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Research article

Cauchy problem for isothermal system in a general nozzle with space-dependent friction

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Abstract: In this paper, we study the Cauchy problem of the isothermal system in a general nozzle with space-dependent friction $\alpha(x)$. First, by using the maximum principle, we obtain the uniform bound $\rho^{\delta,\varepsilon,\tau} \le M$, $|m^{\delta,\varepsilon,\tau}| \le M$, independent of the time, of the viscosity-flux approximation solutions; Second, by using the compensated compactness method coupled with the convergence framework given in [5], we prove that the limit, (ρ, m) of $(\rho^{\delta,\varepsilon,\tau}, m^{\delta,\varepsilon,\tau})$, as $\varepsilon, \delta, \tau$ go to zero, is a uniformly bounded entropy solution.

Keywords: global solution; isothermal system; friction terms; viscosity-flux approximation; compensated compactness
Mathematics Subject Classification: 35L65, 76N10

1. Introduction

The following isentropic gas dynamics system in a general nozzle with friction, whose physical phenomena called "choking or choked flow",

$$\begin{cases} \rho_t + (\rho u)_x = -\frac{a'(x)}{a(x)}\rho u, \\ (\rho u)_t + (\rho u^2 + P(\rho))_x = -\frac{a'(x)}{a(x)}\rho u^2 - \alpha(x)\rho u|u|, \end{cases} \tag{1.1}$$

is of interest because resonance occurs. This means there is a coincidence of wave speeds from different families of waves (see [2,4,6,7,16] and the references cited therein for the details). Here ρ is the density of gas, u the velocity, $P = P(\rho)$ the pressure, $a(x)$ is a slowly variable cross section area at x in the nozzle and $\alpha(x)$ denotes a friction function. For the polytropic gas, P takes the special form $P(\rho) = \frac{1}{\gamma}\rho^{\gamma}$, where $\gamma > 1$ is the adiabatic exponent and for the isothermal gas, $\gamma = 1$.

The Cauchy problem of system (1.1) with bounded initial data

$$((\rho(x,0), u(x,0)) = (\rho_0(x), u(x)), \quad \rho_0(x) \geq 0, \tag{1.2}$$

in the simplest divergent nozzle (with respect to $a'(x) \geq 0$) was first obtained in [19] for the usual gases $1 < \gamma \leq \frac{5}{3}$, and later, extended in [8] to the case of $\gamma > 1$, provided that the initial data are bounded and satisfy the very special condition $z(\rho_0(x), u_0(x)) \leq 0$.

When $\gamma = 1$, the global existence of symmetrical weak solutions of the isothermal gas dynamics system (1.1) without a friction ($\alpha = 0$) in the Lagrangian coordinates was well studied in [12,13,20,21] by using the Glimm scheme method [3, 15]; and in the Euler coordinates studied in [1, 9] by using the compensated compactness theory [5, 14, 18]. The global existence of weak solutions of the isothermal gas dynamics system (1.1) with a constant friction was studied in [10], where, the maximum principle was used directly to obtain the a-priori dependent-time L^{∞} estimate $0 \leq \rho \leq M(T)$, $|u| \leq M(T)$ under the conditions $|A(x)| = \left|\frac{a'(x)}{a(x)}\right| \leq M$ and $\alpha \geq 0$.

In this paper, by carefully applying the maximum principle and the viscosity-flux approximation method introduced in [11], under the more general conditions $A(x) \in L^1$, $\alpha(x) \in L^1$, we improve the above time-dependent bound $M(T)$ to a constant bound M, which ensures that the entropy solutions of the Cauchy problem (1.1) and (1.2) we obtained are stable.

The main result is given in the following

Theorem 1.1. *Let $P(\rho) = \rho$, $0 < a_L \leq a(x) \leq A_L$ for x in any compact set $x \in (-L, L)$, $A(x) = -\frac{a'(x)}{a(x)} \in L^1(\mathbb{R})$ and $\alpha(x) \in L^1(\mathbb{R})$, where A_L, a_L are positive constants, but could depend on L. Moreover, if*

$$|A(x)|_{L^1(\mathbb{R})} \leq \frac{1}{12}, \qquad |\alpha(x)|_{L^1(\mathbb{R})} \leq \frac{1}{12} \tag{1.3}$$

and the bounded initial data satisfy

$$\begin{cases} \ln(\rho_0(x)a(x)) - u_0(x) < M - 3(|A(x)|_{L^1(\mathbb{R})} + |\alpha(x)|_{L^1(\mathbb{R})}), \\ \ln(\rho_0(x)a(x)) + u_0(x) < M, \end{cases} \tag{1.4}$$

where $M > 1$ is a constant, then the Cauchy problem (1.1) *and* (1.2) *have a bounded weak solution (ρ, u), which has the following uniform bound*

$$\begin{cases} \ln(\rho a(x)) - u \leq M, \\ \ln(\rho a(x)) + u \leq M - 3(|A(x)|_{L^1(\mathbb{R})} + |\alpha(x)|_{L^1(\mathbb{R})}), \end{cases}$$

and satisfies system (1.1) *in the sense of distributions and the following Laxs entropy condition*

$$\int_0^{\infty}\int_{-\infty}^{\infty} \eta(\rho,m)\phi_t + q(\rho,m)\phi_x + (A(x)\eta_\rho \rho u + (A(x)\rho u^2 + \alpha(x)\rho u|u|)\eta_m)\phi dxdt \geq 0, \tag{1.5}$$

where (η, q) is a pair of entropy-entropy flux of system (1.1)*, η is convex, and $\phi \in C_0^{\infty}(\mathbb{R} \times \mathbb{R}^+ - \{t = 0\})$ is a nonnegative function.*

2. Proof of Theorem 1.1

Let $v = \rho a(x)$ and rewrite (1.1) as follows

$$\begin{cases} v_t + (vu)_x = 0, \\ (vu)_t + (vu^2 + v)_x + A(x)v + \alpha vu|u| = 0. \end{cases} \tag{2.1}$$

The two eigenvalues of (2.1) are $\lambda_1 = u - 1$ and $\lambda_2 = u + 1$, with corresponding Riemann invariants

$$z(v, m) = \ln(v) - \frac{m}{v} \quad \text{and} \quad w(v, m) = \ln(v) + \frac{m}{v},$$

where $m = vu$.

First, we add the viscosity parameter $\varepsilon > 0$ and the flux-approximation parameter $\delta > 0$ to system (2.1) to obtain the following parabolic system

$$\begin{cases} v_t + ((v - 2\delta)u)_x = \varepsilon v_{xx}, \\ (vu)_t + ((v - \delta)u^2 + v - 2\delta \ln(v))_x + A^\tau(x)\text{sgn}(A(x))v + \alpha^\tau(x)\text{sgn}(\alpha(x))vu|u| = \varepsilon(vu)_{xx}, \end{cases} \tag{2.2}$$

with initial data

$$(v(x, 0), u(x, 0)) = (v_0^\delta(x), u_0^\delta(x)), \tag{2.3}$$

where

$$(v_0^\delta(x), u_0^\delta(x)) = (a(x)\rho_0(x) + 2\delta, u_0(x)) * G^\delta, \qquad (A^\tau(x), \alpha^\tau(x)) = (|A(x)|, |\alpha(x)|) * G_1^\tau,$$

and G^δ, G_1^τ are two mollifiers and $\tau > 0$ is the regularity parameter. Then by the conditions given in Theorem 1.1, we have

$$(v_0^\delta(x), u_0^\delta(x)) \in C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}), \qquad v_0^\delta(x) \le 2\delta, \qquad v_0^\delta(x) + |u_0^\delta(x)| \le M$$

and

$$\begin{cases} 0 \le A^\tau(x) \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \qquad 0 \le \alpha^\tau(x) \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \\ |A^\tau(x)| \le M, \quad \tau \left|\frac{dA^\tau(x)}{dx}\right| \le M, \quad |\alpha^\tau(x)| \le M, \quad \tau \left|\frac{d\alpha^\tau(x)}{dx}\right| \le M. \end{cases}$$

Second, we multiply (2.2) by (w_v, w_m) and (z_v, z_m), respectively, to obtain

$$\begin{aligned} z_t + \lambda_1^\delta z_x - A^\tau(x)\text{sgn}(A(x)) - \alpha^\tau(x)\text{sgn}(\alpha(x))u|u| &= \varepsilon z_{xx} - \varepsilon(z_{vv}v_x^2 + 2z_{vm}v_x m_x + z_{mm}m_x^2) \\ &= \varepsilon z_{xx} + \frac{2\varepsilon}{v} v_x z_x - \frac{\varepsilon v_x^2}{v^2} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} w_t + \lambda_2^\delta w_x + A^\tau(x)\text{sgn}(A(x)) + \alpha^\tau(x)\text{sgn}(\alpha(x))u|u| &= \varepsilon w_{xx} - \varepsilon(w_{vv}v_x^2 + 2w_{vm}v_x m_x + w_{mm}m_x^2) \\ &= \varepsilon w_{xx} + \frac{2\varepsilon}{v} v_x w_x - \frac{\varepsilon v_x^2}{v^2}, \end{aligned} \tag{2.5}$$

where $\lambda_1^\delta = u - \frac{v-2\delta}{v}$ and $\lambda_2^\delta = u + \frac{v-2\delta}{v}$.

Let $X(x) = 3(A^\tau(x) + \alpha^\tau(x))$, then $|X(x)|_{L^1(\mathbb{R})} \leq \frac{1}{2}$ by the condition (1.3). Making the transformations of $z = z_1 + B(x)$, $w = w_1 + C(x)$, where

$$B(x) = M - \int_{-\infty}^{x} X(s)ds > \frac{1}{2}, \qquad C(x) = M + \int_{-\infty}^{x} X(s)ds > \frac{1}{2},$$

for a positive constant $M > 1$, we have from (2.4) and (2.5) that

$$\begin{aligned} & z_{1t} + \lambda_1^\delta z_{1x} - B'(x)z_1 - B'(x)B(x) + B'(x)\ln(v) \\ & \quad - B'(x)\frac{v-2\delta}{v} - A^\tau(x)\mathrm{sgn}(A(x)) - \alpha^\tau(x)\mathrm{sgn}(\alpha(x))u|u| \\ & \qquad = \varepsilon z_{1xx} + \varepsilon B''(x) + \frac{2\varepsilon}{v}v_x z_{1x} + \frac{2\varepsilon}{v}v_x B'(x) - \frac{\varepsilon v_x^2}{v^2} \\ & \qquad = \varepsilon z_{1xx} + \varepsilon B''(x) + \frac{2\varepsilon}{v}v_x z_{1x} - \varepsilon(\frac{v_x}{v} - B'(x))^2 + \varepsilon B'^2(x) \\ & \qquad \leq \varepsilon z_{1xx} + \varepsilon B''(x) + \frac{2\varepsilon}{v}v_x z_{1x} + \varepsilon B'^2(x) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} & w_{1t} + \lambda_2^\delta w_{1x} + C'(x)w_1 + C'(x)C(x) - C'(x)\ln(v) \\ & \quad + C'(x)\frac{v-2\delta}{v} + A^\tau(x)\mathrm{sgn}(A(x)) + \alpha^\tau(x)\mathrm{sgn}(\alpha(x))u|u| \\ & \qquad = \varepsilon w_{1xx} + \varepsilon C''(x) + \frac{2\varepsilon}{v}v_x w_{1x} + \frac{2\varepsilon}{v}v_x C'(x) - \frac{\varepsilon v_x^2}{v^2} \\ & \qquad = \varepsilon w_{1xx} + \varepsilon C''(x) + \frac{2\varepsilon}{v}v_x w_{1x} - \varepsilon(\frac{v_x}{v} - C'(x))^2 + \varepsilon C'^2(x) \\ & \qquad \leq \varepsilon w_{1xx} + \varepsilon C''(x) + \frac{2\varepsilon}{v}v_x w_{1x} + \varepsilon C'^2(x). \end{aligned} \tag{2.7}$$

Clearly, we can choose a suitable small positive constant ε_1 and $\varepsilon = o(\varepsilon_1)$, $\tau = o(\varepsilon_1)$ such that the following terms in (2.6) and (2.7) satisfy

$$\begin{cases} -\varepsilon_1 B'(x)B(x) - \varepsilon B''(x) - \varepsilon B'^2(x) = \varepsilon_1 X(x) + \varepsilon X'(x) - \varepsilon X^2(x) \\ \qquad\qquad \geq \varepsilon_1 X(x) - \varepsilon\tau M X(x) - \varepsilon M X(x) \geq 0, \\ \varepsilon_1 C'(x)C(x) - \varepsilon C''(x) - \varepsilon C'^2(x) = \varepsilon_1 X(x) - \varepsilon X'(x) - \varepsilon X^2(x) \\ \qquad\qquad \geq \varepsilon_1 X(x) - \varepsilon\tau M X(x) - \varepsilon M X(x) \geq 0. \end{cases} \tag{2.8}$$

Since the initial data $v_0^\delta(x) \geq 2\delta$, we may obtain the a priori estimate $v^{\delta,\varepsilon,\tau}(x) \geq 2\delta$ by applying the maximum principle to the first equation in (2.2) (see the proof of Lemma 2.2 in [17]).

Now, under the conditions in Theorem 1.1, by using (2.6)–(2.8), we prove the following inequalities

$$\begin{cases} z_{1t} + b_1(x,t)z_{1x} + b_2(x,t)z_1 + b_3(x,t)w_1 \leq \varepsilon z_{1xx}, \\ w_{1t} + c_1(x,t)w_{1x} + c_2(x,t)w_1 + c_3(x,t)z_1 \leq \varepsilon w_{1xx}, \end{cases} \tag{2.9}$$

where $b_i(x,t)$, $c_i(x,t)$, $i = 1,2,3$, are suitable functions satisfying the necessary conditions $b_3(x,t) \leq 0$, $c_3(x,t) \leq 0$.

Proof of (2.9). We prove (2.9) in several cases for two different groups of points (x,t), where $\alpha(x) \geq 0$ or $\alpha(x) \leq 0$.

We separate $B'(x)B(x) = (1-\varepsilon_1)B'(x)B(x) + \varepsilon_1 B'(x)B(x)$ and let the following terms in (2.6)

$$I_1 := -(1-\varepsilon_1)B'(x)B(x) + B'(x)\ln(v) - B'(x)\frac{v-2\delta}{v} - A^\tau(x)\mathrm{sgn}(A(x)) - \alpha^\tau(x)\mathrm{sgn}(\alpha(x))u|u|.$$

Case I. At the points (x,t), where $\alpha(x) \geq 0$, $v(x,t) \leq 1$ and $w_1 + 2\int_{-\infty}^x X(s)ds \leq 0$, we have

$$\begin{aligned} I_1 \geq& (1-\varepsilon_1)X(x)\left(M - \int_{-\infty}^x X(s)ds\right) \\ &- \frac{1}{3}X(x) - \frac{1}{4}\alpha^\tau(x)\left(w_1 - z_1 + 2\int_{-\infty}^x X(s)ds\right)\left|w_1 - z_1 + 2\int_{-\infty}^x X(s)ds\right| \\ \geq& -\frac{1}{4}\alpha^\tau(x)\left(w_1 - z_1 + 2\int_{-\infty}^x X(s)ds\right)\left|w_1 - z_1 + 2\int_{-\infty}^x X(s)ds\right| \\ \geq& \frac{1}{4}\alpha^\tau(x)\left|w_1 - z_1 + 2\int_{-\infty}^x X(s)ds\right| z_1. \end{aligned}$$

Case II. At the points (x,t), where $\alpha(x) \geq 0$, $v(x,t) \leq 1$ and $w_1 + 2\int_{-\infty}^x X(s)ds \geq 0$,

$$\begin{aligned} I_1 \geq& (1-\varepsilon_1)X(x)\left(M - \int_{-\infty}^x X(s)ds\right) - \frac{1}{3}X(x) + \frac{1}{4}\alpha^\tau(x)\left|w_1 - z_1 + 2\int_{-\infty}^x X(s)ds\right| z_1 \\ &- \frac{1}{4}\alpha^\tau(x)\left(w_1 + 2\int_{-\infty}^x X(s)ds\right)|z_1| - \frac{1}{4}\alpha^\tau(x)\left(w_1 + 2\int_{-\infty}^x X(s)ds\right)^2 \\ =& (1-\varepsilon_1)X(x)\left(M - \int_{-\infty}^x X(s)ds\right) - \frac{1}{3}X(x) - \alpha^\tau(x)\left(\int_{-\infty}^x X(s)ds\right)^2 \\ &+ d(x,t)z_1 + e(x,t)w_1 \geq d(x,t)z_1 + e(x,t)w_1 \end{aligned} \tag{2.10}$$

where $e(x,t) = -\frac{1}{4}\alpha^\tau(x)\left(w_1 + 4\int_{-\infty}^x X(s)ds\right) \leq 0$, because

$$\begin{aligned} &(1-\varepsilon_1)X(x)\left(M - \int_{-\infty}^x X(s)ds\right) - \frac{1}{3}X(x) - \alpha^\tau(x)\left(\int_{-\infty}^x X(s)ds\right)^2 \\ &\qquad \geq \frac{1}{2}(1-\varepsilon_1)X(x) - \frac{1}{3}X(x) - \frac{1}{12}X(x) \geq 0. \end{aligned}$$

Case III. At the points (x,t), where $\alpha(x) \geq 0$, $v(x,t) > 1$ and $w_1 + 2\int_{-\infty}^x X(s)ds \leq 0$, we have $\frac{v-2\delta}{v} \geq 1 - \varepsilon_2 > 0$ for a small $\varepsilon_2 > 0$, and $B'(x)\ln(v) = -X(x)\left(\frac{1}{2}(w_1 + z_1) + M\right)$. Then,

$$\begin{aligned} I_1 \geq& (1-\varepsilon_1)X(x)\left(M - \int_{-\infty}^x X(s)ds\right) - \frac{1}{2}(w_1 + z_1)X(x) - MX(x) + (1-\varepsilon_2)X(x) - \frac{1}{3}X(x) \\ &+ \alpha^\tau(x)\left|w_1 - z_1 + 2\int_{-\infty}^x X(s)ds\right| z_1 \geq -\frac{1}{2}(w_1 + z_1)X(x) + \alpha^\tau(x)\left|w_1 - z_1 + 2\int_{-\infty}^x X(s)ds\right| z_1 \end{aligned}$$

because

$$(1-\varepsilon_1)X(x)\left(M-\int_{-\infty}^{x}X(s)ds\right)-MX(x)+(1-\varepsilon_2)X(x)-\frac{1}{3}X(x)$$
$$\geq X(x)\left(1-\varepsilon_2-\varepsilon_1 M-\frac{1}{2}-\frac{1}{3}\right)\geq 0$$

for small ε_1 and ε_2.

Case IV. At the points (x,t), where $\alpha(x)\geq 0$, $v(x,t)>1$ and $w_1+2\int_{-\infty}^{x}X(s)ds\geq 0$,

$$I_1\geq(1-\varepsilon_1)X(x)\left(M-\int_{-\infty}^{x}X(s)ds\right)-\frac{1}{2}(w_1+z_1)X(x)-MX(x)+(1-\varepsilon_2)X(x)-\frac{1}{3}X(x)$$
$$-\alpha^\tau(x)\left(\int_{-\infty}^{x}X(s)ds\right)^2+d(x,t)z_1+e(x,t)w_1\geq -\frac{1}{2}(w_1+z_1)X(x)+d(x,t)z_1+e(x,t)w_1,$$

because

$$(1-\varepsilon_1)X(x)\left(M-\int_{-\infty}^{x}X(s)ds\right)-MX(x)+(1-\varepsilon_2)X(x)-\frac{1}{3}X(x)-\alpha^\tau(x)\left(\int_{-\infty}^{x}X(s)ds\right)^2$$
$$\geq X(x)\left(1-\varepsilon_2-\varepsilon_1 M-\frac{1}{2}-\frac{1}{3}-\frac{1}{12}\right)\geq 0,$$

where $d(x,t)$, $e(x,t)$ are given in (2.10). Thus we obtain the proof of the first inequality in (2.9) at the points (x,t), where $\alpha(x)\geq 0$.

Now we prove the second inequality in (2.9). Let the following terms in (2.7),

$$I_2:=(1-\varepsilon_2)C'(x)C(x)-C'(x)\ln(v)+C'(x)\frac{v-2\delta}{v}+A^\tau(x)+\alpha^\tau(x)u|u|.$$

At the points (x,t), where $\alpha(x)\geq 0$ and $v(x,t)\leq 1$, we have

$$I_2\geq(1-\varepsilon_1)X(x)\left(M+\int_{-\infty}^{x}X(s)ds\right)-\frac{1}{3}X(x)+\frac{1}{4}\alpha^\tau(x)\left(w_1-z_1+2\int_{-\infty}^{x}X(s)ds\right)\left|w_1-z_1+2\int_{-\infty}^{x}X(s)ds\right|$$
$$\geq\frac{1}{4}\alpha^\tau(x)\,(w_1-z_1)\left|w_1-z_1+2\int_{-\infty}^{x}X(s)ds\right|;$$

at the points (x,t), where $\alpha(x)\geq 0$ and $v(x,t)>1$,

$$I_2\geq(1-\varepsilon_1)X(x)\left(M+\int_{-\infty}^{x}X(s)ds\right)-\frac{1}{2}(w_1+z_1)X(x)-MX(x)+(1-\varepsilon_2)X(x)-\frac{1}{3}X(x)$$
$$+\frac{1}{4}\alpha^\tau(x)\,(w_1-z_1)\left|w_1-z_1+2\int_{-\infty}^{x}X(s)ds\right|$$
$$\geq-\frac{1}{2}(w_1+z_1)X(x)+\frac{1}{4}\alpha^\tau(x)\,(w_1-z_1)\left|w_1-z_1+2\int_{-\infty}^{x}X(s)ds\right|.$$

Thus we obtain the proof of (2.9) at the points (x,t), where $\alpha(x)\geq 0$. Similarly, we may prove (2.9) also at the points (x,t), where $\alpha(x)\leq 0$.

We now return to the proof of the theorem. Under the conditions given in (1.4), it is clear that $z_1(x,0) \leq 0$, $w_1(x,0) \leq 0$, so, we may apply the maximum principle to (2.9) to obtain the estimates (see [9] for the details)

$$2\delta \leq v^{\delta,\varepsilon,\tau} \leq M_1, \qquad \ln(v^{\delta,\varepsilon,\tau}) - M_2 \leq u^{\delta,\varepsilon,\tau} \leq M_2 - \ln(v^{\delta,\varepsilon,\tau}), \qquad |m^{\delta,\varepsilon,\tau}| \leq M_3, \tag{2.11}$$

where M_i, $i = 1,2,3$ are suitable positive constants, independent of ε, δ, τ and the time t.

By applying the general contracting mapping principle to an integral representation of (2.2), with the help of the lower, positive estimate and the L^∞ estimates given in (2.11), we can obtain the existence and uniqueness of smooth solution of the Cauchy problem (2.2) and (2.3). Applying the convergence frame given in [5] we have the pointwise convergence

$$(v^{\delta,\varepsilon,\tau}(x,t), m^{\delta,\varepsilon,\tau}(x,t)) \to (v(x,t), m(x,t)) \text{ a.e.}, \quad \text{as } \varepsilon, \delta, \tau \to 0$$

or

$$(\rho^{\delta,\varepsilon,\tau}(x,t), (\rho^{\delta,\varepsilon,\tau} u^{\delta,\varepsilon,\tau})(x,t)) \to (\rho(x,t), (\rho u)(x,t)) \text{ a.e.}, \quad \text{as } \varepsilon, \delta, \tau \to 0.$$

Furthermore, in a similar way as given in [10], we may prove that the limit $(\rho(x,t), u(x,t))$ satisfies system (1.1) in the sense of distributions and the Lax entropy condition (1.5). So, we complete the proof of Theorem 1.1.

3. Conclusions

In this paper, we only study the Cauchy problem of the isothermal system, which is corresponding to the adiabatic exponent $\gamma = 1$, in a general nozzle with space-dependent friction $\alpha(x)$. It is more interesting and difficult to study the general adiabatic exponent $\gamma > 1$. We will come back to this topic in a coming article.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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