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Research article

Solutions of a non-classical Stefan problem with nonlinear thermal coefficients and a Robin boundary condition

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Abstract: Solutions of similarity-type for a nonlinear non-classical Stefan problem with temperature-dependent thermal conductivity and a Robin boundary condition are obtained. The analysis of several particular cases are given when the thermal conductivity $L(f)$ and specific heat $N(f)$ are linear in temperature such that $L(f) = \alpha + \delta f$ with $N(f) = \beta + \gamma f$. Existence of a similarity type solution also obtained for the general problem by proving the lower and upper bounds of the solution.

Keywords: boundary value problem; Stefan problem; explicit solution; existence of solutions
Mathematics Subject Classification: 35R35, 80A22

1. Introduction

In [1], the authors introduced the following Stefan problem which is governed by a non-classical and nonlinear heat equation with heat source F, thermal coefficients which depend on the temperature and a convective boundary condition at fixed face $x = 0$ (see also [2–4]):

$$\begin{cases} \rho(T)c(T)\frac{\partial T}{\partial t} &= \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) - F(Z(t),t),\ 0 < x < s(t),\ t > 0, \\ k\left(T(0,t)\right)\frac{\partial T(0,t)}{\partial x} &= \frac{h}{\sqrt{t}}\left(T(0,t) - T^*\right),\ h > 0, \\ T(s(t),t) &= T_m, \\ k\left(T(s(t),t)\right)\frac{\partial T(s(t),t)}{\partial x} &= -\rho_0 l s'(t), \\ s(0) &= 0, \end{cases} \tag{1.1}$$

where $\rho(T)$, $c(T)$ and $k(T)$ are the density of the material, its specific heat, and its thermal conductivity, respectively; T_m is the phase-change temperature, $\rho_0 > 0$ its the constant density of mass at the melting temperature; $l > 0$ is the latent heat of fusion by unity of mass and $s(t)$ is the position of phase change location. Also, F depends on the evolution of the heat flux at the boundary $x = 0$ and assumed that

$$F(Z(t), t) = F(Z(t), t) = F\left(\frac{\partial T(0,t)}{\partial x}, t\right) = \frac{\lambda}{\sqrt{t}}\frac{\partial T(0,t)}{\partial x},\ \lambda > 0. \tag{1.2}$$

From the second equation in Pr.(1.1), we see that $T_x(0, t) < 0$. This can be physically justified since F in this particular study represents a heat sink due to "melting ice/water phase change".

To obtain a similarity solution, we introduce the new independent variable

$$\xi = \frac{x}{2\sqrt{\alpha_0 t}} \tag{1.3}$$

and the dimension-free dependent variable, defined by

$$f(\xi) = \frac{T(x,t) - T^*}{T_m - T^*}. \tag{1.4}$$

In terms of these new variables, the problem takes the following nonlinear boundary value problem [1]

$$\begin{cases} [L(f(\xi))f'(\xi)]' + 2\xi N(f(\xi))f'(\xi) &= Af'(0),\ 0 < \xi < \xi_0, \\ L(f(0))f'(0) &= pf(0), \\ f(\xi_0) &= 1, \\ f'(\xi_0) &= M\xi_0, \end{cases} \tag{1.5}$$

where the nonlinear terms $L(f(\xi))$ and $N(f(\xi))$ are given by

$$L(f(\xi)) = \frac{k\left((T_m - T^*)f(\xi) + T^*\right)}{k_0},\ N(f(\xi)) = \frac{\rho c\left((T_m - T^*)f(\xi) + T^*\right)}{\rho_0 c_0} \tag{1.6}$$

and

$$A = \frac{2\lambda}{\rho_0 c_0 k_0},\ p = \frac{2\sqrt{\alpha_0 h_0}}{k_0} > 0,\ M = \frac{2k_0}{kT_m Ste} \tag{1.7}$$

with $Ste = \frac{c(T^* - T_m)}{l} > 0$ (Stefan number) and the system parameters k_0, ρ_0, c_0 and $\alpha_0 = \frac{k_0}{\rho_0 c_0}$ are the reference thermal conductivity, density of mass, specific heat and thermal diffusivity, respectively.

Recently, the authors [1] proved the existence and uniqueness of the similarity solution of Pr.(1.1) under the restrictive Lipschitz conditions on the parameters and using techniques of functional analysis.

In this paper, explicit solutions of Pr.(1.1) are obtained when the thermal conductivity and specific heat are linear in temperature. Also, existence solutions of similarity-type are proved for this non-classical Stefan problem with nonlinear thermal conductivity and specific heat by using a technique based on lower and upper bounds of the solution. Our approach is simpler than the approach used in [1] and is more accessible to readers since it doesn't assume Lipschitz conditions on the parameters and doesn't involve tools of functional analysis.

2. Linear conductivity

We consider the case where thermal conductivity and specific heat are linear in temperature [2–4]. This case has been largely discussed in the literature since it provides a good approximation of the actual values for some material such as water (see for example [5–9]). We also assume that T is continuous differentiable, which is a natural assumption.

2.1. Case 1

Let

$$L(f(\xi)) = \frac{kT^*}{k_0} + \frac{k(T_m - T^*)}{k_0} f(\xi),\ N(f(\xi)) = \frac{\rho c T^*}{\rho_0 c_0} + \frac{\rho c(T_m - T^*)}{\rho_0 c_0} f(\xi). \tag{2.1}$$

Writing the nonlinear second-order ODE of Pr.(1.5) in a simple form

$$[(\alpha + \delta f(\xi))f'(\xi)]' + 2\xi(\beta + \gamma f(\xi))f'(\xi) = Af'(0),\ 0 < \xi < \xi_0, \tag{2.2}$$

where $\alpha = \frac{kT^*}{k_0}, \beta = \frac{\rho c T^*}{\rho_0 c_0}, \delta = \frac{k(T_m - T^*)}{k_0}$ and $\gamma = \frac{\rho c(T_m - T^*)}{\rho_0 c_0}$.
Multiplying both sides of Eq (2.2) by $\delta\gamma$, we obtain

$$\delta\left[(\alpha\gamma + \delta\gamma f(\xi))f'(\xi)\right]' + 2\gamma\xi(\beta\delta + \delta\gamma f(\xi))f'(\xi) = Af'(0)\delta\gamma,\ 0 < \xi < \xi_0, \tag{2.3}$$

or

$$[(\alpha\gamma + \delta\gamma f(\xi))f'(\xi)]' + 2\frac{\gamma}{\delta}\xi(\beta\delta + \delta\gamma f(\xi))f'(\xi) = Af'(0)\gamma,\ 0 < \xi < \xi_0. \tag{2.4}$$

Since $\alpha\gamma = \beta\delta = \frac{kT^*}{k_0}\frac{\rho c(T_m - T^*)}{\rho_0 c_0}$. Then Eq (2.4) can be written as

$$[(a + bf(\xi))f'(\xi)]' + 2\frac{\gamma}{\delta}\xi(a + bf(\xi))f'(\xi) = Af'(0)\gamma,\ 0 < \xi < \xi_0, \tag{2.5}$$

where $a = \alpha\gamma$ and $b = \delta\gamma$.
By the change of variable

$$z(\xi) = (a + bf(\xi))f'(\xi). \tag{2.6}$$

Eq (2.5) becomes

$$z'(\xi) + 2\frac{\gamma}{\delta}\xi z(\xi) = Af'(0)\gamma. \tag{2.7}$$

The solution of this linear first-order equation is given by

$$z(\xi) = Af'(0)\gamma\varphi(\xi)\exp\left(-\frac{\gamma}{\delta}\xi^2\right) + z(0)\exp\left(-\frac{\gamma}{\delta}\xi^2\right), \tag{2.8}$$

where $\varphi(\xi)$ is *the imaginary error function*, that is $\varphi(\xi) = \int_0^\xi \exp\left(\frac{\gamma}{\delta}t^2\right)dt$ and $z(0)$ can be determined by using $z(0) = (a + bf(0))f'(0) = \gamma(\alpha + \delta f(0))f'(0)$ and the first initial condition of Pr.(1.5) from which we can readily obtain

$$z(0) = \gamma p f(0). \tag{2.9}$$

Hence

$$z(\xi) = Af'(0)\gamma\varphi(\xi)\exp\left(-\frac{\gamma}{\delta}\xi^2\right) + \gamma p f(0)\exp\left(-\frac{\gamma}{\delta}\xi^2\right). \tag{2.10}$$

From Eq (2.6), we have

$$(a + bf(\xi))^2 = 2b\int_0^{\xi} z(t)dt + (a + bf(0))^2. \tag{2.11}$$

Substituting $z(\xi)$ into this equation, we obtain

$$a + bf(\xi) = \sqrt{2bA\gamma f'(0)\int_0^{\xi}\varphi(t)\exp\left(-\frac{\gamma}{\delta}t^2\right)dt + 2b\gamma pf(0)\sqrt{\frac{\delta}{\gamma}}\,\mathrm{erf}\left(\sqrt{\frac{\gamma}{\delta}}\xi\right) + (a + bf(0))^2}. \tag{2.12}$$

Employing now the two boundary conditions of Pr.(1.5): $f(\xi_0) = 1$ and $f'(\xi_0) = M\xi_0$ to obtain

$$z(\xi_0) = (a + bf(\xi_0))f'(\xi_0). \tag{2.13}$$

Thus, the condition on ξ_0 is given by

$$(a + b)M\xi_0 = [A\gamma f'(0)\varphi(\xi_0) + \gamma pf(0)]\exp\left(-\frac{\gamma}{\delta}\xi_0^2\right). \tag{2.14}$$

Thus

Theorem 2.1. *The exact solution of Pr. (1.5) is given by*

$$f(\xi;\xi_0) = \frac{1}{b}\sqrt{2bA\gamma f'(0)\int_0^{\xi}\varphi(t)\exp\left(-\frac{\gamma}{\delta}t^2\right)dt + 2b\gamma pf(0)\sqrt{\frac{\delta}{\gamma}}\,\mathrm{erf}\left(\sqrt{\frac{\gamma}{\delta}}\xi\right) + (a + bf(0))^2} - \frac{a}{b} \tag{2.15}$$

subject to the condition (2.14).

Remark 2.2. *From* $f(\xi) = \frac{T(x,t)-T^*}{T_m-T^*}$, *it can be readily seen that* $0 < f(0) < 1$. *Furthermore, for arbitrary* $\epsilon > 0$, *it can also be shown using classical chain rule on this relation with* $\xi = \frac{x}{2\sqrt{\alpha_0 t}}$ *that*

$$f'(\epsilon) = \frac{2\sqrt{\alpha_0 t}}{T_m - T^*}T_x(2\sqrt{\alpha_0 t}\epsilon, t) > 0. \tag{2.16}$$

This gives $0 \le f'(0) = \frac{2\sqrt{\alpha_0 t}}{T_m-T^*}T_x(0,t)$. *Using the second equation in Pr.(1.1) we obtain* $0 \le f'(0) < \frac{2\sqrt{\alpha_0}h}{k(T(0,t))}$. *Therefore, from (2.14) we conclude that* $\xi_0 > 0$.

Consequently,

Theorem 2.3. *The exact solution of the original Pr.(1.1) is given by*

$$T(x,t) = (T_m - T^*)f(\xi;\xi_0) + T^*,\ \xi = \frac{x}{2\sqrt{\alpha_0 t}} \tag{2.17}$$

subject to the condition (2.14) and the free boundary is given by $s(t) = 2\xi_0\sqrt{\alpha_0 t}$.

2.2. *Case 2*

Another important case is $L(f(\xi)) = 1 + \delta f(\xi)$ and $N(f(\xi)) = 1$ [1]. For this case Pr.(1.5) becomes

$$\begin{cases} [(1+\delta f(\xi))f'(\xi)]' + 2\xi f'(\xi) &= Af'(0),\ 0 < \xi < \xi_0, \\ (1+\delta f(0))f'(0) &= pf(0), \\ f(\xi_0) &= 1, \\ f'(\xi_0) &= \frac{2\xi_0}{(1+\delta)Ste}. \end{cases} \tag{2.18}$$

A result on the existence and uniqueness of solution to the nonlinear boundary value problem Pr.(2.18) when $A = 0$ was proved in [8], where the solution was treated as a *Generalized Modified Error*.

Writing the nonlinear ODE of Pr.(2.18) as

$$((1+\delta f(\xi))(1+\delta f(\xi))')' + 2\xi(1+\delta f(\xi))' = Af'(0)\delta,\ 0 < \xi < \xi_0. \tag{2.19}$$

The transformation $u(\xi) = (1+\delta f(\xi))^2$ leads to

$$u'' + 2\xi\frac{u'}{\sqrt{u}} = 2Af'(0)\delta,\ 0 < \xi < \xi_0 \tag{2.20}$$

subject to the initial-boundary conditions

$$\begin{cases} u'(0) &= 2p(\sqrt{u(0)} - 1), \\ u(\xi_0) &= (1+\delta)^2, \\ u'(\xi_0) &= \frac{4\xi_0\delta}{Ste}. \end{cases} \tag{2.21}$$

2.2.1. Solutions

In the special case $A = 0$, Eq (2.20) is transformed to a homogeneous equation

$$u'' + 2\xi\frac{u'}{\sqrt{u}} = 0,\ 0 < \xi < \xi_0. \tag{2.22}$$

Consider the following transformation [9]

$$v = \xi\frac{u'}{u},\ w = -2\frac{\xi^2}{\sqrt{u}}. \tag{2.23}$$

Hence

$$\frac{dw}{d\xi} = \frac{dw}{dv}\frac{dv}{d\xi} = \frac{dw}{dv}\left(\frac{u'}{u} + x\frac{u''}{u} - x(\frac{u'}{u})^2\right) \text{ and } \frac{dw}{d\xi} = -4\frac{\xi}{\sqrt{u}} + \xi^2\frac{u'}{u}\frac{1}{\sqrt{u}}. \tag{2.24}$$

The substitution of these into (2.22) gives

$$(w - v + 1)\frac{dw}{dv} = \left(\frac{2}{v} - \frac{1}{2}\right)w. \tag{2.25}$$

The substitution $\chi = w - v + 1$ brings also Eq (2.25) to the *Abel equation of the second kind*

$$\chi\chi_v = \Phi(v)\chi + \Psi(v), \tag{2.26}$$

where

$$\Phi(v) = \left(\frac{2}{v} - \frac{3}{2}\right) \text{ and } \Psi(v) = \left(\frac{3}{2} - \frac{v}{2} - \frac{2}{v}\right), \tag{2.27}$$

which is in general very difficult to handle. Since the explicit solution cannot be found, then we estimate the solution by finding upper and lower bounds of u.

2.2.2. Upper and lower bounds of the solution

If we assume that $u'(\xi) \geq 0$ then

$$u(0) \leq u(\xi) \leq u(\xi_0),\ \xi \in [0, \xi_0]. \tag{2.28}$$

So that

$$\frac{2\xi u'(\xi)}{\sqrt{u(\xi_0)}} \leq \frac{2\xi u'(\xi)}{\sqrt{u(\xi)}} \leq \frac{2\xi u'(\xi)}{\sqrt{u(0)}}. \tag{2.29}$$

From (2.22), we have

$$\frac{2\xi u'(\xi)}{\sqrt{u(\xi_0)}} \leq -u''(\xi) \leq \frac{2\xi u'(\xi)}{\sqrt{u(0)}}. \tag{2.30}$$

Thus

$$\frac{2\xi}{\sqrt{u(0)}} \leq \frac{u''(\xi)}{u'(\xi)} \leq -\frac{2\xi}{\sqrt{u(\xi_0)}}. \tag{2.31}$$

Integrating both sides of (2.31) from ξ to ξ_0, we obtain

$$u'(\xi_0) e^{\frac{\xi_0^2}{\sqrt{u(\xi_0)}}} e^{-\frac{\xi^2}{\sqrt{u(\xi_0)}}} \leq u'(\xi) \leq u'(\xi_0) e^{\frac{\xi_0^2}{\sqrt{u(0)}}} e^{-\frac{\xi^2}{\sqrt{u(0)}}}. \tag{2.32}$$

Integrating again (2.32) from ξ to ξ_0, we obtain

$$u(\xi_0) - u'(\xi_0) e^{\frac{\xi_0^2}{\sqrt{u(0)}}} \int_{\xi}^{\xi_0} e^{-\frac{\eta^2}{\sqrt{u(0)}}} d\eta \leq u(\xi) \leq u(\xi_0) - u'(\xi_0) e^{\frac{\xi_0^2}{\sqrt{u(\xi_0)}}} \int_{\xi}^{\xi_0} e^{-\frac{\eta^2}{\sqrt{u(\xi_0)}}} d\eta, \tag{2.33}$$

where $u(\xi_0) = (1+\delta)^2$ and $u'(\xi_0) = \frac{4\xi_0 \delta}{Ste}$.

The constants $u(0)$ and $u'(0)$ can be estimated from the mixed condition $u'(0) = 2p(\sqrt{u(0)} - 1)$ and (2.32) (or (2.33)) to find

$$\frac{4\xi_0 \delta}{Ste} e^{\frac{\xi_0^2}{1+\delta}} \leq u'(0) \leq \frac{4\xi_0 \delta}{Ste} e^{\frac{\xi_0^2}{\sqrt{u(0)}}}. \tag{2.34}$$

The case $u'(\xi) \leq 0$ follows in a similar fashion.

Based on this, and in view of the condition $u'(0) = 2p(\sqrt{u(0)} - 1)$, we have

Lemma 2.4. *1). If $A = 0$ and $u'(\xi) \geq 0,\ \xi \in [0, \xi_0]$, then there is at least one solution $u(\xi)$ of Pr.(2.20) and (2.21) such that*

$$(1+\delta)^2 - \frac{4\xi_0 \delta}{Ste} e^{\frac{\xi_0^2}{\sqrt{u(0)}}} \int_{\xi}^{\xi_0} e^{-\frac{\eta^2}{\sqrt{u(0)}}} d\eta \leq u(\xi) \leq (1+\delta)^2 - \frac{4\xi_0 \delta}{Ste} e^{\frac{\xi_0^2}{1+\delta}} \int_{\xi}^{\xi_0} e^{-\frac{\eta^2}{1+\delta}} d\eta \tag{2.35}$$

subject to the following conditions

$$u'(0) \geq \frac{4\xi_0 \delta}{Ste} e^{\frac{\xi_0^2}{1+\delta}} \text{ and } u'(0) e^{-\frac{\xi_0^2}{\frac{u'(0)}{2p}+1}} \leq \frac{4\xi_0 \delta}{Ste}. \tag{2.36}$$

2). If $A = 0$ and $u'(\xi) \leq 0,\ \xi \in [0, \xi_0]$, then there is at least one solution $u(\xi)$ of Pr.(2.20) and (2.21) such that

$$(1+\delta)^2 - \frac{4\xi_0\delta}{Ste} e^{\frac{\xi_0^2}{1+\delta}} \int_{\xi}^{\xi_0} e^{-\frac{\eta^2}{1+\delta}} d\eta \leq u(\xi) \leq (1+\delta)^2 - \frac{4\xi_0\delta}{Ste} e^{\frac{\xi_0^2}{\sqrt{u(0)}}} \int_{\xi}^{\xi_0} e^{-\frac{\eta^2}{\sqrt{u(0)}}} d\eta \tag{2.37}$$

subject to the following conditions

$$u'(0) \leq \frac{4\xi_0\delta}{Ste} e^{\frac{\xi_0^2}{1+\delta}} \text{ and } u'(0) e^{-\frac{\xi_0^2}{\frac{u'(0)}{2p}+1}} \geq \frac{4\xi_0\delta}{Ste}. \tag{2.38}$$

3. Nonlinear conductivity

Let the thermal conductivity $k(\theta)$ given by $k(\theta) = \frac{\rho_0 c_0}{(a+b\theta)^2}$ (see [10]) for a list of references that considered this case.
Thus

$$L(f(\xi)) = \frac{k(f(\xi))}{k_0} = \frac{\rho_0 c_0}{k_0(a+bf(\xi))^2},\ N(f(\xi)) = k^*, \tag{3.1}$$

where a, b, k^* are positive constants. Hence the nonlinear ODE of Pr.(1.5) can be written as

$$\left(\frac{\rho_0 c_0}{k_0(a+bf(\xi))^2} f'(\xi)\right)' + 2k^*\xi f'(\xi) = Af'(0),\ 0 < \xi < \xi_0 \tag{3.2}$$

or in the equivalent form

$$k_1 \left(\frac{(a+bf(\xi))'}{(a+bf(\xi))^2}\right)' + 2k_2\xi(a+bf(\xi))' = Af'(0),\ 0 < \xi < \xi_0, \tag{3.3}$$

where $k_1 = \frac{\rho_0 c_0}{k_0 b}$ and $k_2 = \frac{k^*}{b}$.
The change of variable $z(\xi) = a + bf(\xi)$ leads to an equation of the form

$$k_1 \left(\frac{z'(\xi)}{z^2(\xi)}\right)' + 2k_2\xi z'(\xi) = Af'(0),\ 0 < \xi < \xi_0. \tag{3.4}$$

In turn, Eq (3.4) can be reduced, by the introduction of the new independent variable $u(\xi) = \frac{1}{z(\xi)}$, where $u'(\xi) = -\frac{z'(\xi)}{z^2(\xi)}$ and $z'(\xi) = -\frac{u'(\xi)}{u^2(\xi)}$ to the following form

$$k_1 u''(\xi) + 2k_2\xi \frac{u'(\xi)}{u^2(\xi)} = -Af'(0),\ 0 < \xi < \xi_0. \tag{3.5}$$

The case $A = 0$, brings Eq (3.5) to the simpler form

$$u'' + 2\frac{k_2}{k_1}\xi\frac{u'}{u^2} = 0,\ 0 < \xi < \xi_0. \tag{3.6}$$

3.1. Solutions

Consider now the following transformation

$$v = \xi \frac{u'}{u},\ w = -2\frac{\xi^2}{u^2}. \tag{3.7}$$

A simple computation leads to

$$\frac{dw}{d\xi} = \frac{dw}{dv}\frac{dv}{d\xi} = \frac{dw}{dv}\left(\frac{u'}{u} + x\frac{u''}{u} - x(\frac{u'}{u})^2\right) \text{ and } \frac{dw}{d\xi} = -4\frac{\xi}{u^2} + 4\xi^2\frac{u'}{u^3}. \tag{3.8}$$

The substitution of these into (3.6) gives

$$(k_1(v-1) - k_2 w)\frac{dw}{dv} + 2k_1 w\left(\frac{1}{v} - 1\right) = 0. \tag{3.9}$$

The following transformation $\chi = w - v + 1$ reduces Eq (3.9) into the known *Abel equation of the second kind* (2.26), where $\Phi(v) = k_1 - 2(\frac{1}{v} - 1)$ and $\Psi(v) = -2k_1^2\frac{(v-1)^2}{v}$, which is in general very difficult to handle. So that, we will find the lower and upper bounds of the solution $u(x)$.

3.2. Upper and lower bounds of the solution

Proceeding as before, we obtain

Lemma 3.1. *1). If $u'(\xi) \geq 0,\ \xi \in [0, \xi_0]$ then there is at least one solution $u(\xi)$) of Eq (3.6) subject to (2.21) such that*

$$(1+\delta)^2 - \frac{4\xi_0\delta}{Ste}e^{\frac{k_2\xi_0^2}{k_1 u^2(0)}}\int_\xi^{\xi_0} e^{-\frac{k_2\eta^2}{k_1 u^2(0)}}d\eta \leq u(\xi) \leq (1+\delta)^2 - \frac{4\xi_0\delta}{Ste}e^{\frac{k_2\xi_0^2}{k_1(1+\delta)^4}}\int_\xi^{\xi_0} e^{-\frac{k_2\eta^2}{k_1(1+\delta)^4}}d\eta \tag{3.10}$$

subject to the following conditions

$$u'(0) \geq \frac{4\xi_0\delta}{Ste}e^{\frac{k_2\xi_0^2}{k_1(1+\delta)^4}} \text{ and } u'(0)e^{-\frac{k_2\xi_0^2}{k_1(\frac{u'(0)}{2p}+1)^4}} \leq \frac{4\xi_0\delta}{Ste}. \tag{3.11}$$

2). If $u'(\xi) \leq 0,\ \xi \in [0, \xi_0]$ then there is at least one solution $u(\xi)$) of Eq (3.6) subject to (2.21) such that

$$(1+\delta)^2 - \frac{4\xi_0\delta}{Ste}e^{\frac{k_2\xi_0^2}{k_1(1+\delta)^4}}\int_\xi^{\xi_0} e^{-\frac{k_2\eta^2}{k_1(1+\delta)^4}}d\eta \leq u(\xi) \leq (1+\delta)^2 - \frac{4\xi_0\delta}{Ste}e^{\frac{k_2\xi_0^2}{k_1 u^2(0)}}\int_\xi^{\xi_0} e^{-\frac{k_2\eta^2}{k_1 u^2(0)}}d\eta \tag{3.12}$$

subject to the following conditions

$$u'(0) \leq \frac{4\xi_0\delta}{Ste}e^{\frac{k_2\xi_0^2}{k_1(1+\delta)^4}} \text{ and } u'(0)e^{-\frac{k_2\xi_0^2}{k_1(\frac{u'(0)}{2p}+1)^4}} \geq \frac{4\xi_0\delta}{Ste}. \tag{3.13}$$

4. The general case: $L(f(\xi))$ and $N(f(\xi))$ are nonlinear

Rewrite the general nonlinear equation

$$[L(f(\xi))f'(\xi)]' + 2\xi N(f(\xi))f'(\xi) = Af'(0),\ 0 < \xi < \xi_0 \tag{4.1}$$

in the form

$$[L(f(\xi))f'(\xi)]' + 2\xi\frac{N(f(\xi))}{L(f(\xi))}\left[L(f(\xi))f'(\xi)\right] = Af'(0),\ 0 < \xi < \xi_0. \tag{4.2}$$

Let $F(\xi) = L(f(\xi))f'(\xi)$. Thus

$$F'(\xi) + 2\xi\frac{N(f(\xi))}{L(f(\xi))}F(\xi) = Af'(0),\ 0 < \xi < \xi_0. \tag{4.3}$$

Multiplying both sides of Eq (4.3) by $\exp\left(2\int_0^\xi \frac{\eta N(f(\eta))}{L(f(\eta))}d\eta\right)$, we obtain

$$\left[F(\xi)\exp\left(2\int_0^\xi \frac{N(f(\eta))}{L(f(\eta))}d\eta\right)\right]' = Af'(0)\exp\left(2\int_0^\xi \frac{\eta N(f(\eta))}{L(f(\eta))}d\eta\right). \tag{4.4}$$

Thus

$$F(\xi)\exp\left(2\int_0^\xi \frac{\eta N(f(\eta))}{L(f(\eta))}d\eta\right) = Af'(0)\int_0^\xi \exp\left(2\int_0^\mu \frac{\eta N(f(\eta))}{L(f(\eta))}d\eta\right)d\mu + C, \tag{4.5}$$

where C is a constant of integration and can be found from the condition $F(0) = L(f(0))f'(0) = pf(0)$, that is $C = pf(0)$.

Hence $f(\xi)$ can be immediately expressed by

$$\begin{aligned} f(\xi) = f(0) + pf(0)\int_0^\xi \frac{\exp\left(-2\int_0^\mu \frac{\eta N(f(\eta))}{L(f(\eta))}d\eta\right)}{L(f(\mu))}d\mu \\ +Af'(0)\int_0^\xi \frac{\exp\left(-2\int_0^\mu \frac{\eta N(f(\eta))}{L(f(\eta))}d\eta\right)}{L(f(\mu))}\int_0^\mu \exp\left(2\int_0^\theta \frac{\theta N(f(\theta))}{L(f(\theta))}d\theta\right)d\mu. \end{aligned} \tag{4.6}$$

If we assume that

$$0 < N_m \le N(f(\eta)) \le N_M \text{ and } 0 < L_m \le L(f(\eta)) \le L_M. \tag{4.7}$$

Then a simple computation leads to

$$\frac{1}{L_M}\int_0^\xi \exp\left(-\frac{N_M}{L_m}\eta^2\right)d\eta \le \int_0^\xi \frac{\exp\left(-2\int_0^\mu \frac{\eta N(f(\eta))}{L(f(\eta))}d\eta\right)}{L(f(\mu))}d\mu \le \frac{1}{L_m}\int_0^\xi \exp\left(-\frac{N_m}{L_M}\eta^2\right)d\eta \tag{4.8}$$

and

$$\int_0^\mu \exp\left(\frac{N_m}{L_M}\theta^2\right)d\theta \le \int_0^\xi \exp\left(2\int_0^\mu \frac{\theta N(f(\theta))}{L(f(\theta))}d\theta\right)d\theta \le \int_0^\mu \exp\left(\frac{N_M}{L_m}\theta^2\right)d\theta, \tag{4.9}$$

where $\int_0^\xi \exp\left(-\frac{N_m}{L_M}\eta^2\right)d\eta = \frac{\sqrt{\pi L_M}}{2\sqrt{N_m}}\operatorname{erf}\left(\frac{\sqrt{N_m}}{L_m\sqrt{L_M}}\xi\right)$. Based on this and in view of the boundary conditions $f(\xi_0) = 1$ and $f'(\xi_0) = M\xi_0$, we have

Theorem 4.1. *If the conditions (4.7) are satisfied, then there exists at least one solution $f(\xi)$ of Pr.(1.5) such that*

$$f_1(\xi) \leq f(\xi) \leq f_2(\xi),\ 0 < \xi < \xi_0, \tag{4.10}$$

where

$$f_2(\xi) = f(0) + p\frac{f(0)}{L_m}\frac{\sqrt{\pi L_M}}{2\sqrt{N_m}}\operatorname{erf}\left(\frac{\sqrt{N_m}}{\sqrt{L_M}}\xi\right) + \frac{Af'(0)}{L_m}\int_0^{\xi}\exp\left(-\frac{N_m}{L_M}\eta^2\right)\int_0^{\eta}\exp\left(\frac{N_M}{L_m}\theta^2\right)d\theta d\eta \tag{4.11}$$

and

$$f_1(\xi) = f(0) + p\frac{f(0)}{L_M}\frac{\sqrt{\pi L_m}}{2\sqrt{N_M}}\operatorname{erf}\left(\frac{\sqrt{N_M}}{\sqrt{L_m}}\xi\right) + \frac{Af'(0)}{L_M}\int_0^{\xi}\exp\left(-\frac{N_M}{L_m}\eta^2\right)\int_0^{\eta}\exp\left(\frac{N_m}{L_M}\theta^2\right)d\theta d\eta \tag{4.12}$$

subject to the following conditions

$$f_1(\xi_0) \leq f(\xi_0) = 1 \leq f_2(\xi_0) \tag{4.13}$$

and

$$f_1'(\xi_0) \leq f'(\xi_0) = M\xi_0 \leq f_2'(\xi_0). \tag{4.14}$$

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Conflict of interest

The authors declare no conflict of interest.

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