Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(6): 6406–6421.
DOI: 10.3934/math.2021376
Received: 29 December 2020
Accepted: 06 April 2021
Published: 13 April 2021

Research article

Some new classes of general quasi variational inequalities

Muhammad Aslam Noor[1,*], **Khalida Inayat Noor**[1] **and Bandar B. Mohsen**[2]

[1] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan
[2] Department of Mathematics, College of Science, King Saud University, Riyadh, Saudi Arabia

* **Correspondence:** Email: noormaslam@gmail.com; Tel: +923454027532.

Abstract: In this paper, we introduce and consider some new classes of general quasi variational inequalities, which provide us with unified, natural, novel and simple framework to consider a wide class of unrelated problems arising in pure and applied sciences. We propose some new inertial projection methods for solving the general quasi variational inequalities and related problems. Convergence analysis is investigated under certain mild conditions. Since the general quasi variational inequalities include quasi variational inequalities, variational inequalities, and related optimization problems as special cases, our results continue to hold for these problems. It is an interesting problem to compare these methods with other technique for solving quasi variational inequalities for further research activities.

Keywords: quasi-variational inequality; nonsymmetric boundary value problems; projection operator; inertial methods; convergence
Mathematics Subject Classification: 49J40, 90C33

1. Introduction

Variational inequality theory, which was introduced by Stampacchia [1] in 1964 in potential theory, provides us with a simple, natural, unified, novel and general framework to study an extensive range of unilateral, obstacle, free, moving and equilibrium problems arising in fluid flow through porous media, elasticity, circuit analysis, transportation, oceanography, operations research, finance, economics, and optimization. It is worth mentioning that the variational inequalities can be viewed as a significant and novel generalization of the Riesz-Frechet representation theorem and Lax-Milgram lemma for linear continuous functionals and bifunction functions. For recent developments, see Noor and Noor [2] and the references therein. We would like to emphasize that the origin of the variational inequality theory can be traced back to Euler, Lagrange, Newton and Bernoulli's brothers. It is very simple fact that the minimum of a differentiable convex functions on the convex sets can be characterized by an inequality, which is called the variational inequality. It is amazing that variational inequalities have influenced various areas of pure and applied sciences and are still continue to influence research, see [1–11].

If the convex set in the variational inequalities depends upon the solution explicitly or implicity, then the variational inequality is called quasi variational inequality. Bensoussan and Lions [10] introduced and studied the quasi variational inequalities in the impulse control theory. Quasi-variational inequalities are being used as a mathematical programming tool in modeling various equilibria in economics, operations research, optimization, and regional and transportation science, see [11–20]. It is worth mentioning that the even order and self-adjoint boundary value problems can be studied in the framework of quasi variational inequalities. To overcome these drawback, Noor [21] introduced quasi variational inequalities involving two operators, which are called general quasi variational inequalities. If the convex set-value map is independent of the solution, then the general quasi variational inequalities are equivalent to the general variational inequalities, introduced and studied by Noor [6, 7]. It turned out that odd-order and nonsymmetric obstacle boundary value problems, which arise in oceanography, package industries and other branches of pure and applied sciences, can be studied in the framework of general quasi variational inequalities. General quasi variational inequalities arise as an optimality condition of the differentiable general functions on the convex-valued sets. For recent developments, see [22–25] and the references therein.

Relevant to the variational inequalities, we have the complementarity problems. It is worth mention that Lemke [26] considered the linear complementarity problems in bimatrix equilibrium in finite dimensional space. Clearly the variational inequalities and complementarity problems are quite different from each other. However, Karamardian [27] introduced the generalized complementarity problems and proved the complementarity problems and variational inequalities are equivalent, if the underlying set is a convex cone. This significant result has influenced the research direction. This interplay between these problems is very useful and has been successfully applied to suggest and analyze iterative algorithms for various classes of complementarity problems, see [9, 26–29]. Noor [23, 24] used this alternative formulation to suggest iterative methods for solving the complementarity problems. One of the most difficult and important problems in variational inequalities is the development of efficient numerical methods. Several numerical methods have been developed for solving the variational inequalities and their variant forms. These methods have been extended and modified in numerous ways. Noor [12] proved that the quasi variational inequalities are equivalent to the fixed point problem. This alternative formulation has allowed to consider the existence of a solution, iterative schemes, sensitivity analysis, merit functions and other aspects of the quasi variational inequalities. Noor [12] used this equivalent form to suggest an iterative projection method for solving a class of quasi variational inequalities. Antipin et al. [19] proposed gradient projection and extra gradient methods for finding the solution of quasi variational inequality when the involved operator is strongly monotone and Lipschitz continuous. Mijajlovic et al. [17] and Antipin [30] introduced a more general gradient projection method with strong convergence for solving the quasi variational inequality in a real Hilbert spaces.

It is very important to develop some efficient iterative methods for solving the quasi variational inequalities. Alvarez et al. [31] used the inertial type projection methods for solving variational inequalities. Noor [7] suggested and investigated inertial type projection methods for solving general variational inequalities. These inertial type methods have been modified in various directions for solving variational inequalities and related optimization problems. Jabeen et al. [20, 24, 25] analyzed some inertial projection methods for some classes of general quasi variational inequalities. Convergence analysis of these inertial type methods has been considered under some mild conditions. For more details see [31–36] and reference therein. Recently, Faraci et al. [37] and Jadamba et al. [38] have considered the theory of stochastic variational inequalities, which is another interesting aspect of variational inequalities. For more details, see [39–44] and the references therein.

Motivated and inspired by the recent research activities, we consider and study some new classes of quasi variational inequalities involving two arbitrary two operators, which are called the general quasi variational inequalities. We have shown that nonsymmetric and odd-order obstacle boundary vale problems can be studied in the framework of general quasi variational inequalities. Some special important cases are also discussed. We have shown that the general quasi variational inequalities are equivalent to the fixed point problems. This equivalence is used to propose some new inertial type methods for solving general quasi variational inequalities. These inertial methods include the extragradient method of Koperlevich [39]and double modified projections of Noor [7]. The convergence of the proposed inertial methods is considered. We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate the efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may be extended for other classes of quasi variational inequalities and related optimization problems.

2. Basic definitions and results

Let $\mathtt{K}$ be a set in a real Hilbert space $\mathtt{H}$ with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$.
Let $\mathtt{T},\mathtt{g}:\mathtt{H}\longrightarrow\mathtt{H}$ be nonlinear operators in $\mathtt{H}$. Let $\mathtt{K}:\mathtt{H}\longrightarrow\mathtt{H}$ be a set-valued mapping which, for any element $\mu\in\mathtt{H}$, associates a convex-valued and closed set $\mathtt{K}(\mu)\subset\mathtt{H}$.

We also need the following concepts.

For the nonsymmetric and odd-order problems, many methods have developed by several authors to construct the energy functional of type (2.4) by introducing the‘concept of g-symmetry and g-positivity of the operator g.

Definition 2.1. [45,46]. $\forall u,v\in H$, the operator $T:H\longrightarrow H$ is said to be :
(a). *g-symmetric ,* if and only if,

$$\langle Tu, g(v)\rangle = \langle g(u), Tv\rangle.$$

(b). g-positive, if and only if,

$$\langle Tu, g(u)\rangle \geq 0.$$

(c). *g-coercive (g-elliptic,* if there exists a constant $\alpha>0$ such that

$$\langle Tu, g(u)\rangle \geq \alpha\|g(u)\|^2.$$

Note that g-coercivity implies g-positivity, but the converse is not true. It is also worth mentioning that there are operators which are not g-symmetric but g-positive. On the other hand, there are g-positive, but not g-symmetric operators.

We consider the problem of finding $\mu \in H : g(\mu) \in K(\mu)$, such that

$$\langle T\mu, g(\nu) - g(\mu) \rangle \geq 0, \quad \forall\, \nu \in H : g(\nu) \in K(\mu), \tag{2.1}$$

which is called the general quasi variational inequality.

Applications

To convey an idea of the applications of the quasi variational inequality, we consider the third-order implicit obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -u'''(x(\geq f(x) & \text{on } \Omega = [a,b] \\ u(x) \geq M(u) & \text{on } \Omega = [a,b] \\ [-u'''(x) - f(x)][u - M(u)] = 0 & \text{on } \Omega = [a,b] \\ u(a) = 0, \quad u'(a) = 0, \quad u'(b) = 0. & \end{array}\right\} \tag{2.2}$$

where $f(x)$ is a continuous function and $M(u)$ is the cost (obstacle) function. The prototype encountered is

$$M(u) = k + \inf_i \{u^i\}. \tag{2.3}$$

In (2.3), k represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator M provides the coupling between the unknowns $u = (u^1, u^2, \ldots, u^i)$. We study the problem (2.2) in the framework of general quasi variational inequality approach. To do so, we first define the set K as

$$K(u) = \{v : v \in H_0^2(\Omega) : v \geq M(u), \quad \text{on } \Omega\},$$

which is a closed convex-valued set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev (Hilbert) space, see [4,5,45]. One can easily show that the energy functional associated with the problem (2.2) is

$$\begin{aligned} I[v] &= -\int_a^b \left(\frac{d^2v}{dx^2}\right)\frac{dv}{dx}dx - 2\int_a^b f(x)\frac{dv}{dx})dx, \quad \forall \frac{dv}{dx} \in K(u) \\ &= \int_a^b \left(\frac{dv}{dx}\right)^2 dx - 2\int_a^b f(x)\frac{dv}{dx}dx \\ &= \langle Tv, \frac{dv}{dx}\rangle - 2\langle f, \frac{dv}{dx}\rangle \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \langle Tu, v\rangle &= -\int_a^b \left(\frac{d^3u}{dx^3}\right)\left(\frac{dv}{dx}\right)dx = \int_a^b \frac{du}{dx}\frac{dv}{dx}dx \\ \langle f, \frac{dv}{dx}\rangle &= \int_a^b f(x)\frac{dv}{dx}dx. \end{aligned} \tag{2.5}$$

It is clear that the operator T defined by (2.5) is linear, nonsymmetric and g-positive. Using the technique of Noor [7, 47], one can show that the minimum of the functional $I[v]$ defined by (2.4) associated with the problem (2.2) on the closed convex-valued set $K(u)$ can be characterized by the inequality of type

$$\langle Tu, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in K(u), \tag{2.6}$$

which is exactly the quasi variational inequality (2.1).

Special cases

We now discuss some special cases of general quasi variational inequalities (2.1)

1. If $\mathtt{K}(\mu) = \mathtt{K}$, then problem (2.1) is equivalent to finding $\mu \in \mathtt{H} : \mathtt{g}(\mu) \in \mathtt{K}$ such that

$$\langle\, \mathtt{T}\mu\,,\ \mathtt{g}(\nu) - \mathtt{g}(\mu)\,\rangle \geq 0, \qquad \forall\ \nu \in \mathtt{H} : \mathtt{g}(\nu) \in \mathtt{K}, \tag{2.7}$$

   which is called the general variational inequality, introduced and studied by Noor [6,7,21,23]. It has been shown a wide class of nonsymmetric and odd-order obstacle boundary value and initial value problems can be studied in the general framework of general variational inequalities (2.6). For the applications, numerical methods, sensitivity analysis, dynamical system, merit functions and other aspects of general variational inequalities, see, Noor at al. [6,7,21,23,43,44,47] and the references therein.
2. If $K^*(u) = \{u \in H : \langle u, v\rangle \geq 0, \forall v \in K(u)\}$ is a polar (dual) cone of a convex cone $K(u)$ in H, then problem (2.7) is equivalent to finding $u \in H$ such that

$$g(u) \in K(u), \quad Tu \in K^*(u) \quad \text{and} \quad \langle Tu, g(u)\rangle = 0, \tag{2.8}$$

   which is known as the general quasi complementarity problem and appears to be a new one. If $g = I$, then problem 2.8 is called the implicit complementarity problem, introduced and studied by Noor [13]. For $g(u) = m(u) + K$, where m is a point-to-point mapping, the problem(2.8) is called the implicit (quasi) complementarity problem. If $g \equiv I$, then problem (2.8) is known as the generalized complementarity problems. Such problems have been studied extensively in recent years.
3. For $\mathtt{g} = I$, problem (2.1) is equivalent to finding $\mu \in \mathtt{K}(\mu)$ such that

$$\langle\, \mathtt{T}\mu\,,\ \nu - \mu\,\rangle \geq 0, \qquad \forall\ \nu \in \mathtt{K}(\mu) \tag{2.9}$$

   is known as quasi variational inequality introduced and studied by Bensoussan and Lions [10].
4. If we take $\mathtt{K}(\mu) = \mathtt{K}$, and $\mathtt{g} = I$, the identity operator, then problem (2.1) reduces to the variational inequality: That is, finding $\mu \in \mathtt{K}$, such that

$$\langle\, \mathtt{T}\mu\,,\ \nu - \mu\,\rangle \geq 0, \qquad \forall\ \nu\ \in\ \mathtt{K}. \tag{2.10}$$

   is classical variational inequality. It was introduced and studied by Stampacchia [1].

For a different and appropriate choice of the operators and spaces, one can obtain several known and new classes of variational inequalities and related problems. This clearly shows that the problem (2.1) considered in this paper is more general and unifying one.

We need the following well-known definitions and results in obtaining our results.

Definition 2.2. Let $\mathtt{T} : \mathtt{H} \longrightarrow \mathtt{H}$ be a given mapping.

i. The mapping $\mathtt{T}$ is called r-strongly monotone ($\mathtt{r} \geq 0$), if

$$\langle\, \mathtt{T}\mu - \mathtt{T}\nu\,,\ \mu - \nu\,\rangle \geq \mathtt{r}\,\|\mu - \nu\|^2, \qquad \forall\ \mu,\ \nu\ \in\ \mathtt{H}.$$

ii. The mapping $\mathtt{T}$ is called $\xi-$ cocoercive ($\xi > 0$), if

$$\langle \mathtt{T}\mu - \mathtt{T}\nu\,,\ \mu - \nu \rangle \geq \xi \| \mathtt{T}\mu - \mathtt{T}\nu \|^2, \qquad \forall\, \mu,\ \nu\ \in\ \mathtt{H}.$$

iii. The mapping $\mathtt{T}$ is called relaxed (ξ, $\mathtt{r}$)-cocoercive ($\mathtt{r} > 0,\ \xi > 0$), if

$$\langle \mathtt{T}\mu - \mathtt{T}\nu\,,\ \mu - \nu \rangle \geq\ -\xi \| \mathtt{T}\mu - \mathtt{T}\nu \|^2 + \mathtt{r}\, \|\mu - \nu\|^2, \qquad \forall\, \mu,\ \nu\ \in\ \mathtt{H}.$$

For $\xi = 0$, $\mathtt{T}$ is $\mathtt{r}$-strongly monotone. The class of relaxed (ξ, $\mathtt{r}$)$-$cocoercive mapping is the generalized class than the $r-$strongly monotone mapping and $\xi-$cocoercive.

iv. The mapping $\mathtt{T}$ is called $\eta-$Lipschitz continuous ($\eta > 0$), if

$$\| \mathtt{T}\mu - \mathtt{T}\nu \| \leq\ \eta\, \|\mu - \nu\|, \qquad \forall\, \mu,\ \nu\ \in\ \mathtt{H}.$$

The following projection result plays an indispensable role in achieving our results.

Lemma 2.1. *[12, 14] For a given $\omega \in \mathtt{H}$, find $\mu \in \mathtt{K}(\mu)$, such that*

$$\langle \mu - \omega, \nu - \mu \rangle \geq 0, \quad \forall\, \nu \in \mathtt{K}(\mu)\,,$$

if and only if

$$\mu = \Pi_{\mathtt{K}(\mu)}[\omega]\,,$$

where $\Pi_{\mathtt{K}(\mu)}$ is the implicit projection of $\mathtt{H}$ onto the closed convex-valued set $\mathtt{K}(\mu)$ in $\mathtt{H}$.

The implicit projection operator $\Pi_{\mathtt{K}(\mu)}$ is nonexpansive and has the following characterization.

Assumpstion 2.1. [14] The implicit projection operator $\Pi_{\mathtt{K}(\mu)}$, satisfies the condition

$$\| \Pi_{\mathtt{K}(\mu)}[\omega] - \Pi_{\mathtt{K}(\nu)}[\omega] \| \leq\ \upsilon\ \| \mu - \nu \| \qquad \forall\, \mu,\ \nu\,,\ \omega\ \in\ \mathtt{H}, \tag{2.11}$$

where $\upsilon > 0$, is a constant.

In many important applications, the convex-valued set $K(u)$ is of the form

$$K(u) = m(u) + K,$$

where m is a point-to-point mapping and K is a closed convex set. In this case,

$$P_{K(u)}w = P_{m(u)+K}(w) = m(u) + P_K[w - m(u)], \quad \forall u, w \in H.$$

If m is a Lipschitz continuous with constant ν, then

$$\begin{aligned} \|P_{K(u)}w - P_{K(v)}w\| &= \|m(u) - m(v) + P_K[w - m(u)] - P_K[w - m(v)]\| \\ &\leq \|m(u) - m(v)\| + \|P_K[w - m(u)] - P_K[w - m(v)]\| \\ &\leq 2\|m(u) - m(v)\| \leq 2\nu. \end{aligned}$$

This show that the Assumption 2.1 holds.

Lemma 2.2. *[48] Consider a sequence of non negative real numbers $\{\varrho_n\}$, satisfying*

$$\varrho_{n+1} \leq (1 - \Upsilon_n)\varrho_n + \Upsilon_n \sigma_n + \varsigma_n, \quad \forall\, n \geq 1,$$

where

- ***i.*** $\{\Upsilon_n\} \subset [0,\, 1]$, $\sum_{n=1}^{\infty} \Upsilon_n = \infty$;
- ***ii.*** $\limsup \sigma_n \leq 0$;
- ***iii.*** $\varsigma_n \geq 0\ (n \geq 1)$, $\sum_{n=1}^{\infty} \varsigma_n < \infty$.

Then, $\varrho_n \longrightarrow 0$ as $n \longrightarrow \infty$.

3. Inertial projection methods

In this section, we suggest some new inertial-type approximation schemes for solving the general quasi variational inequality (2.1). One can prove that the general quasi variational inequality (2.1) is equivalent to fixed point problem by using Lemma 2.1.

Lemma 3.1. *The function $\mu \in H : g(\mu) \in K(\mu)$ is solution of general quasi variational inequality (2.1) if and only if $\mu \in H : g(\mu) \in K(\mu)$ satisfies the relation*

$$g(\mu) = \Pi_{K(\mu)}\,[g(\mu) - \rho T\mu]\,, \tag{3.1}$$

where $\rho > 0$ is a constant and $\Pi_{K(\mu)}$ is the projection of H into $K(\mu)$.

Lemma 3.1 implies that the problem (2.1) is equivalent to a fixed point problem (3.1). This alternate form is very useful from both numerical and theoretical point of views.

Using the result (3.1), we can propose some iterative approximation schemes for solving the general quasi variational inequality (2.1).

Algorithm 3.1. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$g(\mu_{n+1}) = \Pi_{K(\mu_n)}\,[g(\mu_n) - \rho T\mu_n]\,, \quad n = 1, 2, \ldots,$$

which is known projection method and has been studied extensively.

Algorithm 3.2. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$g(\mu_{n+1}) = \Pi_{K(\mu_n)}\,[g(\mu_n) - \rho T\mu_{n+1}]\,, \quad n = 1, 2, \ldots,$$

which is known as the extragradient projection method in the sense of Koperlevich sense [39] and is equivalent to the following two-step method.

Algorithm 3.3. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} g(\omega_n) &= \Pi_{K(\mu_n)}\,[g(\mu_n) - \rho T\mu_n] \\ g(\mu_{n+1}) &= \Pi_{K(\mu_n)}\,[g(\mu_n) - \rho T\omega_n]\,, \quad n = 1, 2, \ldots, \end{aligned}$$

which is called the predictor-corrector method.

Using the Eq (3.1), Noor [7] suggested the following double projection method:

Algorithm 3.4. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$g(\mu_{n+1}) = \Pi_{K(\mu_n)}[g(\mu_{n+1}) - \rho T\mu_{n+1}], \quad n = 1, 2, \dots,$$

Using the predictor-corrector technique, Algorithm 3.4 be written in the following form

Algorithm 3.5. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} g(\omega_n) &= \Pi_{K(\mu_n)}[g(\mu_n) - \rho T\mu_n] \\ g(\mu_{n+1}) &= \Pi_{K(\mu_n)}[g(\omega_n) - \rho T\omega_n], \quad n = 1, 2, \dots, \end{aligned}$$

Algorithm 3.5 appeared to be a new two-step method for solving general quasi variational inequality (2.1).

We can rewrite (3.1) as

$$\begin{aligned} g(\mu) &= \Pi_{K(\mu)}\{g(\mu) - \rho T\mu - \Theta(g(\mu) - g(\mu))\}, \\ &= \Pi_{K(\mu)}\{(1 - \Theta)g(\mu) + \Theta g(\mu) - \rho T\mu\}, \end{aligned}$$

where $\Theta_n \in [0, 1]$, $\forall n \geq 1$.

This fixed point formulation is used to suggest the following two-step method for solving general quasi variational inequality (2.1) using the ideas of Alvarez [21], Alvarez et al. [32] and Noor [7].

Algorithm 3.6. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ g(\mu_{n+1}) &= \Pi_{K(\mu_n)}[g(\omega_n) - \rho T\mu_n], \quad n = 1, 2, \dots, \end{aligned}$$

where $\Theta_n \in [0, 1]$, $\forall\, n \geq 1$.

Such type of inertial projection methods for solving general variational inequalities have been considered by Noor [7] and Noor et al. [7–9].

Using this technique, we can suggest the following inertial type methods for solving general quasi variational inequalities (2.1).

Algorithm 3.7. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ g(\mu_{n+1}) &= \Pi_{K(\omega_n)}[g(\omega_n) - \rho T\omega_n], \quad n = 1, 2, \dots, \end{aligned}$$

where $\Theta_n \in [0, 1]$, $for\ all\ n \geq 1$.

Algorithm 3.7 is known as modified inertial method for solving inequality (2.1).

Algorithm 3.8. For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation

$$\begin{aligned} \omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}) \\ g(y_n) &= \Pi_{K(\omega_n)}[g(\omega_n) - \rho T\omega_n], \\ g(\mu_{n+1}) &= \Pi_{K(y_n)}[g(y_n) - \rho Ty_n], \quad n = 1, 2, \dots, \end{aligned}$$

where $\Theta_n \in [0, 1], \forall n \geq 1$.

Algorithm 3.8 is a three-step modified inertial method for solving inequality (2.1).
We now suggest a four-step inertial method for solving the general quasi variational inequality (2.1).

Algorithm 3.9. For given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the recurrence relation

$$\omega_n = \mu_n - \Theta_n (\mu_n - \mu_{n-1}), \tag{3.2}$$
$$\mathtt{x}_n = (1-\gamma_n)\mu_n + \gamma_n\{\omega_n - \mathtt{g}(\omega_n) + \Pi_{\mathtt{K}(\omega_n)}[\mathtt{g}(\omega_n) - \rho\mathtt{T}\omega_n]\}, \tag{3.3}$$
$$\mathtt{y}_n = (1-\beta_n)\mu_n + \beta_n\{\mathtt{x}_n - \mathtt{g}(\mathtt{x}_n) + \Pi_{\mathtt{K}(\mathtt{x}_n)}[\mathtt{g}(\mathtt{x}_n) - \rho\mathtt{T}\mathtt{x}_n]\}, \tag{3.4}$$
$$\mu_{n+1} = (1-\alpha_n)\mu_n + \alpha_n\{\mathtt{y}_n - \mathtt{g}(\mathtt{y}_n) + \Pi_{\mathtt{K}(\mathtt{y}_n)}[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n]\}, \quad n = 1,2,\ldots, \tag{3.5}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1.$

If $g = I$, the identity, then Algorithm (3.9) reduces to:

Algorithm 3.10. For given $\mu_0, \mu_1 \in \mathtt{H}$, compute μ_{n+1} by the recurrence relation

$$\begin{aligned}
\omega_n &= \mu_n - \Theta_n (\mu_n - \mu_{n-1}), \\
\mathtt{x}_n &= (1-\gamma_n)\mu_n + \gamma_n\{\omega_n - (\omega_n) + \Pi_{\mathtt{K}(\omega_n)}[(\omega_n) - \rho\mathtt{T}\omega_n]\}, \\
\mathtt{y}_n &= (1-\beta_n)\mu_n + \beta_n\{\mathtt{x}_n - (\mathtt{x}_n) + \Pi_{\mathtt{K}(\mathtt{x}_n)}[(\mathtt{x}_n) - \rho\mathtt{T}\mathtt{x}_n]\}, \\
\mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{\mathtt{y}_n - (\mathtt{y}_n) + \Pi_{\mathtt{K}(\mathtt{y}_n)}[(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n]\}, \quad n = 1,2,\ldots,
\end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1.$

For a different and suitable choice of operators and spaces in Algorithm (3.9), one can obtain numerous new and previous iterative schemes for solving inequality (2.1) and related problems. This shows that the Algorithm (3.9) is quite flexible and unifying ones.

4. Convergence analysis

In this section, we estimate convergence analysis for Algorithm 3.9 under some mild and appropriate conditions.

Theorem 4.1. *Let the following assumptions be fulfilled:*

i. $\mathtt{K}(\mu) \subset \mathtt{H}$ *be a nonempty, closed and convex-valued subset of Hilbert space* $\mathtt{H}$.

ii. *The operators* $\mathtt{T}, \mathtt{g} : \mathtt{H} \longrightarrow \mathtt{H}$ *be relaxed* $(\xi_1, \mathtt{r}_1), (\xi_2, \mathtt{r}_2)$, *−cocoercive and* η_1, η_2*−Lipschitz continuous, respectively.*

iii. *Assumption 2.1 holds.*

iv. *The parameter* $\rho > 0$ *satisfies the condition*

$$\left|\rho - \frac{(\mathtt{r}_1 - \xi_1\eta_1^2)}{\eta_1^2}\right| < \frac{\sqrt{(\mathtt{r}_1 - \xi_1\eta_1^2)^2 - \eta_1^2\mathtt{k}(2-\mathtt{k})}}{\eta_1^2},$$
$$\mathtt{r}_1 > \xi_1\eta_1^2 + \eta_1\sqrt{\mathtt{k}(2-\mathtt{k})},\ \mathtt{k} < 1, \tag{4.1}$$

where

$$\mathtt{k} = 2\sqrt{1 - 2(\mathtt{r}_2 - \xi_2\eta_2^2) + \eta_2^2} + \upsilon. \tag{4.2}$$

v. *Let* $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1]$, *for all* $n \geq 1$ *such that* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

$\sum_{n=1}^{\infty} \Theta_n \parallel \mu_n - \mu_{n-1} \parallel < \infty$.

Then, for every initial approximation μ_n*, the sequence* $\{\mu_n\}$ *obtained from the iterative scheme defined in Algorithm 3.9 converges to unique solution* $\mu \in \mathtt{H} : \mathtt{g}(\mu) \in \mathtt{K}(\mu)$ *satisfying the general quasi variational inequality (2.1) as* $n \longrightarrow \infty$.

Proof. Let $\mu \in \mathtt{H} : \mathtt{g}(\mu) \in \mathtt{K}(\mu)$ be a solution of (2.1). Then

$$\mu = (1-\alpha_n)\mu + \alpha_n\{\mu - \mathtt{g}(\mu) + \Pi_{\mathtt{K}(\mu)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu]\}, \tag{4.3}$$

$$= (1-\beta_n)\mu + \beta_n\{\mu - \mathtt{g}(\mu) + \Pi_{\mathtt{K}(\mu)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu]\}, \tag{4.4}$$

$$= (1-\gamma_n)\mu + \gamma_n\{\mu - \mathtt{g}(\mu) + \Pi_{\mathtt{K}(\mu)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu]\}, \tag{4.5}$$

where $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1, \forall n \geq 1$, are constants.

Using Assumption (2.1), from (3.6) and (4.3), we have

$$\begin{aligned}
||\mu_{n+1} - \mu|| &= ||(1-\alpha_n)\mu_n + \alpha_n\{\mathtt{y}_n - \mathtt{g}(\mathtt{y}_n) + \Pi_{\mathtt{K}(\mathtt{y}_n)}[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n]\} \\
&\quad - (1-\alpha_n)\mu - \alpha_n\{\mu - \mathtt{g}(\mu) + \Pi_{\mathtt{K}(\mu)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu]\}|| \\
&\leq (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n||\Pi_{\mathtt{K}(\mathtt{y}_n)}[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n] - \Pi_{\mathtt{K}(\mu)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu]|| \\
&\leq (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n||\Pi_{\mathtt{K}(\mathtt{y}_n)}[\mathtt{g}(\mathtt{y}_n) - \rho\mathtt{T}\mathtt{y}_n] - \Pi_{\mathtt{K}(\mathtt{y}_n)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu]|| \\
&\quad + \alpha_n||\Pi_{\mathtt{K}(\mathtt{y}_n)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu] - \Pi_{\mathtt{K}(\mu)}[\mathtt{g}(\mu) - \rho\mathtt{T}\mu]|| \\
&\leq (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n||[\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)] - \rho[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu]|| + \alpha_n\upsilon||\mathtt{y}_n - \mu|| \\
&= (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n|| - (\mathtt{y}_n - \mu) + [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)] + (\mathtt{y}_n - \mu) - \rho[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu]|| \\
&\quad + \alpha_n\upsilon||\mathtt{y}_n - \mu|| \\
&\leq (1-\alpha_n)||\mu_n - \mu|| + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| \\
&\quad + \alpha_n||\mathtt{y}_n - \mu - [\mathtt{g}(\mathtt{y}_n) - \mathtt{g}(\mu)]|| + \alpha_n||\mathtt{y}_n - \mu - \rho[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu]|| \\
&\quad + \alpha_n\upsilon||\mathtt{y}_n - \mu||.
\end{aligned} \tag{4.6}$$

From the relaxed $(\xi_1, \mathtt{r}_1)$−cocoercive and η_1−Lipschitzian definition for operator $\mathtt{T}$, we have

$$\begin{aligned}
&||\mathtt{y}_n - \mu - \rho[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu]||^2 \\
&= ||\mathtt{y}_n - \mu||^2 - 2\rho\langle \mathtt{T}\mathtt{y}_n - \mathtt{T}\mu, \mathtt{y}_n - \mu\rangle + \rho^2||[\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu||^2 \\
&\leq ||\mathtt{y}_n - \mu||^2 + 2\rho\xi_1||\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu||^2 - 2\rho\mathtt{r}_1||\mathtt{y}_n - \mu||^2 + \rho^2||\mathtt{T}\mathtt{y}_n - \mathtt{T}\mu||^2 \\
&\leq ||\mathtt{y}_n - \mu||^2 + 2\rho\xi_1\eta_1^2||\mathtt{y}_n - \mu||^2 - 2\rho\mathtt{r}_1||\mathtt{y}_n - \mu||^2 + \rho^2\eta_1^2||\mathtt{y}_n - \mu||^2 \\
&= (1 - 2\rho(\mathtt{r}_1 - \xi_1\eta_1^2) + \rho^2\eta_1^2)||\mathtt{y}_n - \mu||^2.
\end{aligned} \tag{4.7}$$

Similarly, from the relaxed $(\xi_2, r_2),)-$cocoercive and $\eta_2, -$Lipschitzian definition for operators g, respectively, we have

$$\|y_n - \mu - [g(y_n) - g(\mu)]\|^2 \leq (1 - 2(r_2 - \xi_2\eta_2^2) + \eta_2^2)\|y_n - \mu\|^2, \tag{4.8}$$

From (4.6)–(4.8), , we have

$$\begin{aligned}
&\|\mu_{n+1} - \mu\| \\
&\leq (1 - \alpha_n)\|\mu_n - \mu\| + \alpha_n\Big(\sqrt{1 - 2(r_2 - \xi_2\eta_2^2)} \\
&\quad + \sqrt{1 - 2\rho(r_1 - \xi_1\eta_1^2) + \rho^2\eta_1^2} + \upsilon\Big)\|y_n - \mu\| \\
&= (1 - \alpha_n)\|\mu_n - \mu\| + \alpha_n(k + t(\rho))\|y_n - \mu\| \\
&= (1 - \alpha_n)\|\mu_n - \mu\| + \alpha_n\vartheta\|y_n - \mu\|,
\end{aligned} \tag{4.9}$$

where

$$\vartheta = k + t(\rho) < 1, \quad \text{from condition (4.1)},$$
$$t(\rho) = \sqrt{1 - 2\rho(r_1 - \xi_1\eta_1^2) + \rho^2\eta_1^2}, \text{ and } k \text{ is defined by (4.2).}$$

Similarly, from (3.6) and (4.4), we have

$$\begin{aligned}
\| y_n - \mu \| &= \|(1 - \beta_n)\mu_n + \beta_n\{x_n - g(x_n) + \Pi_{K(x_n)}[g(x_n) - \rho Tx_n]\} \\
&\quad - (1 - \beta_n)\mu - \beta_n\{\mu - g(\mu) + \Pi_{K(\mu)}\| \\
&\leq (1 - \beta_n)\|\mu_n - \mu\| + \beta_n\vartheta\|x_n - \mu\|.
\end{aligned} \tag{4.10}$$

In a similar way, from (3.6) and (4.5), we have

$$\begin{aligned}
\| x_n - \mu \| &= \|(1 - \gamma_n)\mu_n + \gamma_n\{\omega_n - g(\omega_n) + \Pi_{K(\omega_n)}[g(\omega_n) - \rho T\omega_n]\} \\
&\quad - (1 - \gamma_n)\mu - \gamma_n\{\mu - g(\mu) + \Pi_{K(\mu)}[g(\mu) - \rho T\mu]\}\| \\
&\leq (1 - \gamma_n)\|\mu_n - \mu\| + \gamma_n\vartheta\|\omega_n - \mu\|.
\end{aligned} \tag{4.11}$$

From (3.6), we have

$$\begin{aligned}
\|\omega_n - \mu\| &= \|\mu_n - \mu - \Theta_n(\mu_n - \mu_{n-1})\| \\
&\leq \|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|.
\end{aligned} \tag{4.12}$$

From (4.11) and (4.12), we have

$$\begin{aligned}
\| x_n - \mu \| &\leq (1 - \gamma_n)\|\mu_n - \mu\| + \gamma_n\vartheta[\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|] \\
&\leq (1 - \gamma_n)\|\mu_n - \mu\| + \gamma_n\vartheta\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\| \\
&= [1 - \gamma_n(1 - \vartheta)]\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\| \\
&\leq \|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|.
\end{aligned} \tag{4.13}$$

From (4.10) and (4.13), we have

$$\begin{aligned}\| y_n - \mu \| &\le (1-\beta_n)\|\mu_n - \mu\| + \beta_n\vartheta[\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|]\\ &\le (1-\beta_n)\|\mu_n - \mu\| + \beta_n\vartheta\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|\\ &= [1-\beta_n(1-\vartheta)]\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|\\ &\le \|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|. \end{aligned} \tag{4.14}$$

From (4.9) and (4.14), we have

$$\begin{aligned}\| \mu_{n+1} - \mu \| &\le (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\vartheta[\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|]\\ &\le (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\vartheta\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|\\ &= [1-\alpha_n(1-\vartheta)]\|\mu_n - \mu\| + \Theta_n\|\mu_n - \mu_{n-1}\|. \end{aligned}$$

From condition (4.1), we have $\vartheta < 1$. Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, setting $\sigma_n = 0$ and $\varsigma_n = \sum_{n=1}^{\infty}\Theta_n \| \mu_n - \mu_{n-1} \| < \infty$, using Lemma 2.2, we have $\mu_n \longrightarrow \mu$, $n \longrightarrow \infty$. Hence the sequence $\{\mu_n\}$ obtained from Algorithm 3.9 converges to a unique solution $\mu \in H : g(\mu) \in K(\mu)$ satisfying the inequality (2.1), the required result. □

Similarly convergence analysis for other inertial iterative methods can be estimated.

(I.) If $K(\mu) = K$, then the following result can be obtained from Theorem 4.1.

Theorem 4.2. *Let the following assumptions be fulfilled:*

i. K *be a nonempty, closed, and convex set in Hilbert space* H.
ii. *The operators* $T, g : H \longrightarrow H$ *be relaxed* $(\xi_1, r_1), (\xi_2, r_2).)-$*cocoercive and* $\eta_1, \eta_2, -$*Lipschitz continuous, respectively.*
iii. *The parameter* $\rho > 0$ *satisfies the condition*

$$\left|\rho - \frac{(r_1 - \xi_1\eta_1^2)}{\eta_1^2}\right| < \frac{\sqrt{(r_1 - \xi_1\eta_1^2)^2 - \eta_1^2 k(2-k)}}{\eta_1^2},$$
$$r_1 > \xi_1\eta_1^2 + \eta_1\sqrt{k(2-k)},\ k < 1,$$

where

$$k = 2\sqrt{1 - 2(r_2 - \xi_2\eta_2^2) + \eta_2^2}.$$

iv. *Let* $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1]$, *for all* $n \ge 1$ *such that* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

$\sum_{n=1}^{\infty}\Theta_n \| \mu_n - \mu_{n-1} \| < \infty$.

Then, for every initial approximation μ_n, *the sequence* $\{\mu_n\}$ *obtained from*

$$\begin{aligned}
\omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}),\\
x_n &= (1-\gamma_n)\mu_n + \gamma_n\{\omega_n - g(\omega_n) + \Pi_K[g(\omega_n) - \rho T\omega_n]\},\\
y_n &= (1-\beta_n)\mu_n + \beta_n\{x_n - g(x_n) + \Pi_K[g(x_n) - \rho Tx_n]\},\\
\mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{y_n - g(y_n) + \Pi_K[g(y_n) - \rho Ty_n]\}, \quad n = 1, 2, \ldots,
\end{aligned}$$

converges to unique solution $\mu \in H \in K$ *satisfying the extended general variational inequality (2.7) as* $n \longrightarrow \infty$.

(II.) If $K(\mu) = K$ and $g = I$, then we get the following result from Theorem 4.1.

Theorem 4.3. *Let the following assumptions be fulfilled:*

i. K *be a nonempty, closed, and convex set in Hilbert space* H.
ii. *The operator* $T : H \longrightarrow H$ *be relaxed* (ξ_1, r_1)*−cocoercive and* η_1*−Lipschitz continuous, respectively.*
iii. *The parameter* $\rho > 0$ *satisfies the condition*

$$\left|\rho - \frac{(r_1 - \xi_1\eta_1^2)}{\eta_1^2}\right| < \frac{\sqrt{(r_1 - \xi_1\eta_1^2)^2}}{\eta_1^2},$$
$$r_1 > \xi_1\eta_1^2$$

iv. *Let* $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0, 1]$, *for all* $n \geq 1$ *such that* $\sum_{n=1}^{\infty}\alpha_n = \infty$,

$\sum_{n=1}^{\infty}\Theta_n \| \mu_n - \mu_{n-1} \| < \infty.$

Then, for every initial approximation μ_n, *the sequence* $\{\mu_n\}$ *obtained from*

$$\begin{aligned}
\omega_n &= \mu_n - \Theta_n(\mu_n - \mu_{n-1}),\\
x_n &= (1-\gamma_n)\mu_n + \gamma_n\{\omega_n - (\omega_n) + \Pi_K[\omega_n - \rho T\omega_n]\},\\
y_n &= (1-\beta_n)\mu_n + \beta_n\{x_n - (x_n) + \Pi_K[x_n - \rho Tx_n]\},\\
\mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\{y_n - (y_n) + \Pi_K[y_n - \rho Ty_n]\}, \quad n = 1, 2, \ldots,
\end{aligned}$$

converges to unique solution $\mu \in H : \mu \in K$ *satisfying the general variational inequality (2.9) as* $n \longrightarrow \infty$.

Remark 4.1. It is known that the problems (2.1) and (2.8) are equivalent. Consequently, all these results continue to hold for solving the general quasi-complementarity problems and related optimization problems. We would like to point out that very few numerical examples are available for classical quasi variational inequalities due to their complex nature. In spite of these activities, further efforts are needed to develop numerical implementable methods

5. Conclusions

In this paper, several new inertial projection methods have been suggested and analyzed for solving general quasi variational inequalities involving two operators. It is shown that odd-order implicit obstacle problems can be studied in the unified frame work of general quasi variational inequalities. We have proved that the general quasi variational inequalities are equivalent to the implicit fixed point problems. This alternative equivalent formulation is used to suggest a wide class of inertial type iterative methods for solving general quasi variational inequalities. Several important special cases are discussed. Convergence analysis of these proposed inertial projection methods is investigated. It is an interesting problem to compare the efficiency of the proposed methods with other known methods. Similar methods can be suggested for stochastic variational inequalities, which is an interesting and challenging problem. We expect that the ideas and techniques of this paper will motivate and inspire the interested readers to explore its applications in various fields.

Acknowledgments

The authors wish to express their deepest gratitude to our teachers, students, colleagues, collaborators and friends, who have directly or indirectly contributed in the preparation of this paper. The authors extend their appreciation to the International Scientific Partnership Program ISPP at King Saud University for funding this research work through ISPP-125.

Conflict of interest

The authors declare that they have no competing interests.

References

1. G. Stampacchia, Formes bilineaires coercivites sur les ensembles convexes, *C. R. Acad. Sci.*, **258** (1964), 4413–4416.
2. M. A. Noor, K. I. Noor, From representation theorems to variational inequalities, In: N. J. Daras, T. M. Rassiass, *Computational Mathematics and Variational Analysis*, Switzerland: Springer, 2020.
3. M. A. Noor, *On Variational Inequalities,* Ph.D Thesis, Brunel University, London, 1975.
4. R. Glowinski, J. L. Lions, R. Tremolieres, *Numerical Analysis of Variational Inequalities*, New York: North-Holland Publishing Company, 1981.
5. D. Kinderlehrer, G. Stampachia, *An Introduction to Variational Inequalities and Their Applications,* New York: Academic Press, 1980.
6. M. A. Noor, General variational inequalities, *Appl. Math. Lett.*, **1** (1988), 119–122.
7. M. A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.,* **152** (2004), 199–277.
8. M. A. Noor, K. I. Noor, M. Th. Rassias, New trends in general variational inequalities, *Acta Appl. Mathematica*, **170** (2020), 981–1064.
9. M. A. Noor, K. I. Noor, T. M. Rassias, Some aspects of variational inequalities, *J. Comput. Appl. Math.*, **47** (1993), 285–312.

10. A. Bensoussan, J. L. Lions, *Application des Inéqualities Variationnelles en Contrôl Stochastique*, Paris: Bordas(Dunod), 1978.
11. D. Chan, J. Pang, The generalized quasi-variational inequality problem, *Math. Oper. Res.*, **7** (1982), 211–222.
12. M. A. Noor, An iterative scheme for class of quasi variational inequalities, *J. Math. Anal. Appl.*, **110** (1985), 463–468.
13. M. A. Noor, Noor, General quasi complementarity problems, *J. Math. Anal. Appl.*, **120** (1986), 321–327.
14. M. A. Noor, Implicit dynamical systems and quasi variational inequalities, *Appl. Math. Comput.*, **134** (2003), 69–81.
15. M. A. Noor, Merit functions for quasi variational inequalities, *J. Math. Inequal.*, **1** (2007), 259–269.
16. M. Jacimovic, N. Mijajlovic, On a continuous gradient-type method for solving quasi variational inequalities, *Proc. Mont. Acad. Sci Arts*, **19** (2011), 16–27.
17. N. Mijajlovic, J. Milojica, M. A. Noor, Gradient-type projection methods for quasi variational inequalities, *Optim. Lett.*, **13** (2019), 1885–1896.
18. Y. Shehu, A. Gibali, S. Sagratella, Inertial projection-type method for solving quasi variational inequalities in real Hilbert space, *J. Optim. Theory Appl.*, **184** (2020), 877–894.
19. A. S. Antipin, M. Jacimovic, N. Mijajlovic, Extra gradient method for solving quasi variational inequalities, *Optimization*, **67** (2018), 103–112.
20. S. Jabeen, M. A. Noor, K. I. Noor, Some new inertial projection methods for quasi variational inequalities, *Appl. Math. E-Notes*, **2021** (2021).
21. M. A. Noor, Quasi variational inequalities, *Appl. Math. Lett.*, **1** (1988), 367–370.
22. M. A. Noor, W. Oettli, On general nonlinear complementarity problems and quasi equilibria, *Le Mathematiche*, **49** (1994), 313–331.
23. M. A. Noor, On general quasi variational inequalities, *J. King Saud Univ. Sci.*, **24** (2012), 81–88.
24. S. Jabeen, M. A. Noor, K. I. Noor, Inertial iterative methods for general quasi variational inequalities and dynamical systems, *J. Math. Anal.*, **11** (2020), 14–29.
25. S. Jabeen, B. Bin-Mohsen, M. A. Noor, K. I. Noor, Inertial projection methods for solving general quasi-variational inequalities, *AIMS Math.*, **6** (2021), 1075–1086.
26. C. F. Lemke, Bimatrix equilibrium points and mathematical programming, *Manage. Sci.*, **11** (1965), 681–689.
27. R. W. Cottle, J. S. Pang, R. E. Stone, *The Linear Complementarity Problem*, SIAM Publication, 1992.
28. S. Karamardian, Generalized complementarity problem, *J. Optim. Theory Appl.*, **8** (1971), 161–168.
29. K. G. Murty, *Linear Complementarity, Linear and Nonlinear Programming*, Berlin: Heldermann Verlag, 1988.
30. A. S. Antipin, Minimization of convex functions on convex sets by means of differential equations, *Differ. Equations*, **30** (2003), 1365–1357.
31. F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, *SIAM J. Optim.*, **14** (2003), 773–782.

32. F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, *Set-Valued Anal.*, **9** (2001), 3–11.
33. H. Attouch, M. O. Czarnecki, Asymtotic control and stabilization of nonlinear oscillators with non-isolated equilibria, *J. Differ. Equatations*, **179** (2002), 278–310.
34. H. Attouch, X. Goudon, P. Redont, The heavy ball with friction method, I. The continuous dynamical system: Global exploration of the local minima of a real-valued function by asymptotic analysis of a dissipative dynamical system, *Commun. Contemp. Math.*, **2** (2000), 1–34.
35. P. E. Mainge, Regularized and inertial algorithms for common fixed points of nonlinear operators, *J. Math. Anal. Appl.*, **344** (2008), 876–887.
36. Y. Shehu, P. Cholamjiak, Iterative method with inertial for variational inequalities in Hilbert spaces, *Calcolo*, **56** (2019), 4.
37. F. Faraci, B. Jadamba, F. Raciti, On stochastic variational inequalities with mean value constraints, *J. Optim. Theory Appl.*, **171** (2016), 675–693.
38. B. Jadamba, A. A. Khan, F. Raciti, Regularization of stochastic variational inequalities and a comparison of an *Lp* approach and a sample-path approach, *Nonlinear Anal.: Theory Methods Appl.*, **94** (2014), 65–83.
39. G. M. Koperlevich, The extragradient method for finding saddle points and other problems, *Matecom,* **12** (1976), 203–213.
40. B. T. Polyak, Some methods of speeding up the convergence of iterative methods, *USSR Comput. Math. Math. Phys.*, **4** (1964), 1–17.
41. A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, *SIAM J. Imaging Sci.*, **2** (2009), 183–202.
42. A. Bnouhachem, M. A. Noor, Three-step projection method for general variational inequalities, *Int. J. Modern Phys. B*, **26** (2012), 1250066.
43. M. A. Noor, Extended general variational inequalities, *Appl. Math. Lett.*, **22** (2009), 182–185.
44. M. A. Noor, Projection iterative methods for extended general variational inequalities, *J. Appl. Math. Comput.*, **32** (2010), 83–95.
45. V. M. Filippov, *Variational Principles for Nonpotential Operators*, Providence, USA, 1989.
46. E. Tonti, Variational formulation for every nonlinear problem, *Int. J. Eng. Sci.*, **22** (1984), 1343–1371.
47. M. A. Noor, Variational inequalities in physical oceanography, In: M. Rahman, *Ocean Wave Engineering*, Scotland: Printed Bell & Bain Ltd., 1994.
48. H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.*, **66** (2002), 240–256.