1)\,zD_qf\,(z)+2\,[p]_q\,f\,(z)}\right| \\
&= 2\,(k+1)\left|\frac{\sum\limits_{n=p+1}^{\infty}\left([n]_q-[p]_q\right)a_nz^{n-p}}{(1+q)\,[p]_q+\sum\limits_{n=p+1}^{\infty}\left((q-1)\,[n]_q+2\,[p]_q\right)a_nz^{n-p}}\right| \\
&\leqq 2\,(k+1)\,\frac{\sum\limits_{n=p+1}^{\infty}\left([n]_q-[p]_q\right)|a_n|}{(1+q)\,[p]_q-\sum\limits_{n=p+1}^{\infty}\left(q^n+2\,[p]_q-1\right)|a_n|}.
\end{aligned}
$$

The last expression is bounded by 1 if (3.1) holds true. This completes the proof of Theorem 1. □

Corollary 1. *If $f\,(z)\in k\text{-}\mathcal{ST}_{q,p}$, then*

$$
|a_n|\leqq\frac{(1+q)\,[p]_q}{\left\{2\,(k+1)\left([n]_q-[p]_q\right)+q^n+2\,[p]_q-1\right\}}\qquad (n\geqq p+1)\,.
$$

The result is sharp for the function $f(z)$ given by

$$
f\,(z)=z^p+\frac{(1+q)\,[p]_q}{\left\{2\,(k+1)\left([n]_q-[p]_q\right)+q^n+2\,[p]_q-1\right\}}z^n\qquad (n\geqq p+1)\,.
$$

Remark 1. Putting $p=1$ Theorem 1, we obtain the following result which corrects a result of Srivastava *et al.* [50, Theorem 3.1].

Corollary 2. (see Srivastava *et al.* [50, Theorem 3.1]) *If a function $f\in\mathcal{A}$ is of the form* (1.1) (*with* $p=1$) *and satisfies the following condition*:

$$
\sum_{n=2}^{\infty}\left\{2\,(k+1)\left([n]_q-1\right)+q^n+1\right\}|a_n|<(1+q)
$$

then the function $f\in k\text{-}\mathcal{ST}_q$.

Letting $q\to 1-$ in Theorem 1, we obtain the following known result [29, Theorem 1] *with*

$$
\alpha_1=\beta_1=p,\ \ \alpha_i=1(i=2,\cdots,s+1)\quad\text{and}\quad \beta_j=1(j=2,\cdots,s).
$$

Corollary 3. *If a function $f\in\mathcal{A}\,(p)$ is of the form* (1.1) *and satisfies the following condition*:

$$
\sum_{n=p+1}^{\infty}\{(k+1)\,(n-p)+p\}\,|a_n|<p,
$$

then the function $f\in k\text{-}\mathcal{ST}_p$.

Remark 2. Putting $p = 1$ in Corollary 3, we obtain the result obtained by Kanas and Wiśniowska [19, Theorem 2.3].

By using Theorem 1 and (1.18), we obtain the following result.

Theorem 2. *If a function $f \in \mathcal{A}(p)$ is of the form* (1.1) *and satisfies the following condition*:

$$\sum_{n=p+1}^{\infty} \left(\frac{[n]_q}{[p]_q} \right) \left\{ 2(k+1)\left([n]_q - [p]_q\right) + q^n + 2[p]_q - 1 \right\} |a_n| < (1+q)[p]_q ,$$

then the function $f \in k\text{-}\mathcal{UCV}_{q,p}$.

Remark 3. Putting $p = 1$ Theorem 1, we obtain the following result which corrects the result of Srivastava *et al.* [50, Theorem 3.3].

Corollary 4. (see Srivastava *et al.* [50, Theorem 3.3]) *If a function $f \in \mathcal{A}$ is of the form (1.1) (with $p = 1$) and satisfies the following condition*:

$$\sum_{n=2}^{\infty} [n]_q \left\{ 2(k+1)\left([n]_q - 1\right) + q^n + 1 \right\} |a_n| < (1+q) ,$$

then the function $f \in k\text{-}\mathcal{UCV}_q$.

Letting $q \to 1-$ in Theorem 2, we obtain the following corollary (see [29, Theorem 1]) with

$$\alpha_1 = p+1, \;\; \beta_1 = p, \;\; \alpha_\ell = 1 \;\; (\ell = 2, \cdots, s+1) \quad \text{and} \quad \beta_j = 1 (j = 2, \cdots, s).$$

Corollary 5. *If a function $f \in \mathcal{A}(p)$ is of the form* (1.1) *and satisfies the following condition*:

$$\sum_{n=p+1}^{\infty} \left(\frac{n}{p} \right) \{ (k+1)(n-p) + p \} |a_n| < p,$$

then the function $f \in k\text{-}\mathcal{UCV}_p$.

Remark 4. Putting $p = 1$ in Corollary 5, we obtain the following corollary which corrects the result of Kanas and Wiśniowska [18, Theorem 3.3].

Corollary 6. *If a function $f \in \mathcal{A}$ is of the form* (1.1) (*with* $p = 1$) *and satisfies the following condition*:

$$\sum_{n=2}^{\infty} n \{ n(k+1) - k \} |a_n| < 1,$$

then the function $f \in k\text{-}\mathcal{UCV}$.

Theorem 3. *If $f \in k\text{-}\mathcal{ST}_{q,p}$, then*

$$\left| a_{p+1} \right| \leqq \frac{(1+q)[p]_q Q_1}{2q^p [1]_q} \tag{3.2}$$

and, for all $n = 3, 4, 5, \cdots$,

$$\left| a_{n+p-1} \right| \leqq \frac{(1+q)[p]_q Q_1}{2q^p [n-1]_q} \prod_{j=1}^{n-2} \left(1 + \frac{(1+q)[p]_q Q_1}{2q^p [j]_q} \right). \tag{3.3}$$

Proof. Suppose that

$$\frac{zD_q f(z)}{[p]_q f(z)} = \mathfrak{p}(z), \tag{3.4}$$

where

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in k\text{–}\mathcal{P}_q.$$

Eq (3.4) can be written as follows:

$$zD_q f(z) = [p]_q f(z)\,\mathfrak{p}(z),$$

which implies that

$$\sum_{n=p+1}^{\infty} \left([n]_q - [p]_q\right) a_n z^n = [p]_q \left(z^p + \sum_{n=p+1}^{\infty} a_n z^n\right)\left(\sum_{n=1}^{\infty} c_n z^n\right). \tag{3.5}$$

Comparing the coefficients of z^{n+p-1} on both sides of (3.5), we obtain

$$\left([n+p-1]_q - [p]_q\right) a_{n+p-1} = [p]_q \left\{c_{n-1} + a_{p+1}c_{n-2} + \cdots + a_{n+p-2}c_1\right\}.$$

By taking the moduli on both sides and then applying the following coefficient estimates (see [50]):

$$|c_n| \leqq \frac{1}{2}(1+q)\,Q_1 \qquad (n \in \mathbb{N}),$$

we find that

$$\left|a_{n+p-1}\right| \leqq \frac{(1+q)\,[p]_q Q_1}{2q^p\,[n-1]_q} \left\{1 + \left|a_{p+1}\right| + \cdots + \left|a_{n+p-2}\right|\right\}. \tag{3.6}$$

We now apply the principle of mathematical induction on (3.6). Indeed, for $n = 2$, we have

$$\left|a_{p+1}\right| \leqq \frac{(1+q)\,[p]_q Q_1}{2q^p\,[1]_q}, \tag{3.7}$$

which shows that the result is true for $n = 2$. Next, for $n = 3$ in (3.7), we get

$$\left|a_{p+2}\right| \leqq \frac{(1+q)\,[p]_q Q_1}{2q^p\,[2]_q} \left\{1 + \left|a_{p+1}\right|\right\}.$$

By using (3.7), we obtain

$$\left|a_{p+2}\right| \leqq \frac{(1+q)\,[p]_q Q_1}{2q^p\,[2]_q} \left(1 + \frac{(1+q)\,[p]_q Q_1}{2q^p\,[1]_q}\right),$$

which is true for $n = 3$. Let us assume that (3.3) is true for $n = t$ $(t \in \mathbb{N})$, that is,

$$\left|a_{t+p-1}\right| \leqq \frac{(1+q)\,[p]_q\,Q_1}{2q^p\,[t-1]_q} \prod_{j=1}^{t-2} \left(1 + \frac{(1+q)\,[p]_q\,Q_1}{2q^p\,[j]_q}\right).$$

Let us consider

$$\begin{aligned}
\left|a_{t+p}\right| &\leqq \frac{(1+q)\,[p]_q Q_1}{2q^p\,[t]_q}\left\{1+\left|a_{p+1}\right|+\left|a_{p+2}\right|+\cdots+\left|a_{t+p-1}\right|\right\} \\
&\leqq \frac{(1+q)\,[p]_q Q_1}{2q^p\,[t]_q}\left\{1+\frac{(1+q)\,[p]_q Q_1}{2q^p\,[1]_q}+\frac{(1+q)\,[p]_q Q_1}{2q^p\,[2]_q}\left(1+\frac{(1+q)\,[p]_q Q_1}{2q^p\,[1]_q}\right)\right. \\
&\qquad \left.+\cdots+\frac{(1+q)\,[p]_q\,Q_1}{2q^p\,[t-1]_q}\prod_{j=1}^{t-2}\left(1+\frac{(1+q)\,[p]_q\,Q_1}{2q^p\,[j]_q}\right)\right\} \\
&= \frac{(1+q)\,[p]_q Q_1}{2q^p\,[t]_q}\left\{\left(1+\frac{(1+q)\,[p]_q Q_1}{2q^p\,[1]_q}\right)\left(1+\frac{(1+q)\,[p]_q Q_1}{2q^p\,[2]_q}\right)\right. \\
&\qquad \left.\cdots\left(1+\frac{(1+q)\,[p]_q Q_1}{2q^p\,[t-1]_q}\right)\right\} \\
&= \frac{(1+q)\,[p]_q Q_1}{2q^p\,[t]_q}\prod_{j=1}^{t-1}\left(1+\frac{(1+q)\,[p]_q Q_1}{2q^p\,[j]_q}\right)
\end{aligned}$$

Therefore, the result is true for $n = t + 1$. Consequently, by the principle of mathematical induction, we have proved that the result holds true for all $n \quad (n \in \mathbb{N} \setminus \{1\})$. This completes the proof of Theorem 3. □

Similarly, we can prove the following result.

Theorem 4. *If $f \in k\text{-}\mathcal{UCV}_{q,p}$ and is of form* (1.1), *then*

$$\left|a_{p+1}\right| \leqq \frac{(1+q)\,[p]_q^2\,Q_1}{2q^p\,[p+1]_q}$$

and, for all $n = 3, 4, 5, \cdots,$

$$\left|a_{n+p-1}\right| \leqq \frac{(1+q)\,[p]_q^2\,Q_1}{2q^p\,[n+p-1]_q\,[n-1]_q}\prod_{j=1}^{n-2}\left(1+\frac{(1+q)\,[p]_q\,Q_1}{2q^p\,[j]_q}\right).$$

Putting $p = 1$ in Theorems 3 and 4, we obtain the following corollaries.

Corollary 7. *If $f \in k\text{-}\mathcal{ST}_q$, then*

$$|a_2| \leqq \frac{(1+q)\,Q_1}{2q}$$

and, for all $n = 3, 4, 5, \cdots,$

$$|a_n| \leqq \frac{(1+q)\,Q_1}{2q\,[n-1]_q}\prod_{j=1}^{n-2}\left(1+\frac{(1+q)\,Q_1}{2q\,[j]_q}\right).$$

Corollary 8. *If $f \in k\text{-}\mathcal{UCV}_q$, then*

$$|a_2| \leqq \frac{Q_1}{2q}$$

and, for all $n = 3, 4, 5, \cdots$,

$$|a_n| \leqq \frac{(1+q)\,Q_1}{2q\,[n]_q\,[n-1]_q} \prod_{j=1}^{n-2} \left(1 + \frac{(1+q)\,Q_1}{2q\,[j]_q}\right).$$

Theorem 5. *Let* $f \in k\text{-}\mathcal{ST}_{q,p}$. *Then* $f(\mathbb{U})$ *contains an open disk of the radius given by*

$$r = \frac{2q^p}{2(p+1)\,q^p + (1+q)\,[p]_q\,Q_1}.$$

Proof. Let $w_0 \neq 0$ be a complex number such that $f(z) \neq w_0$ for $z \in \mathbb{U}$. Then

$$f_1(z) = \frac{w_0 f(z)}{w_0 - f(z)} = z^{p+1} + \left(a_{p+1} + \frac{1}{w_0}\right) z^{p+1} + \cdots.$$

Since f_1 is univalent, so

$$\left|a_{p+1} + \frac{1}{w_0}\right| \leqq p + 1.$$

Now, using Theorem 3, we have

$$\left|\frac{1}{w_0}\right| \leqq p + 1 + \frac{(1+q)\,[p]_q\,Q_1}{2q^p} = \frac{2q^p(p+1) + (1+q)\,[p]_q\,Q_1}{2q^p}.$$

Hence

$$|w_0| \geqq \frac{2q^p}{2q^p(p+1) + (1+q)\,[p]_q\,Q_1}.$$

This completes the proof of Theorem 5. □

Theorem 6. *Let the function* $f \in k\text{-}\mathcal{ST}_{q,p}$ *be of the form* (1.1). *Then, for a complex number* μ,

$$\begin{aligned} \left|a_{p+2} - \mu a_{p+1}^2\right| \leqq \frac{[p]_q\,Q_1}{2q^p} \max\Bigg\{1, \Bigg|\frac{Q_2}{Q_1} + \frac{\left([p]_q(1+q) - q^p(1-q)\right) Q_1}{2q^p} \\ \cdot \left(1 - \mu\,\frac{(1+q)^2\,[p]_q}{[p]_q(1+q) - q^p(1-q)}\right)\Bigg|\Bigg\}. \end{aligned} \tag{3.8}$$

The result is sharp.

Proof. If $f \in k\text{-}\mathcal{ST}_{q,p}$, we have

$$\frac{zD_q f(z)}{[p]_q\,f(z)} \prec p_{k,q}(z) = \frac{2p_k(z)}{(1+q) + (1-q)\,p_k(z)}.$$

From the definition of the differential subordination, we know that

$$\frac{zD_q f(z)}{[p]_q\,f(z)} = p_{k,q}(w(z)) \qquad (z \in \mathbb{U}), \tag{3.9}$$

where $w(z)$ is a Schwarz function with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$.

Let $h \in \mathcal{P}$ be a function defined by

$$h(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

This gives

$$w(z) = \frac{1}{2} c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots$$

and

$$\begin{aligned} p_{k,q}(w(z)) = 1 + \frac{1+q}{4} c_1 Q_1 z + \frac{1+q}{4} \Bigg\{ Q_1 c_2 + \frac{1}{2}\Bigg(Q_2 - Q_1 \\ - \frac{1-q}{2} Q_1^2 \Bigg) c_1^2 \Bigg\} z^2 + \cdots . \end{aligned} \tag{3.10}$$

Using (3.10) in (3.9), we obtain

$$a_{p+1} = \frac{(1+q)[p]_q c_1 Q_1}{4q^p}$$

and

$$a_{p+2} = \frac{[p]_q Q_1}{4q^p}\left[c_2 - \frac{1}{2}\left(1 - \frac{Q_2}{Q_1} - \frac{[p]_q (1+q) - q^p (1-q)}{2q^p} Q_1\right) c_1^2\right]$$

Now, for any complex number μ, we have

$$\begin{aligned} a_{p+2} - \mu a_{p+1}^2 = \frac{[p]_q Q_1}{4q^p}\left[c_2 - \frac{1}{2}\left(1 - \frac{Q_2}{Q_1} - \frac{[p]_q (1+q) - q^p (1-q)}{2q^p} Q_1\right) c_1^2\right] \\ - \mu \frac{(1+q)^2 [p]_q^2 Q_1^2 c_1^2}{16 q^{2p}}. \end{aligned} \tag{3.11}$$

Then (3.11) can be written as follows:

$$a_{p+2} - \mu a_{p+1}^2 = \frac{[p]_q Q_1}{4q^p}\left\{c_2 - v c_1^2\right\}, \tag{3.12}$$

where

$$\begin{aligned} v = \frac{1}{2}\Bigg[1 - \frac{Q_2}{Q_1} - \frac{\left([p]_q (1+q) - q^p (1-q)\right) Q_1}{2q^p} \\ \cdot \left(1 - \mu \frac{(1+q)^2 [p]_q}{[p]_q (1+q) - q^p (1-q)}\right)\Bigg]. \end{aligned} \tag{3.13}$$

Finally, by taking the moduli on both sides and using Lemma 4, we obtain the required result. The sharpness of (3.8) follows from the sharpness of (2.5). Our demonstration of Theorem 6 is thus completed. □

Similarly, we can prove the following theorem.

Theorem 7. *Let the function $f \in k\text{-}\mathcal{UCV}_{q,p}$ be of the form* (1.1)*. Then, for a complex number μ,*

$$\left|a_{p+2}-\mu a_{p+1}^{2}\right| \leqq \frac{[p]_q^2 Q_1}{2q^p [p+2]_q} \max\left\{1, \left|\frac{Q_2}{Q_1}+\frac{\left((1+q)[p]_q-(1-q)q^p\right)Q_1}{2q^p} \cdot\left(1-\mu\,\frac{[p+2]_q (1+q)^2 [p]_q^2}{\left((1+q)[p]_q-(1-q)q^p\right)[p+1]_q^2}\right)\right|\right\}.$$

The result is sharp.

Putting $p=1$ in Theorems 6 and 7, we obtain the following corollaries.

Corollary 9. *Let the function $f \in k\text{-}\mathcal{ST}_q$ be of the form* (1.1) (*with $p=1$*)*. Then, for a complex number μ,*

$$\left|a_3-\mu a_2^2\right| \leqq \frac{Q_1}{2q} \max\left\{1, \left|\frac{Q_2}{Q_1}+\frac{\left(1+q^2\right)Q_1}{2q}\left(1-\mu\frac{(1+q)^2}{1+q^2}\right)\right|\right\}.$$

The result is sharp.

Corollary 10. *Let the function $f \in k\text{-}\mathcal{UCV}_q$ be of the form* (1.1) (*with $p=1$*)*. Then, for a complex number μ,*

$$\left|a_3-\mu a_2^2\right| \leqq \frac{Q_1}{2q[3]_q} \max\left\{1, \left|\frac{Q_2}{Q_1}+\frac{\left(1+q^2\right)Q_1}{2q}\left(1-\mu\frac{[3]_q}{1+q^2}\right)\right|\right\}.$$

The result is sharp.

Theorem 8. *Let*

$$\sigma_1=\frac{\left([p]_q(1+q)-q^p(1-q)\right)Q_1^2+2q^p(Q_2-Q_1)}{[p]_q(1+q)^2 Q_1^2},$$

$$\sigma_2=\frac{\left([p]_q(1+q)-q^p(1-q)\right)Q_1^2+2q^p(Q_2+Q_1)}{[p]_q(1+q)^2 Q_1^2}$$

and

$$\sigma_3=\frac{\left([p]_q(1+q)-q^p(1-q)\right)Q_1^2+2q^pQ_2}{[p]_q(1+q)^2 Q_1^2}.$$

If the function f given by (1.1) *belongs to the class $k\text{-}\mathcal{ST}_{q,p}$, then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leqq \begin{cases} \frac{[p]_qQ_1}{2q^p}\left\{\frac{Q_2}{Q_1}+\frac{\left([p]_q(1+q)-q^p(1-q)\right)Q_1}{2q^p}\left(1-\mu\frac{(1+q)^2[p]_q}{[p]_q(1+q)-q^p(1-q)}\right)\right\} & (\mu \leqq \sigma_1) \\ \frac{[p]_qQ_1}{2q^p} & (\sigma_1 \leqq \mu \leqq \sigma_2), \\ -\frac{[p]_qQ_1}{2q^p}\left\{\frac{Q_2}{Q_1}+\frac{\left([p]_q(1+q)-q^p(1-q)\right)Q_1}{2q^p}\left(1-\mu\frac{(1+q)^2[p]_q}{[p]_q(1+q)-q^p(1-q)}\right)\right\} & (\mu \geqq \sigma_2). \end{cases}$$

Furthermore, if $\sigma_1 \leqq \mu \leqq \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{2q^p}{(1+q)^2 [p]_q Q_1} \left\{1 - \frac{Q_2}{Q_1} - \frac{\left([p]_q (1+q) - q^p (1-q)\right) Q_1}{2q^p} \cdot \left(1 - \mu \frac{(1+q)^2 [p]_q}{\left([p]_q (1+q) - q^p (1-q)\right)}\right)\right\} \left|a_{p+1}\right|^2 \leqq \frac{[p]_q Q_1}{2q^p}.$$

If $\sigma_3 \leqq \mu \leqq \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{2q^p}{(1+q)^2 [p]_q Q_1} \left\{1 + \frac{Q_2}{Q_1} + \frac{\left([p]_q (1+q) - q^p (1-q)\right) Q_1}{2q^p} \cdot \left(1 - \mu \frac{(1+q)^2 [p]_q}{\left([p]_q (1+q) - q^p (1-q)\right)}\right)\right\} \left|a_{p+1}\right|^2 \leqq \frac{[p]_q Q_1}{2q^p}.$$

Proof. Applying Lemma 4 to (3.12) and (3.13), respectively, we can derive the results asserted by Theorem 8. □

Putting $p = 1$ in Theorem 8, we obtain the following result.

Corollary 11. *Let*

$$\sigma_4 = \frac{\left(1+q^2\right) Q_1^2 + 2q\left(Q_2 - Q_1\right)}{(1+q)^2 Q_1^2},$$

$$\sigma_5 = \frac{\left(1+q^2\right) Q_1^2 + 2q\left(Q_2 + Q_1\right)}{(1+q)^2 Q_1^2}$$

and

$$\sigma_6 = \frac{\left(1+q^2\right) Q_1^2 + 2q Q_2}{(1+q)^2 Q_1^2}.$$

If the function f *given by* (1.1) (*with* $p = 1$) *belongs to the class* $k\text{-}\mathcal{ST}_q$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{Q_1}{2q}\left\{\frac{Q_2}{Q_1} + \frac{\left(1+q^2\right) Q_1}{2q}\left(1 - \mu \frac{(1+q)^2}{1+q^2}\right)\right\} & (\mu \leqq \sigma_4) \\ \frac{Q_1}{2q} & (\sigma_4 \leqq \mu \leqq \sigma_5) \\ -\frac{Q_1}{2q}\left\{\frac{Q_2}{Q_1} + \frac{\left(1+q^2\right) Q_1}{2q}\left(1 - \mu \frac{(1+q)^2}{1+q^2}\right)\right\} & (\mu \geqq \sigma_5). \end{cases}$$

Furthermore, if $\sigma_4 \leqq \mu \leqq \sigma_6$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2q}{(1+q)^2 Q_1}\left\{1 - \frac{Q_2}{Q_1} - \frac{\left(1+q^2\right)Q_1}{2q}\left(1 - \mu\frac{(1+q)^2}{1+q^2}\right)\right\}|a_2|^2 \leqq \frac{Q_1}{2q}.$$

If $\sigma_3 \leqq \mu \leqq \sigma_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2q}{(1+q)^2 Q_1}\left\{1 + \frac{Q_2}{Q_1} + \frac{\left(1+q^2\right)Q_1}{2q}\left(1 - \mu\frac{(1+q)^2}{1+q^2}\right)\right\}|a_2|^2 \leqq \frac{Q_1}{2q}.$$

Similarly, we can prove the following result.

Theorem 9. *Let*

$$\eta_1 = \frac{\left[\left((1+q)[p]_q - (1-q)q^p\right)Q_1^2 + 2q^p(Q_2 - Q_1)\right][p+1]_q^2}{[p]_q^2[p+2]_q(1+q)^2 Q_1^2},$$

$$\eta_2 = \frac{\left[\left((1+q)[p]_q - (1-q)q^p\right)Q_1^2 + 2q^p(Q_2 + Q_1)\right][p+1]_q^2}{[p]_q^2[p+2]_q(1+q)^2 Q_1^2}$$

and

$$\eta_3 = \frac{\left[\left((1+q)[p]_q - (1-q)q^p\right)Q_1^2 + 2q^p Q_2\right][p+1]_q^2}{[p]_q^2[p+2]_q(1+q)^2 Q_1^2}.$$

If the function f *given by* (1.1) *belongs to the class* $k\text{-}\mathcal{UCV}_{q,p}$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqq \begin{cases} \frac{[p]_q^2 Q_1}{2q^p[p+2]_q}\left\{\frac{Q_2}{Q_1} + \frac{\left((1+q)[p]_q-(1-q)q^p\right)Q_1}{2q^p}\left(1 - \frac{[p+2]_q(1+q)^2[p]_q^2\,\mu}{\left((1+q)[p]_q-(1-q)q^p\right)[p+1]_q^2}\right)\right\} & (\mu \leqq \eta_1) \\ \frac{[p]_q^2 Q_1}{2q^p[p+2]_q} & (\eta_1 \leqq \mu \leqq \eta_2) \\ -\frac{[p]_q^2 Q_1}{2q^p[p+2]_q}\left\{\frac{Q_2}{Q_1} + \frac{\left((1+q)[p]_q-(1-q)q^p\right)Q_1}{2q^p}\left(1 - \frac{[p+2]_q(1+q)^2[p]_q^2\,\mu}{\left((1+q)[p]_q-(1-q)q^p\right)[p+1]_q^2}\right)\right\} & (\mu \geqq \eta_2). \end{cases}$$

Furthermore, if $\eta_1 \leqq \mu \leqq \eta_3$, *then*

$$\begin{aligned}\left|a_{p+2} - \mu a_{p+1}^2\right| &+ \frac{2q^p[p+1]_q^2 Q_1}{[p+2]_q(1+q)^2[p]_q^2 Q_1^2}\left\{1 - \frac{Q_2}{Q_1} - \frac{\left((1+q)[p]_q - (1-q)q^p\right)Q_1}{2q^p}\right.\\ &\left.\cdot\left(1 - \mu\,\frac{[p+2]_q(1+q)^2[p]_q^2}{\left((1+q)[p]_q - (1-q)q^p\right)[p+1]_q^2}\right)\right\}\left|a_{p+1}\right|^2\end{aligned}$$

$$\leqq \frac{[p]_q^2 Q_1}{2q^p [p+2]_q}.$$

If $\eta_3 \leqq \mu \leqq \eta_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{2q^p [p+1]_q^2 Q_1}{[p+2]_q (1+q)^2 [p]_q^2 Q_1^2} \left\{1 + \frac{Q_2}{Q_1} + \frac{\left((1+q)[p]_q - (1-q) q^p\right) Q_1}{2q^p} \cdot \left(1 - \mu \frac{[p+2]_q (1+q)^2 [p]_q^2}{\left((1+q)[p]_q - (1-q) q^p\right) [p+1]_q^2}\right)\right\} \left|a_{p+1}\right|^2$$
$$\leqq \frac{[p]_q^2 Q_1}{2q^p [p+2]_q}.$$

Putting $p = 1$ in Theorem 9, we obtain the following result.

Corollary 12. *Let*

$$\eta_4 = \frac{\left(1+q^2\right) Q_1^2 + 2q (Q_2 - Q_1)}{[3]_q Q_1^2},$$

$$\eta_5 = \frac{\left(1+q^2\right) Q_1^2 + 2q (Q_2 + Q_1)}{[3]_q Q_1^2}$$

and

$$\eta_6 = \frac{\left(1+q^2\right) Q_1^2 + 2q Q_2}{[3]_q Q_1^2}.$$

If the function f *given by* (1.1) (*with* $p = 1$) *belongs to the class* k-$\mathcal{UCV}_q$, *then*

$$\left|a_3 - \mu a_2^2\right| \leqq \begin{cases} \frac{Q_1}{2q[3]_q} \left\{\frac{Q_2}{Q_1} + \frac{\left(1+q^2\right)Q_1}{2q} \left(1 - \mu \frac{[3]_q}{1+q^2}\right)\right\} & (\mu \leqq \eta_4) \\ \frac{Q_1}{2q[3]_q} & (\eta_4 \leqq \mu \leqq \eta_5) \\ -\frac{Q_1}{2q[3]_q} \left\{\frac{Q_2}{Q_1} + \frac{\left(1+q^2\right)Q_1}{2q} \left(1 - \mu \frac{[3]_q}{1+q^2}\right)\right\} & (\mu \geqq \eta_5). \end{cases}$$

Furthermore, if $\eta_4 \leqq \mu \leqq \eta_6$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2q}{[3]_q Q_1} \left\{1 - \frac{Q_2}{Q_1} - \frac{\left(1+q^2\right) Q_1}{2q} \left(1 - \mu \frac{[3]_q}{1+q^2}\right)\right\} |a_2|^2 \leqq \frac{Q_1}{2q [3]_q}.$$

If $\eta_3 \leqq \mu \leqq \eta_2$, *then*

$$\left|a_3 - \mu a_2^2\right| + \frac{2q}{[3]_q Q_1} \left\{1 + \frac{Q_2}{Q_1} + \frac{\left(1+q^2\right) Q_1}{2q} \left(1 - \mu \frac{[3]_q}{1+q^2}\right)\right\} |a_2|^2 \leqq \frac{Q_1}{2q [3]_q}.$$

4. Concluding remarks and observations

In our present investigation, we have applied the concept of the basic (or q-) calculus and a generalized conic domain, which was introduced and studied earlier by Srivastava *et al.* (see, for example, [43, 50]). By using this concept, we have defined two subclasses of normalized multivalent functions which map the open unit disk:

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\}$$

onto this generalized conic domain. We have derived a number of useful properties including (for example) the coefficient estimates and the Fekete-Szegö inequalities for each of these multivalent function classes. Our results are connected with those in several earlier works which are related to this field of Geometric Function Theory of Complex Analysis.

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas [see, for example, [48, pp. 350–351]. Moreover, as we remarked in the introductory Section 1 above, in the recently-published survey-cum-expository review article by Srivastava [42], the so-called (p, q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant or superfluous (see, for details, [42, p. 340]). This observation by Srivastava [42] will indeed apply to any attempt to produce the rather straightforward (p, q)-variations of the results which we have presented in this paper.

In conclusion, with a view mainly to encouraging and motivating further researches on applications of the basic (or q-) analysis and the basic (or q-) calculus in Geometric Function Theory of Complex Analysis along the lines of our present investigation, we choose to cite a number of recently-published works (see, for details, [25,47,51,53,56] on the Fekete-Szegö problem; see also [20–24,27,28,35–37, 40,44,46,49,52,55,57] dealing with various aspects of the usages of the q-derivative operator and some other operators in Geometric Function Theory of Complex Analysis). Indeed, as it is expected, each of these publications contains references to many earlier works which would offer further incentive and motivation for considering some of these worthwhile lines of future researches.

Conflicts of interest:

The authors declare no conflicts of interest.

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