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Research article

Partial synchronization in community networks based on the intra-community connections

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Abstract: In this paper, we propose a novel criterion on the partial synchronization in a generalized linearly coupled network by employing Lyapunov stability theory and linear matrix inequality. The obtained criterion is only dependent on intra-community connections, and the information of inter-community connections is not necessary. Therefore, it provides more convenience in reducing network sizes in practice. Compared with the previous classical criterion, the threshold derived from the obtained criterion is no less than the classical threshold. We give some particular cases in which the obtained threshold is equal to the classical threshold. Finally, we show numerical simulations to verify the validity of the proposed criteria and comparisons.

Keywords: community network; partial synchronization; invariant manifold; intra-community connection
Mathematics Subject Classification: 05C82, 34D06

1. Introduction

Complex networks have been observed in a wide range of application domains, such as neural networks [1], social interacting species [2], multi-agent systems [3], and so forth. Since the nodes in a complex network are interconnected, communicating and interacting with each other, it is not surprisingly that the collective behavior (e.g., synchronization and consensus) widely exists in complex networks [4–6]. During the past decades, the studies of synchronization have been extensively explored, and many types of synchronization phenomena have been proposed such as complete synchronization [7], partial synchronization [8–10], lag synchronization [11, 12], exponential synchronization [13, 14]. Recently the studies of consensus dynamics with additive stochastic disturbances have attracted increasing attention where the consensus (named as network coherence) was characterized by the spectra of Laplacian matrix [15, 16]. It has been already shown that the coupling needed to realize complete synchronization is inversely proportional to the nonzero eigenvalue of the coupling graph [17]. And synchronization patterns can be easily detected based on the eigenvalues of the original networks in networks of chaotic systems with time-delayed couplings [18].

In this paper, we consider partial synchronization of linearly coupled complex networks, which is also called cluster synchronization [9, 10]. Roughly speaking, partial synchronization is the phenomenon in which the nodes split into several communities, where the nodes synchronize with each other in the same community, however synchronization doesn't occur among different communities. Partial synchronization widely exists in biological systems, cyber physical systems, social Systems, and so on. On the one hand, it is obvious that there is a close interplay between partial synchronization and network topologies. Therefore, various control schemes depending heavily on community structures of the network topologies were proposed to realize partial synchronization. However, partial synchronization is also observed in real networks without cluster structures, and many researches have been carried out to study this topic [19]. Recently, an effective adaptive aperiodically intermittent pinning control scheme was developed to realize partial synchronization for colored community networks [20].

Recent studies have shown that partial synchronization can be realized via two schemes. The first scheme is partial synchronization induced by the intrinsic structure and mutual couplings of the network. Ma et al. observed that the nodes in the same community only have cooperative connections [21]. Later, Wu and Chen showed geometrically that partial synchronization amounts to the global attractiveness of the corresponding invariant synchronization manifold, and they obtained several meaningful criteria through a series of topological analysis on the invariant synchronization manifold [10]. Stuer et al. have shown that certain symmetries of network topology could identify partial synchronization manifolds, and sufficient conditions for its asymptotically stability were also given in networks consisting of diffusive time-delay coupled oscillatory units [22]. By developing a modified model with inter-cluster co-competition balance, Zhang et al. have obtained a criterion for partial synchronization, and proved that the corresponding cluster synchronous pattern formation is robust [8]. Note that partial synchronization can be realized in case that the coupling matrix is constructed reasonably and effectively.

For the second scheme, a recent research realized partial synchronization in a linearly coupled network via a generalized pinning control strategy [23]. Later, it was proposed and rigorously proved that partial synchronization can also be realized by adding some external controllers on just partial clusters [24]. By using an adaptive pinning-control scheme including adaptive strategy on both coupling strengths and feedback gains, it was shown that a network can realize partial synchronization under weak coupling strengths and small feedback gains [25]. Recently, the partial synchronization induced external control has attracted increasing attention [26–28].

Motivated by the above discussions, this paper further focuses on partial synchronization induced by the intrinsic structure and mutual couplings of the network. To the best of our knowledge, the global information of the network topology is a necessary condition in past studies on partial synchronization. In the coming era of big data, complex networks in reality are composed of massive nodes and edges. The global data of the network topology is usually very large, which brings about great difficulties in data analysis. In order to simplify the criterion on partial synchronization of complex networks, we decompose the whole network into several communities, and establish a brief criterion by neglecting the inter-community couplings. By employing Lyapunov stability theory and linear matrix inequalities, we prove that appropriate intra-community couplings are also sufficient to realize partial synchronization. The novel criterion doesn't depend on the inter-community couplings, which greatly reduces the amount of calculation for the data analysis. Furthermore, we also make a comparison with one of previous classical criteria through rigorous theoretical analysis. As the result of neglecting the inter-community couplings, the obtained threshold is larger than or equal to that obtained by the classical criterion. However, the obtained criterion drastically reduces the matrix of network topology. It should be flexible, convenient and efficient in practice, especially for large-scale networks with a mass of nodes.

The outline of this paper is as follows. In Section 2, we provide some preliminary definitions, assumptions, and existing theoretical results on partial synchronization. In Section 3, we give a series of stability analysis on the partial synchronization manifold, and make some rigorous theoretical comparisons with the previous classical criteria. In Section 4, some numerical examples are presented to verify our theoretical results. Finally, Section 5 concludes the paper.

2. Preliminaries

In this section, we introduce some basic concepts and some theoretical results on partial synchronization.

Suppose a general network consisting of N dynamical nodes labeled as $1, \ldots, N$, which are divided into n communities of various sizes $G_1 = \{1, \ldots, N_1\}, G_2 = \{N_1+1, \ldots, N_2\}, \cdots, G_n = \{N_{n-1}+1, \ldots, N\}$. For convenience, we denote $N_0 = 0$, $N_n = N$, $n_p = N_p - N_{p-1}$, $\mathfrak{n} = \{1, \ldots, n\}$, and $\mathbb{N} = \{1, \ldots, N\}$. Then, the community $G_p = \{N_{p-1} + 1, \ldots, N_p\}$ contains n_p nodes for any $p \in \mathfrak{n}$.

Consider the following dynamical network, which is composed of ordinary differential equations coupled linearly and symmetrically,

$$\dot{x}_i(t) = f(x_i(t), t) + \varepsilon \sum_{p=1}^{n} \sum_{j \in G_p} c_{ij} \Gamma x_j(t), \tag{2.1}$$

where $x_i(t) = (x_i^1(t), \ldots, x_i^m(t))^\top$ is the state vector of the node i, m is the dimension of $x_i(t)$, $f : R^m \times [0, +\infty) \to R^m$ is a continuous function, $\varepsilon > 0$ is the coupling strength, $C = (c_{ij})_{N\times N}$ is the network adjacency matrix with $c_{ij} = c_{ji} \geq 0$ for $i \neq j$, $\sum_{j=1}^{N} c_{ij} = 0$, and $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_m)$ is a nonzero matrix with $\gamma_k \geq 0$, $i, j \in \mathbb{N}, k \in \mathfrak{m} = \{1, \ldots, m\}$. Here, the symmetric matrix C represents the topological structure of the network, and the diagonal matrix Γ represents the inner coupling components of each node.

It has been shown that the partial synchronization of the network (2.1) is equivalent to the global attractiveness of the corresponding invariant manifold corresponding to the partition $G = \{G_1, \ldots, G_n\}$ of the dynamical network (2.1).

Definition 2.1. *The set*

$$\mathbb{S} = \left\{ (x_1^\top, \ldots, x_N^\top)^\top \in R^{mN} \;\middle|\; x_i = x_j,\ i, j \in G_p, p \in \mathfrak{n} \right\}.$$

is called the partial synchronization manifold correspond to the partition $G = \{G_1, \ldots, G_n\}$.

Definition 2.2. *The partial synchronization manifold $\mathbb{S}$ is globally attractive for the system (2.1), or, partial synchronization corresponding to the partition $G = \{G_1, \ldots, G_n\}$ occurs, if*

$$\lim_{t\to+\infty} \|x_i(t) - x_j(t)\| = 0$$

holds for arbitrary initial values and for all $i, j \in G_p$, $p \in \mathfrak{n}$.

Here, as a generalization of the global network synchronization, it is not specifically required the states in different communities to be eventually separated from each other, namely, $\|x_i(t) - x_j(t)\| \geq 0$ as $t \to +\infty$ for any $x_i \in G_p$ and $x_j \in G_q$ with $p \neq q$.

For convenience, we rewrite the coupling weight matrix C to be in a block form based on the partition G as follows

$$C = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}. \tag{2.2}$$

It has been shown that the partial synchronous manifold $\mathbb{S}$ of the network (2.1) is invariant if and only if every sub-matrix C_{pq} in the form (2.2) has equal row-sums for all $p, q \in \mathfrak{n}$ [10]. In the study of partial synchronization of complex networks, it is always supposed that the corresponding partial synchronization manifolds are invariant manifolds. Therefore, the following hypotheses were usually supposed to hold in previous works.

(H_1) The partial synchronization manifold $\mathbb{S}$ is an invariant manifold of the dynamical network (2.1).

(H_2) Let $P = \mathrm{diag}(p_1, \ldots, p_m)$ be a positive definite diagonal matrix, $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_m)$ be a diagonal matrix, and $E_N \in R^{N\times N}$ be the identity matrix. There exists a constant $\epsilon > 0$ such that the inequality

$$(u - v)^\top P\{[f(u,t) - f(v,t)] - \Delta(u - v)\} \leq -\epsilon(u - v)^\top (u - v)$$

holds for any $u, v \in R^m$ and $t \geq 0$.

With the help of the preliminaries mentioned above, a recent research proposed the following valuable criterion about the occurrence of partial synchronization with increasing coupling strength.

Lemma 2.3. *[10, Theorem 2] Let hypotheses (H_1) and (H_2) be satisfied. Assume that dynamical network (2.1) satisfies*

(H_3) *the inequality*

$$\varepsilon\gamma_k\lambda_{\max}^{\bar{\mathbb{S}}}(C) + \delta_k \leq 0, \quad k \in \mathfrak{m}, \tag{2.3}$$

holds, where $\lambda_{\max}^{\bar{\mathbb{S}}}(C) = \max\{\lambda \in \sigma(C) : \Xi_C(\lambda) \nsubseteq \mathbb{S}\}$, $\sigma(C)$ is the set of all eigenvalues of C, and $\Xi_C(\lambda)$ is the eigenspace of C corresponding to eigenvalue λ.

Then the synchronization manifold $\mathbb{S}$ is globally attractive for dynamical network (2.1).

This criterion investigated the relationship between the partial synchronization problem and the coupling matrix of the whole network, which implies that partial synchronization can be realized by increasing coupling strength. However, it is very difficult to calculate the eigenvalue of the whole network matrix $\lambda_{\max}^{\bar{\mathbb{S}}}(C)$, especially the eigenspace $\Xi_C(\lambda)$ corresponding to each eigenvalue. In the next section, we carry out another criterion to simplify the complicated calculation questions, which may have certain theoretical value and practical significance.

3. Main results

3.1. A brief criterion on partial synchronization

The results in Lemma 2.3 focused on the topology of the whole network, the size of which may be very large and it is tedious to obtain the parameter $\lambda^{\bar{\mathbb{S}}}_{\max}(C)$. In this section, we point out that the results on partial synchronization can be ensured merely by the intra-community connections. That is to say, it is irrelevant to the inter-community couplings. Therefore, great amounts of calculations on the topology of the whole network could be avoided, and a brief criterion is obtained as follows.

Theorem 3.1. *Let hypotheses* (H_1) *and* (H_2) *hold. Suppose that dynamical network* (2.1) *satisfies*

(H'_3) *the inequality*

$$\varepsilon\gamma_k\lambda_2(\tilde{C}_p) + \delta_k \leq 0, \quad k \in \mathfrak{m} \tag{3.1}$$

holds for any given $p \in \mathfrak{n}$*, where* $\tilde{C}_p = (\tilde{c}_{ij})_{n_p\times n_p}$*,* $i, j \in G_p$*,*

$$\tilde{c}_{ij} = \begin{cases} c_{ij}, & i \neq j, \\ -\sum_{k\in G_p, k\neq i} c_{ik}, & i = j, \end{cases}$$

and $\lambda_2(\tilde{C}_p)$ *is the second-largest eigenvalue of the matrix* $\tilde{C}_p$*.*

Then the synchronization manifold $\mathbb{S}$ *is globally attractive for dynamical network* (2.1).

Similar to the proof of Lemma 2.3, we can prove Theorem 3.1 based on the geometrical analysis of the partial synchronization manifold. Different slightly from the proof of Lemma 2.3, we should selectively analyze the intra-community connections and neglect the analysis on the inter-community couplings. Notice that the parameter $\lambda^{\bar{\mathbb{S}}}_{\max}(C)$ represents the adjacency matrix of the whole network, the parameters $\lambda_2(\tilde{C}_p)$, $p \in \mathfrak{n}$, should be easier to calculate. Therefore, Theorem 3.1 is more convenient in practical applications, especially for networks consisting of a great mount of nodes.

Remark. In the coming era of big data, the global information of the network topology is usually very large, which is a necessary condition in the previous results on partial synchronization. In order to explore a more concise and more convenient criterion, Theorem 3.1 builds a novel criterion independent of the inter-community connections. This criterion has shown that partial synchronization can be ensured only by the intra-community connections, and the information of inter-community connections is not necessary. Therefore, it may provide more convenience in reducing network sizes in practice, especially for networks consisting of a great amount of nodes.

3.2. Comparisons between Lemma 2.3 and Theorem 3.1

It is noted that Theorem 3.1 provides us a novel index of partial synchronizability by ignoring inter-community connections, and Lemma 2.3 was derived based on the analysis of the adjacency matrix of the whole network. In this subsection, a rigorous theoretical proof is carried out to show that the threshold obtained by Lemma 2.3 is more accurate than the one obtained by Theorem 3.1, and the conditions of Theorem 3.1 are much weaker than that of Lemma 2.3.

Theorem 3.2. *Assume that hypotheses* (H_1) *and* (H_2) *are satisfied. Then the following conclusions hold.*

(*i*) *If hypothesis* (H_3') *holds, then hypothesis* (H_3) *is satisfied.*

(*ii*) *Denote the threshold for dynamical network* (2.1) *to realize partial synchronization derived from Lemma 2.3 as*

$$\varepsilon_0 = \frac{\max\{\delta_k/\gamma_k : k \in \mathfrak{m}\}}{|\lambda_{\max}^{\bar{\mathbb{S}}}(C)|}, \tag{3.2}$$

and denote the one derived from Theorem 3.1 as

$$\varepsilon_0' = \frac{\max\{\delta_k/\gamma_k : k \in \mathfrak{m}\}}{\min\{|\lambda_2(\tilde{C}_p)| : p \in \mathfrak{n}\}}. \tag{3.3}$$

Then $\varepsilon_0 \le \varepsilon_0'$.

Proof. (*i*) Notice that inequality (3.1) holds, it is easy to see that for any $u_p \in R^{n_p}$ satisfying $u_p \neq (\alpha, \alpha, \ldots, \alpha)^\top \in R^{n_p}, \alpha \in R$, there holds

$$u_p^\top(\varepsilon\gamma_k\tilde{C}_p + \delta_k E_{n_p})u_p \le 0, \quad k \in \mathfrak{m}, p \in \mathfrak{n},$$

or

$$\sum_{i\in G_p} u_i^\top\Big[\varepsilon\gamma_k \sum_{j\in G_p} \tilde{c}_{ij}u_j + \delta_k u_i\Big] \le 0, \quad k \in \mathfrak{m}, p \in \mathfrak{n}. \tag{3.4}$$

Therefore, we conclude that for any $z = (z_1, \ldots, z_N) \notin \mathbb{S}$, $k \in \mathfrak{m}$, there holds

$$\begin{aligned}
& z^\top(\varepsilon\gamma_k C + \delta_k E_N)z \\
= \;& z^\top(\varepsilon\gamma_k \sum_{j=1}^N c_{1j}z_j + \delta_k z_1, \cdots, \varepsilon\gamma_k \sum_{j=1}^N c_{Nj}z_j + \delta_k z_N)^\top \\
= \;& \sum_{i=1}^N z_i^\top\Big[\varepsilon\gamma_k \sum_{j=1}^N c_{ij}z_j + \delta_k z_i\Big] \\
= \;& \sum_{p=1}^n \sum_{i\in G_p} z_i^\top\Big[\varepsilon\gamma_k \sum_{q=1}^n \sum_{j\in G_q} c_{ij}z_j + \delta_k z_i\Big] \\
= \;& \sum_{p=1}^n \sum_{i\in G_p} z_i^\top\Big[\varepsilon\gamma_k \sum_{j\in G_p} c_{ij}z_j + \delta_k z_i + \varepsilon\gamma_k \sum_{q=1,q\neq p}^n \sum_{j\in G_q} c_{ij}z_j\Big].
\end{aligned}$$

Based on the relationship between c_{ij} and $\tilde{c}_{ij}$ defined in hypothesis (H_3'), we have

$$\begin{aligned}
& z^\top(\varepsilon\gamma_k C + \delta_k E_N)z \\
= \;& \sum_{p=1}^n \sum_{i\in G_p} z_i^\top\Big[\varepsilon\gamma_k \sum_{j\in G_p} \tilde{c}_{ij}z_j + \delta_k z_i\Big] + \varepsilon\gamma_k \sum_{p=1}^n \sum_{i\in G_p} \sum_{q=1,q\neq p}^n \sum_{j\in G_q} c_{ij}z_i^\top(z_j - z_i),
\end{aligned}$$

where $z = (z_1, \ldots, z_N) \notin \mathbb{S}$, $k \in \mathfrak{m}$. Taking into account that inequality (3.4) and $z = (z_1, \ldots, z_N) \notin \mathbb{S}$, we obtain

$$\begin{aligned}
& z^\top(\varepsilon\gamma_k C + \delta_k E_N)z \\
\le \;& \varepsilon\gamma_k \sum_{p=1}^n \sum_{i\in G_p} \sum_{q=1,q\neq p}^n \sum_{j\in G_q} c_{ij}z_i^\top(z_j - z_i) \\
= \;& \varepsilon\gamma_k \sum_{p=1}^{n-1} \sum_{q=p+1}^n \sum_{i\in G_p} \sum_{j\in G_q} c_{ij}z_i^\top(z_j - z_i) + \varepsilon\gamma_k \sum_{q=1}^{n-1} \sum_{p=q+1}^n \sum_{i\in G_p} \sum_{j\in G_q} c_{ij}z_i^\top(z_j - z_i).
\end{aligned}$$

Renaming in the second term p by q, i by j and vice versa, and utilizing the symmetry of c_{ij}, one gets

$$\begin{aligned} & z^\top(\varepsilon\gamma_k C + \delta_k E_N)z \\ = & \ \varepsilon\gamma_k \sum_{p=1}^{n-1}\sum_{q=p+1}^{n}\sum_{i\in G_p}\sum_{j\in G_q} c_{ij} z_i^\top (z_j - z_i) + \varepsilon\gamma_k \sum_{p=1}^{n-1}\sum_{q=p+1}^{n}\sum_{j\in G_q}\sum_{i\in G_p} c_{ji} z_j^\top (z_i - z_j) \\ = & \ -\varepsilon\gamma_k \sum_{p=1}^{n-1}\sum_{q=p+1}^{n}\sum_{i\in G_p}\sum_{j\in G_q} c_{ij} (z_j - z_i)^\top (z_j - z_i) \\ \leq & \ 0. \end{aligned}$$

Thus, one has $z^\top(\varepsilon\gamma_k C + \delta_k E_N)z \leq 0$ for any $z = (z_1, \ldots, z_N) \notin \mathbb{S}$. Therefore,

$$\varepsilon\gamma_k \lambda_{\max}^{\bar{\mathbb{S}}}(C) + \delta_k \leq 0, \quad j \in \mathfrak{m}, p \in \mathfrak{n}.$$

(ii) Since the coupling weight matrix satisfies $c_{ij} = c_{ji} \geq 0$ for $i \neq j$ and $\sum_{j=1}^N c_{ij} = 0$, it is not difficult to see that all eigenvalues of the matrix C (or $\tilde{C}_p,\ p \in \mathfrak{n}$) are negative except that the biggest eigenvalue equals to zero. Combining the property with the statement (i) proved above, we can conclude that the statement (ii) is proved. □

Theorem 3.2 gives a comparison of Lemma 2.3 and Theorem 3.1. It can be seen that no extra conditions should be satisfied for the inter- or intra-community connections for Theorem 3.1. Because the inter-community connections are ignored, our threshold is rougher than that of Lemma 2.3. That is to say, our result requires higher coupling strength to realize partial synchronization. This is the disadvantage of our result. But in many common cases, our threshold is equal to that of Lemma 2.3. Then, the superiority of our results is shown.

As a direct conclusion of the statement (ii), it is straightforward to prove that $\lambda_{\max}^{\bar{\mathbb{S}}}(C) \leq \max\{\lambda_2(\tilde{C}_p) : p \in \mathfrak{n}\}$. Then a question arises naturally: under what conditions the parameters satisfy that $\lambda_{\max}^{\bar{\mathbb{S}}}(C) = \max\{\lambda_2(\tilde{C}_p) : p \in \mathfrak{n}\}$? In order to answer the question mentioned above, we carry out the next subsection.

3.3. Partial synchronization of networks consisting of identical communities

In this subsection, we will first introduce a lemma for the eigenvalues of a class of matrices with special structures. Based on the two criteria mentioned in Lemma 2.3 and Theorem 3.1, the thresholds for a class of networks with special structures to realize partial synchronization are deduced and compared.

Corollary 3.3. *For any matrix $C, C_i \in R^{d\times d}$, $d = N/n$, define $\bar{C}_i = \sum_{p=1,p\neq i}^n C_p$, $\bar{C}_0 = \sum_{p=1}^n C_p$, $i \in \mathfrak{n}$, and the matrix*

$$C = \begin{pmatrix} C - \bar{C}_1 & C_2 & \cdots & C_n \\ C_1 & C - \bar{C}_2 & \cdots & C_n \\ \cdots & \cdots & \cdots & \cdots \\ C_1 & C_2 & \cdots & C - \bar{C}_n \end{pmatrix}. \tag{3.5}$$

Then the following conclusions hold.

(i) *The matrix C has simple eigenvalues $\lambda_1 = 0, \lambda_2, \cdots, \lambda_d$, and $(n-1)$-multiple eigenvalues $\lambda_{d+1}, \lambda_{d+2}, \cdots, \lambda_{2d}$, where $\lambda_i, i = 1, 2, \ldots, d$, are eigenvalues of the matrix C, $\lambda_{d+i}, i = 1, 2, \ldots, d$, are eigenvalues of the matrix $C - \bar{C}_0$.*

(*ii*) *In particular, if* $C_i = \theta_i E_d$, $i \in \mathfrak{n}$, *then the matrix C has simple eigenvalues* $\lambda_1 = 0, \lambda_2, \cdots, \lambda_d$, *and* $(n-1)$*-multiple eigenvalues* $\lambda_i - \sum_{p=1}^{n} \theta_p, i = 1, 2, \ldots, d$.

Proof. (i) It is direct to obtain the eigenpolynomial of the coupling matrix C, that is,

$$\begin{aligned}
&|C - \lambda E_N| \\
= &\begin{vmatrix} E_d & C_1 & C_2 & \cdots & C_n \\ 0 & C - \bar{C}_1 - \lambda E_d & C_2 & \cdots & C_n \\ 0 & C_1 & C - \bar{C}_2 - \lambda E_d & \cdots & C_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & C_1 & C_2 & \cdots & C - \bar{C}_n - \lambda E_d \end{vmatrix} \\
= &\begin{vmatrix} E_d + \sum\limits_{q=1}^{n} C_q [C - \bar{C}_0 - \lambda E_d]^{-1} & 0 & \cdots & 0 \\ -E_d & C - \bar{C}_0 - \lambda E_d & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -E_d & 0 & \cdots & C - \bar{C}_0 - \lambda E_d \end{vmatrix} \\
= &|C - \lambda E_d| \left| C - \bar{C}_0 - \lambda E_d \right|^{n-1}
\end{aligned}$$

Therefore, the matrix C has simple eigenvalues λ_i and $(n-1)$-multiple eigenvalues λ_{d+i}, $i = 1, 2, \ldots, d$.

(ii) In particular, if $C_p = \theta_p E_d$, $p \in \mathfrak{n}$, then $\bar{C}_0 = \sum_{p=1}^{n} \theta_p E_d$ and the matrix $C - \bar{C}_0$ has simple eigenvalues $\lambda_{d+i} = \lambda_i - \sum_{p=1}^{n} \theta_p$, $i \in \mathfrak{n}$. Based on the item (i), the validity of the item (ii) is confirmed. □

The coupling weight matrix of many complex networks in real world can be rewritten as the form of the matrix (3.5), which implies that all communities consisted in the network have the same community sizes and the same coupling topologies. For instance, the student network in a school consists of many identical classes, the size of each class is identical and all the students in the same class are coupled globally. Therefore, partial synchronization of networks consisting of identical communities is worth studying, and there may be some potential applications. Now, we consider the problem of partial synchronization in a network with the coupling matrix (3.5). Here, we take the intra-community connection matrix $C_i = E_d$, $i \in \mathfrak{n}$ as a special case, which implies that the nodes in any two different communities are one-to-one coupled, then we obtain the following theorem with the help of Corollary 3.3.

Corollary 3.4. *Consider dynamical network* (2.1) *with the coupling matrix* (3.5)*, where* $C_p = E_d$, $p \in \mathfrak{n}$, *and suppose that hypotheses* (H_1) *and* (H_2) *are satisfied. Then the following conclusions hold.*

(*i*) $\lambda_{\max}^{\bar{\mathbb{S}}} = \max\{\lambda_2(\tilde{C}_p) : p \in \mathfrak{n}\} < 0$.

(*ii*) *Hypothesis* (H_3') *holds if and only if hypothesis* (H_3) *is satisfied.*

(*iii*) *Dynamical network* (2.1) *realizes partial synchronization if* $\varepsilon \geq \varepsilon_0 = \varepsilon_0'$.

The proof of this theorem is not particularly difficult, and so is omitted.

4. Numerical examples

In this section, we provide several numerical examples to verify the obtained theoretical results for partial synchronization.

4.1. Partial synchronization in a network with different communities

Consider a complex network consisting of 9 nodes with three different communities, which is shown in Figure 1. The network adjacency matrix is

$$C = \begin{pmatrix} -\beta-1 & 1 & 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ 1 & -\beta-2 & 1 & 0 & \beta & 0 & 0 & 0 & 0 \\ 0 & 1 & -\beta-1 & 0 & 0 & \beta & 0 & 0 & 0 \\ \beta & 0 & 0 & -2\beta-2 & 1 & 1 & \beta & 0 & 0 \\ 0 & \beta & 0 & 1 & -2\beta-2 & 1 & 0 & \beta & 0 \\ 0 & 0 & \beta & 1 & 1 & -2\beta-2 & 0 & 0 & \beta \\ 0 & 0 & 0 & \beta & 0 & 0 & -\beta-1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 & 1 & -\beta-2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \beta & 0 & 1 & -\beta-1 \end{pmatrix},$$

where β is a nonnegative constant.

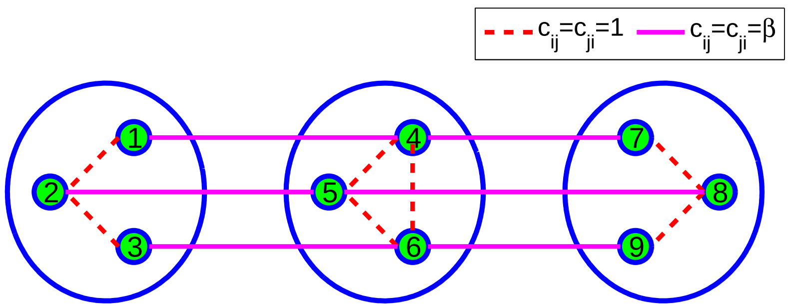

Figure 1. Topology structure of a complex network consisting of 9 nodes with three different communities.

By further calculations, one obtains that the eigenvalue of the topology matrix mentioned in Lemma 2.3 is as follows

$$\lambda_{\max}^{\bar{\mathbb{S}}}(C) = -(3\beta + 4 - \sqrt{9\beta^2 + 4\beta + 4})/2 \leq -1,$$

and the eigenvalue mentioned in Theorem 3.1 is $\max\{\lambda_2(\tilde{C}_p), p = 1, 2, 3\} = -1$. Therefore, Theorem 3.2 holds for any $\beta \geq 0$.

We choose the node dynamics of the network as the well-known neural networks

$$\dot{x}_i = -Dx_i + Tg(x_i) + \varepsilon \sum_{j=1}^{m} c_{ij} H x_j, \quad i = 1, \ldots, m, \tag{4.1}$$

where $x_i \in R^3, D = H = E_3$, $g(x_i) = (g(x_i^1), g(x_i^2), g(x_i^3))^\top$, $g(s) = (|s+1| - |s-1|)/2$, and

$$T = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & -4.4 & 1.0 \end{pmatrix}.$$

By using Matlab LMI Control Toolbox, one derives that the matrix $\Delta = 5.5685E_3$ satisfies condition (H_2). Taking $\Gamma = E_3$ and $\beta = 0.5$, the network with randomly chosen initial conditions reaches partial synchronization. Denote the complete synchronization error $e_1 = \frac{1}{9}\sum_{p=1}^{9}\|x_j - x_1\|$ and the partial synchronization error $e_2 = \frac{1}{9}\sum_{p=1}^{9}\|x_j - x_{\hat{j}}\|$, where $\hat{1} = \hat{2} = \hat{3} = 1$, $\hat{4} = \hat{5} = \hat{6} = 4$ and $\hat{7} = \hat{8} = \hat{9} = 7$. The performance is shown in Figure 2, which indicates the variation of synchronization errors with respect to the coupling strength. Based on Theorem 3.2, we calculate the coupling strength threshold for partial synchronization $\varepsilon_0' = 5.5685$. By Lemma 2.3, the coupling strength threshold should be $\varepsilon_0 = 4.2259$.

As Figure 2 shows, the blue line denoting the errors between nodes in a same community tends to zero when the coupling strength $\varepsilon > 2$. The red line denoting the errors of the whole network also tends to zero when the coupling strength $\varepsilon > 8$. The evolution trends of the two lines implies that both partial synchronization and complete synchronization are achieved. However, there is no contradiction. Based on the definition of partial synchronization in page 3, if the network realizes complete synchronization, the network is also considered to realize partial synchronization as a special case. Therefore, Figure 2 verifies the validity of Theorem 3.1.

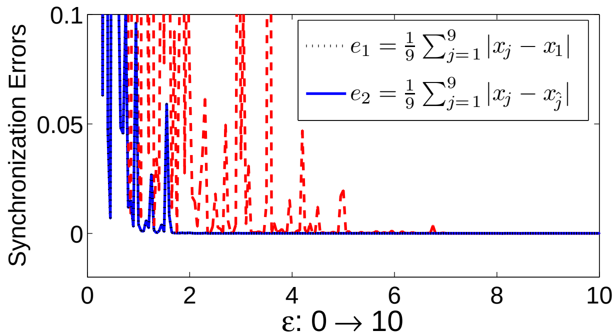

Figure 2. The variation of synchronization errors with respect to the coupling strength. The dashed line represents the complete synchronization errors, and the solid line represents the partial synchronization errors.

4.2. *Partial synchronization in a network with identical communities*

This subsection considers a complex network consisting of 9 nodes with three identical communities, the topological structure of which is shown in Figure 3. The network adjacency matrix is

$$C = \begin{pmatrix} -2\beta-1 & 1 & 0 & \beta & 0 & 0 & \beta & 0 & 0 \\ 1 & -2\beta-2 & 1 & 0 & \beta & 0 & 0 & \beta & 0 \\ 0 & 1 & -2\beta-1 & 0 & 0 & \beta & 0 & 0 & \beta \\ \beta & 0 & 0 & -2\beta-1 & 1 & 0 & \beta & 0 & 0 \\ 0 & \beta & 0 & 1 & -2\beta-2 & 1 & 0 & \beta & 0 \\ 0 & 0 & \beta & 0 & 1 & -2\beta-1 & 0 & 0 & \beta \\ \beta & 0 & 0 & \beta & 0 & 0 & -2\beta-1 & 1 & 0 \\ 0 & \beta & 0 & 0 & \beta & 0 & 1 & -2\beta-2 & 1 \\ 0 & 0 & \beta & 0 & 0 & \beta & 0 & 1 & -2\beta-1 \end{pmatrix},$$

where β is a nonnegative constant.

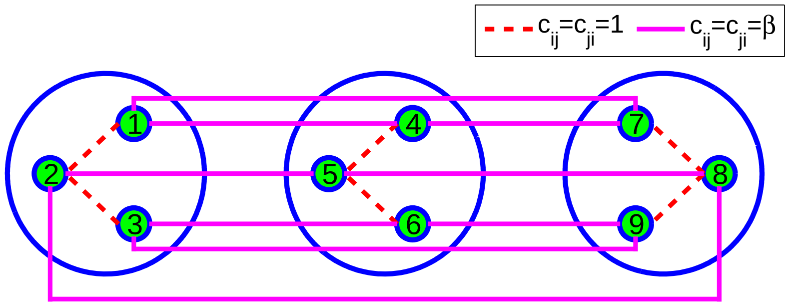

Figure 3. Topology structure of a complex network consisting of 9 nodes with three identical communities.

It is easy to obtain that the eigenvalues of the topology matrix are listed as $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = -3, \lambda_{4,5} = -3\beta, \lambda_{6,7} = -1 - 3\beta, \lambda_{8,9} = -3 - 3\beta$. Note that the eigenvectors of $\lambda_{4,5} = -3\beta$ are $(0, 0, 0, -1, -1, -1, 1, 1, 1)^\top$ and $(1, 1, 1, -1, -1, -1, 0, 0, 0)^\top$, one has $\lambda^{\bar{\mathbb{S}}}_{\max} = -1 = \max\{\lambda_2(\tilde{C}_p) : p \in \mathfrak{n}\}$. Time evolutions of the errors between nodes in a same community are shown in Figure 4. It can be seen that all the nodes in the same community behave in the same synchronous fashion. Thus, the numerical example confirms the effectiveness of corollaries 3.3 and 3.4.

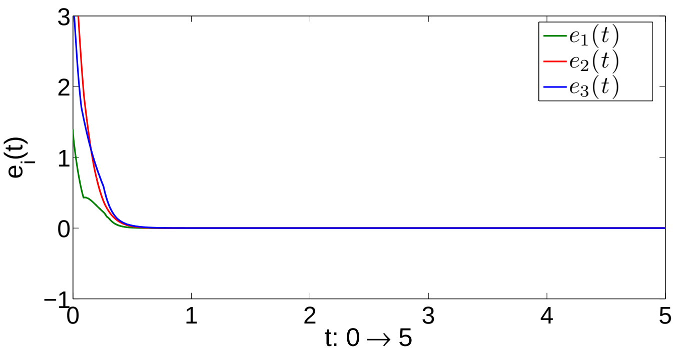

Figure 4. Time evolutions of the errors between nodes in a same community, where $e_1 = \frac{1}{2} \sum_{j=2,3} |x_j - x_1|, e_2 = \frac{1}{2} \sum_{j=5,6} |x_j - x_4|, e_3 = \frac{1}{2} \sum_{j=8,9} |x_j - x_7|$.

5. Conclusions

This paper provides a new perspective to study partial synchronization of a generalized linearly coupled network. Compared to previous results, the obtained criteria show that partial synchronization is ensured by intra-community connections, which is in agreement with the definition of partial synchronization intuitively. With the view of practical application, new criteria need just the topology information of the intra-community structure, a great deal of information on the inter-community connections has been streamlined. It is shown that new criteria could provide the same threshold for partial synchronization as the previous results under certain circumstances.

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Conflict of interest

The authors declare that there is no conflict of interest.

References

1. X. S. Yang, J. D. Cao, J. Q. Lu, Synchronization of randomly coupled neural networks with Markovian jumping and time-delay, *IEEE T. Circuits-I*, **60** (2013), 363–376.
2. Y. Zhu, B. Xu, X. H. Shi, A survey of social-based routing in delay tolerant networks: positive and negative social effects, *IEEE Commun. Surv. Tut.*, **15** (2013), 387–401.
3. W. W. Yu, W. Ren, W. X. Zheng, G. R. Chen, J. H. Lv, Distributed control gains design for consensus in multi-agent systems with second-order nonlinear dynamics, *Automatica*, **49** (2013), 2107–2115.
4. J. Gómez-Gardenes, Y. Moreno, A. Arenas, Paths to synchronization on complex networks, *Phys. Rev. Lett.*, **98** (2007), 034101.
5. T. F. Weng, H. J. Yang, C. G. Gu, J. Zhang, M. Small, Synchronization of chaotic systems and their machine-learning models, *Phys. Rev. E*, **99** (2019), 042203.
6. Z. Yao, J. Ma, Y. G. Yao, C. N. Wang, Synchronization realization between two nonlinear circuits via an induction coil coupling, *Nonlinear Dynam.*, **96** (2019), 205–217.
7. L. Pecora, T. Carroll, Synchronization in chaotic systems, *Phys. Rev. Lett.*, **64** (1990), 821–824.
8. J. B. Zhang, Z. J. Ma, G. R. Chen, Robustness of cluster synchronous patterns in small-world networks with inter-cluster co-competition balance, *Chaos*, **24** (2014), 023111.
9. J. B. Zhang, A. C. Zhang, J. D. Cao, J. L. Qiu, F. E. Alsaadi, Adaptive outer synchronization between two delayed oscillator networks with cross couplings, *Sci. China Inform. Sci.*, **63** (2020), 209204.
10. W. Wu, T. P. Chen, Partial synchronization in linearly and symmetrically coupled ordinary differential systems, *Physica D*, **238** (2009), 355–364.
11. L. Z. Zhang, Y. Q. Yang, F. Wang, X. Sui, Lag synchronization for fractional-order memristive neural networks with time delay via switching jumps mismatch, *J. Franklin I.*, **355** (2018), 1217–1240.
12. S. M. Cai, F. L. Zhou, Q. B. He, Fixed-time cluster lag synchronization in directed heterogeneous community networks, *Physica A*, **525** (2019), 128–142.

13. Q. T. Gan, Exponential synchronization of generalized neural networks with mixed time-varying delays and reaction-diffusion terms via aperiodically intermittent control, *Chaos*, **27** (2017), 013113.
14. C. Chen, K. Xie, F. L. Lewis, S. Xie, A. Davoudi, Fully distributed resilience for adaptive exponential synchronization of heterogeneous multiagent systems against actuator faults, *IEEE T. Automat. Contr.*, **64** (2019), 3347–3354.
15. W. G. Sun, Q. Y. Ding, J. B. Zhang, F. Y. Chen, Coherence in a family of tree networks with an application of Laplacian Spectrum, *Chaos*, **24** (2014), 043112.
16. J. K. Ni, P. Shi, Adaptive Neural Network Fixed-Time Leader-Follower Consensus for Multiagent Systems With Constraints and Disturbances, *IEEE T. Cybernetics*, **99** (2020), 1–14.
17. C. W. Wu, L. O. Chua, On a conjecture regarding the synchronization in an array of linearly coupled dynamical systems, *IEEE T. Circuits-I*, **43** (1996), 161–165.
18. K. Oooka, T. Oguchi, Estimation of synchronization patterns of chaotic systems in cartesian product networks with delay couplings, *Int. J. Bifurcat. Chaos*, **26** (2016), 1630028.
19. J. B. Zhang, Z. J. Ma, G. Zhang, Cluster synchronization induced by one-node clusters in networks with asymmetric negative couplings, *Chaos*, **23** (2013), 043128.
20. P. P. Zhou, S. M. Cai, J. W. Shen, Z. R. Liu, Adaptive exponential cluster synchronization in colored community networks via aperiodically intermittent pinning control, *Nonlinear Dynam.*, **92** (2018), 905–921.
21. Z. J. Ma, Z. R. Liu, G. Zhang, A new method to realize cluster synchronization in connected chaotic networks, *Chaos*, **16** (2006), 023103.
22. E. Steur, T. Oguchi, C. Leeuwen, H. Nijmeijer, Partial synchronization in diffusively time-delay coupled oscillator networks, *Chaos*, **22** (2012), 043144.
23. C. B. Yu, J. H. Qin, H. J. Gao, Cluster synchronization in directed networks of partial-state coupled linear systems under pinning control, *Automatica*, **50** (2014), 2341–2349.
24. S. M. Cai, Q. Jia, Z. R. Liu, Cluster synchronization for directed heterogeneous dynamical networks via decentralized adaptive intermittent pinning control, *Nonlinear Dynam.*, **82** (2015), 689–702.
25. H. S. Su, Z. H. Rong, M. Z. Q. Chen, X. F. Wang, G. R. Chen, H. W. Wang, Decentralized adaptive pinning control for cluster synchronization of complex dynamical networks, *IEEE T. Cybernetics*, **43** (2013), 394–399.
26. F. D. Rossa, L. Pecora, K. Blaha, A. Shirin, I. Klickstein, F. Sorrentino, Symmetries and cluster synchronization in multilayer networks, *Nat. Commun.*, **11** (2020), 3179.
27. L. V. Gambuzza, M. Frasca, A criterion for stability of cluster synchronization in networks with external equitable partitions, *Automatica*, **100** (2019), 212–218.
28. H. B. Chen, P. Shi, C. C. Lim, Cluster Synchronization for Neutral Stochastic Delay Networks via Intermittent Adaptive Control, *IEEE T. Neur. Net. Lear.*, **30** (2019), 3246–3259.

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