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Research article

Ordering results of second order statistics from random and non-random number of random variables with Archimedean copulas

Bin Lu and Rongfang Yan*

College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China

* **Correspondence:** Email: yanrf@nwnu.edu.cn.

Abstract: In this paper, we investigate stochastic comparisons of the second largest order statistics of homogeneous samples coupled by Archimedean copula, and we establish the reversed hazard rate and likelihood ratio orders, and we further generalize the corresponding results to the case of random sample size. Also, we derive some results for relative ageing between parallel systems and 2-out-of-n/2-out-of-$(n + 1)$ systems. Finally, some examples are given to illustrate the obtained results.

Keywords: stochastic orders; archimedean copula; order statistics; random sample size; relative ageing
Mathematics Subject Classification: 60E15, 60K10, 90B25

1. Introduction

Order statistics play an important role in reliability theory, auction theory, operations research, and many applied probability areas. Let $X_{k:n}$ denotes the kth smallest of random variables $X_1, \ldots, X_n$, $k = 1, \ldots, n$. In reliability theory, $X_{k:n}$ characterizes the lifetime of a $(n-k+1)$-out-of-n system, which works if at least $(n-k+1)$ of all the n components function normally. Specifically, $X_{1:n}$ and $X_{n:n}$ denote the lifetimes of series and parallel systems, $X_{2:n}$ and $X_{n-1:n}$ characterize the lifetime of the fail-safe system and 2-out-of-n system, respectively(see Barlow and Proschan [1]). In auction theory, $X_{1:n}$ and $X_{2:n}$ represent the final price of the first-price and second-price procurement auction, $X_{n-1:n}$ and $X_{n:n}$ represent the final price of the first-price and second-price sealed-bid auction (see Fang and Li [2]), respectively.

The groundbreaking work by Boland et al. [3] on the sample from i.i.d. random variables was to study ordering properties between $X_{i-1:n}$ and $X_{i:n+1}$. In the context of $X_k \leq_{hr} X_{n+1}$, they obtained the hazard rate order $X_{i-1:n} \leq_{hr} X_{i:n+1}$, and also proved that $X_{n+1} \leq_{hr} X_k$ implies that $X_{i:n} \geq_{hr} X_{i:n+1}$. Raqab and Amin [4] established the likelihood ratio order between order statistics from samples of different sizes. Bapat and Kochar [5] proved the likelihood ratio ordering between order statistics from a sample with observations arranged in the likelihood ration order. For homogeneous random variables with an Archimedean (survival) copula, Li and Fang [6] derived the hazard rate, the reversed hazard rate and the likelihood ratio ordering of the extramel values and its adjacent order statistics. Subsequently, Fang and Li [7] further developed the reversed hazard rate order and the hazard rate order on sample extremes in the context of proportional reversed hazard models and proportional hazard models, respectively. For heterogenous random variables connected with an Archimedean copula, Mesfioui et al. [8] obtained the ordering properties of the maximum order statistic of the sample and its two adjacent order statistics. Barmalzan et al. [9] established the hazard rate order and reversed hazard rate order of series and parallel systems with dependent components following either modified proportional reversed hazard models or modified proportional hazard models under Archimedean copula. In fact, Pledger and Proschan [10] were the first to deal with the problem of comparing order statistics from heterogeneous exponential random variables. Subsequently, many researchers devoted themselves to stochastic comparisons of order statistics from heterogeneous independent or dependent samples, to name a few, see [11–22].

The notions of relative ageing describe the rate at which one component or system is aging relative to the other. Various partial orders describing relative ageing of two life distributions have been introduced in the literature. Kalashnikov and Rachev [23] introduced a relative aging notion based on the increasing ratio of two hazard rate functions. Rezaei et al. [24] further studied the relative ageing by considering the ratio of two reversed hazard rate functions. Lai and Xie [25] showed that the parallel system with additional redundant components ageing faster in terms of the increasing hazard ratio. Li and Li [26] studied the effect of heterogeneity among independent components on the relative ageing of the series and parallel systems. Ding and Zhang [27] further investigated the effects of Arichimedean dependence and heterogeneity among components on the relative ageing of series and parallel system. For more research on relative ageing, one may refer to [28–32].

In reliability theory, actuarial science and survival analysis, some observations may be lost for unavoidable reasons, and thus it may be impossible to obtain a fixed sample size. Sometimes the sample size may depend on the occurrence of some events, which makes the sample size always random. For example, if a common dose of radiation is given to a sample of animals, then the interest often is in the times that the first and the last expire (see, Consul [33]). There is quite rich literature on the sample with a random size, for example [34–36].

To the best of our knowledge, Li and Fang [6] were the first to study stochastic comparison among $X_{n:n}$ and $X_{n+1:n+1}$ from homogeneous random variables with an Archimedean copula. And the most existing research are focusing on the extreme order statistics, while, there are few works on stochastic comparisons among the second order statistics. In this paper, we will focus on stochastic comparisons of the second largest order statistics from the random and non-random number of homogeneous samples coupled by Archimedean copula, and investigate the impact of sample size and dependence on the the second largest order statistics. Also, some ordering results are derived for relative ageing between parallel systems and 2-out-of-n/2-out-of-$(n+1)$ systems in terms of increasing reversed hazard ratio.

The remainder of this paper is organized as follows: Section 2 recalls some concepts and notations used in this paper. Section 3 presents the results for the case of non-random sample size. Section 4 establishes the results for the case of random sample size. Section 5 provides the application of our main results. Section 6 summarizes our research findings.

2. Preliminaries

In this section, let us first recall some important concepts and notations related to the main results of this article.

For random variable X with support $\mathbb{R}_+ = [0, +\infty)$, let $F_X(x)$ be distribution function ($f_X(x)$ be densities when absolutely continuous), and denote $\bar{F}_X(x) = 1 - F_X(x)$ the reliability function. Let $h_X(x) = f_X(x)/\bar{F}_X(x)$ and $r_X(x) = f_X(x)/F_X(x)$ be the hazard rate function and reversed hazard rate function of X, respectively. The Laplace-Stieltjes transform of X is given by

$$L_X(x) = \int_0^\infty e^{-xt} \mathrm{d}F_X(t).$$

Definition 1. *For two nonnegative random variables X and Y, X is said to be smaller than Y in the*
(i) stochastic order (denoted by $X \leq_{\mathrm{st}} Y$*) if* $\bar{F}_X(x) \leq \bar{F}_Y(x)$ *for all* $x \in \mathbb{R}_+$*;*
(ii) hazard rate order (denoted by $X \leq_{\mathrm{hr}} Y$*) if* $\bar{F}_Y(x)/\bar{F}_X(x)$ *is increasing in* $x \in \mathbb{R}_+$*;*
(iii) reversed hazard rate order (denoted by $X \leq_{\mathrm{rh}} Y$*) if* $F_Y(x)/F_X(x)$ *is increasing in* $x \in \mathbb{R}_+$*;*
(iv) likelihood ratio order (denoted by $X \leq_{\mathrm{lr}} Y$*) if* $f_Y(x)/f_X(x)$ *is increasing in* $x \in \mathbb{R}_+$*;*
(v) Laplace transform ratio order (denoted by $X \leq_{\text{Lt-r}} Y$*) if* $L_Y(x)/L_X(x)$ *is decreasing in* $x \in \mathbb{R}_+$*.*

It is well known that the above stochastic orders have the following relations:

$$X \leq_{st} [\leq_{Lt-r}] Y \Longleftarrow X \leq_{rh} Y \Longleftarrow X \leq_{lr} Y \Longrightarrow X \leq_{hr} Y \Longrightarrow X \leq_{st} Y.$$

For more comprehensive discussions on stochastic orders, please refer to Shaked and Shanthikumar [37], Li and Li [38] and Belzunce et al. [39].

Next, we introduce the concept of relative ageing.

Definition 2. *X is said to be ageing faster than Y in the reversed failure rate, denoted by* $X \prec_b Y$*, if* $r_X(x)/r_Y(x)$ *is decreasing in* $x \in \mathbb{R}_+$*.*

For more details on ageing, one may refer to Lai and Xie [40].

Now, let us review the concept of Archimedean Copulas.

Definition 3. *For a decreasing and continuous function* $\psi : [0, +\infty) \mapsto [0, 1]$ *such that* $\psi(0) = 1$ *and* $\psi(+\infty) = 0$*, let* $\phi = \psi^{-1}$ *be the pseudo-inverse of* ψ*. Then*

$$C_\psi(u_1, ..., u_n) = \psi(\phi(u_1) + ... + \phi(u_n)), \qquad u_i \in [0, 1], \quad i \in \mathcal{I}_n,$$

is said to be an Archimedean copula with generator ψ *if* $(-1)^k \psi^{(k)}(x) \geq 0$ *for* $k = 0, \ldots, n-2$ *and* $(-1)^{n-2}\psi^{(n-2)}(x)$ *is decreasing and convex.*

Copula is used to describe the dependence among random variables and plays an important role in constructing joint distribution through marginal distribution as it does not contain any information of marginal distributions. Archimedean copulas are rather popular because of the mathematical tractability and the capability of capturing wide ranges of dependence. It is well known that the Archimedean family contains a great many useful copulas, including the well-known independence (product) copula, the Clayton copula and the Ali– Mikhail–Haq (AMH) copula. For detailed discussions on copulas and its applications, one may refer to Nelsen [41].

A real-valued function $g : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be supermodular(submodular) if the following inequality

$$g(x_1 \wedge y_1, \ldots, x_n \wedge y_n) + g(x_1 \vee y_1, \ldots, x_n \vee y_n) \geq (\leq) g(x_1, \ldots, x_n) + g(y_1, \ldots, y_n)$$

holds for all $x_i, y_i \in \mathbb{R}$, where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. In particular, a function g with finite second partial derivatives on $\mathbb{R}^n$ is supermodular(submodular) if and only if $\partial^2 g(\boldsymbol{x})/(\partial x_i \partial x_j) \geq (\leq) 0$ for all $1 \leq i \neq j \leq n$ and $\boldsymbol{x} \in \mathbb{R}^n$.

Throughout the manuscript, all concerned random variables are assumed to be absolutely continuous and nonnegative, and the terms increasing and decreasing stand for non-decreasing and non-increasing, respectively. "$\overset{\text{sgn}}{=}$" means equality of sign.

3. Comparison results for the case of non-random sample size

In this section, we study the ordering results of the second largest order statistics of homogeneous sample coupled by Archimedean copula. First, we present the comparison result in the sense of the reversed hazard rate order between $X_{n-1:n}$ and $X_{n:n+1}$.

Theorem 4. *Suppose homogeneous random variables $X_1, X_2, \ldots, X_{n+1}$ having an Archimedean copula with generator ψ. If $t\psi'(t)/\psi(t)$ is convex and $t\psi''(t)/\psi'(t)$ is decreasing, then*

$$X_{n-1:n} \leq_{rh} X_{n:n+1}.$$

Proof. The distribution functions of $X_{n-1:n}$ and $X_{n:n+1}$ can be expressed as

$$F_{X_{n-1:n}}(x) = n\psi((n-1)\phi(F(x))) - (n-1)\psi(n\phi(F(x)))$$

and

$$F_{X_{n:n+1}}(x) = (n+1)\psi(n\phi(F(x))) - n\psi((n+1)\phi(F(x))),$$

respectively. Let $u = F(x)$, to obtain the desired result, it suffices to show that

$$A_1(u) = \frac{F_{X_{n:n+1}}(x)}{F_{X_{n-1:n}}(x)} = \frac{(n+1)\psi(n\phi(u)) - n\psi((n+1)\phi(u))}{n\psi((n-1)\phi(u)) - (n-1)\psi(n\phi(u))}$$

is increasing in $u \in [0, 1]$. Taking the derivative of $A_1(u)$, we have

$$\begin{aligned} A_1'(u) &\overset{\text{sgn}}{=} \phi'(u)\Big((n+1)n\Big(\psi'(n\phi(u)) - \psi'((n+1)\phi(u))\Big)\Big(n\psi((n-1)\phi(u)) - (n-1)\psi(n\phi(u))\Big) \\ &\quad -(n-1)n\Big(\psi'((n-1)\phi(u)) - \psi'(n\phi(u))\Big)\Big((n+1)\psi(n\phi(u)) - n\psi((n+1)\phi(u))\Big)\Big) \\ &= n\phi'(u)\Big((n+1)\Big(n\psi'(n\phi(u))\psi((n-1)\phi(u)) - (n-1)\psi'((n-1)\phi(u))\psi(n\phi(u))\Big) \\ &\quad +(n-1)\Big((n+1)\psi'((n+1)\phi(u))\psi(n\phi(u)) - (n-1)\psi'(n\phi(u))\psi((n+1)\phi(u))\Big) \\ &\quad +n\Big((n-1)\psi'((n-1)\phi(u))\psi((n+1)\phi(u)) - (n+1)\psi'((n+1)\phi(u))\psi((n-1)\phi(u))\Big)\Big) \end{aligned}$$

$$= \quad n\phi'(u)\Big((n+1)\Delta_1(n-1,n;u)+(n-1)\Delta_1(n,n+1;u)-n\Delta_1(n-1,n+1;u)\Big), \qquad (3.1)$$

where, for all (i, j)

$$\Delta_1(i,j;u) = j\psi'(j\phi(u))\psi(i\phi(u)) - i\psi'(i\phi(u))\psi(j\phi(u)).$$

As $t\psi''(t)/\psi'(t)$ is decreasing implies that $t\psi'(t)/\psi(t)$ decreases (c.f. Theorem 3.1 of [6]), then for $i < j$,

$$\Delta_1(i,j;u) \overset{\text{sgn}}{=} \frac{j\phi(u)\psi'(j\phi(u))}{\psi(j\phi(u))} - \frac{i\phi(u)\psi'(i\phi(u))}{\psi(i\phi(u))} < 0.$$

Note that ψ is decreasing and $t\psi'(t)/\psi(t)$ is convex, we have

$$\begin{aligned}
u\Delta_1(n-1,n;u) &= \psi((n-1)\phi(u))\psi(n\phi(u))\left[\frac{n\phi(u)\psi'(n\phi(u))}{\psi(n\phi(u))} - \frac{(n-1)\phi(u)\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))}\right] \\
&\leq \psi((n+1)\phi(u))\psi(n\phi(u))\left[\frac{n\phi(u)\psi'(n\phi(u))}{\psi(n\phi(u))} - \frac{(n-1)\phi(u)\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))}\right] \\
&\leq \psi((n+1)\phi(u))\psi(n\phi(u))\left[\frac{(n+1)\phi(u)\psi'((n+1)\phi(u))}{\psi((n+1)\phi(u))} - \frac{n\phi(u)\psi'(n\phi(u))}{\psi(n\phi(u))}\right] \\
&= u\Delta_1(n,n+1;u).
\end{aligned}$$

Let $l_1(x,y) = \mathbb{I}(x \geq y)(x\psi'(x)\psi(y) - y\psi'(y)\psi(x))$. As $t\psi''(t)/\psi'(t)$ is decreasing, then for $x \geq y$, we have

$$\frac{\partial^2 l_1(x,y)}{\partial x \partial y} = \psi'(x)\psi'(y)\left(\frac{x\psi''(x)}{\psi'(x)} - \frac{y\psi''(y)}{\psi'(y)}\right) \leq 0,$$

which implies that $l_1(x,y)$ is submodular. Therefore, we have

$$\begin{aligned}
&\phi(u)(\Delta_1(n-1,n;u) + \Delta_1(n,n+1;u)) \\
=\ & l_1(nu,(n-1)u) + l_1((n+1)u,nu) \\
\leq\ & l_1(nu,nu) + l_1((n+1)u,(n-1)u) = \phi(u)\Delta_1(n-1,n+1;u).
\end{aligned}$$

Thus,

$$\begin{aligned}
&(n+1)\Delta_1(n-1,n;u) + (n-1)\Delta_1(n,n+1;u) - n\Delta_1(n-1,n+1;u) \\
=\ & (n+1)\Big(\Delta_1(n-1,n;u) - \frac{1}{2}\Delta_1(n-1,n+1;u)\Big) \\
& +(n-1)\Big(\Delta_1(n,n+1;u) - \frac{1}{2}\Delta_1(n-1,n+1;u)\Big) \\
\leq\ & (n-1)\Big(\Delta_1(n-1,n;u) + \Delta_1(n,n+1;u) - \Delta_1(n-1,n+1;u)\Big) \leq 0.
\end{aligned}$$

Then, in combination with the decreasing property of ψ, (3.1) is nonnegative, which implies that $F_{X_{n:n+1}}(x)/F_{X_{n-1:n}}(x)$ is increasing in x. Hence we complete the proof. □

In the following, we establish the likelihood ratio order of the second largest order statistics from n and $n+1$ homogeneous observations, respectively.

Theorem 5. *Suppose homogeneous random variables $X_1, X_2, \ldots, X_{n+1}$ having an Archimedean copula with generator ψ. If $t\psi'''(t)/\psi''(t)$ is decreasing, then*

$$X_{n-1:n} \leq_{lr} X_{n:n+1}.$$

Proof. The density functions of $X_{n-1:n}$ and $X_{n:n+1}$ can be expressed as

$$f_{X_{n-1:n}}(x) = n(n-1)\phi'(F(x))f(x)\Big(\psi'((n-1)\phi(F(x))) - \psi'(n\phi(F(x)))\Big)$$

and

$$f_{X_{n:n+1}}(x) = n(n+1)\phi'(F(x))f(x)\Big(\psi'(n\phi(F(x))) - \psi'((n+1)\phi(F(x)))\Big),$$

respectively. Let $u = F(x)$, it is sufficient to show that

$$A_2(u) \overset{\text{sgn}}{=} \frac{f_{X_{n-1:n}}(x)}{f_{X_{n:n+1}}(x)} = \frac{\psi'((n-1)\phi(u)) - \psi'(n\phi(u))}{\psi'(n\phi(u)) - \psi'((n+1)\phi(u))}$$

is decreasing in $u \in [0, 1]$. Taking the derivative of $A_2(u)$, we have

$$\begin{aligned} A_2'(u) \overset{\text{sgn}}{=}\ & \phi'(u)\Big((n-1)\psi''((n-1)\phi(u))\psi'(n\phi(u)) - n\psi''(n\phi(u))\psi'((n-1)\phi(u))\Big) \\ & +(n+1)\psi''((n+1)\phi(u))\psi'((n-1)\phi(u)) - (n-1)\psi''((n-1)\phi(u))\psi'((n+1)\phi(u))\Big) \\ & +n\psi''(n\phi(u))\psi'((n-1)\phi(u)) - (n+1)\psi''((n+1)\phi(u))\psi'(n\phi(u))\Big) \\ =\ & \phi'(u)\Big(\Delta_2(n-1, n; u) + \Delta_2(n, n+1; u) - \Delta_2(n-1, n+1; u)\Big), \end{aligned} \tag{3.2}$$

where, for all (i, j)

$$\Delta_2(i, j; u) = i\psi''(i\phi(u))\psi'(j\phi(u)) - j\psi''(j\phi(u))\psi'(i\phi(u)).$$

Let $l_2(x, y) = \mathbb{I}(x \leq y)(x\psi''(x)\psi'(y) - y\psi''(y)\psi'(x))$. As $t\psi'''(t)/\psi''(t)$ is decreasing, then for $x \leq y$, we have

$$\frac{\partial^2 l_2(x, y)}{\partial x \partial y} = \psi''(x)\psi''(y)\Big(\frac{x\psi'''(x)}{\psi''(x)} - \frac{y\psi'''(y)}{\psi''(y)}\Big) \geq 0,$$

which implies that $l_2(x, y)$ is supermodular. Then,

$$\begin{aligned} & \phi(u)(\Delta_2(n-1, n; u) + \Delta_2(n, n+1; u)) \\ = \ & l_2((n-1)u, nu) + l_2(nu, (n+1)u) \\ \geq \ & l_2((n-1)u, (n+1)u) + l_2(nu, nu) = \phi(u)\Delta_2(n-1, n+1; u). \end{aligned}$$

That is,

$$\Delta_2(n-1, n; u) + \Delta_2(n, n+1; u) - \Delta_2(n-1, n+1; u) \geq 0.$$

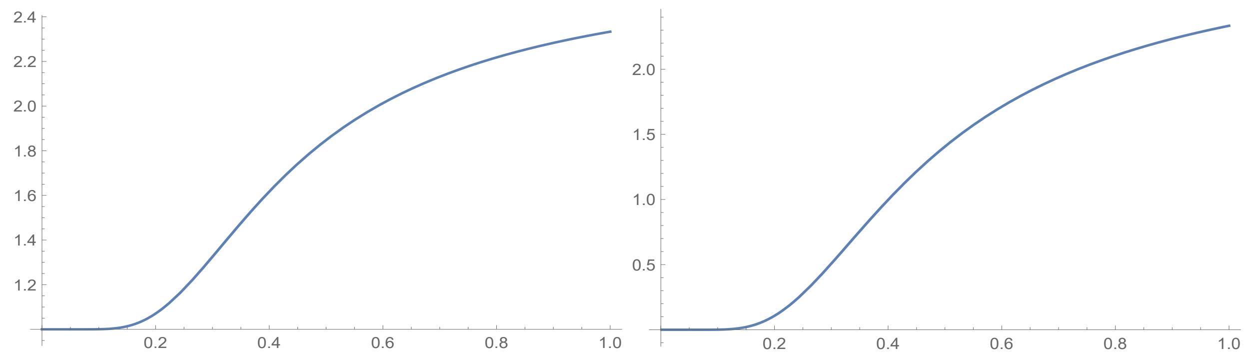
Thus, (3.2) is non-positive, which implies that $f_{X_{n-1:n}}(x)/f_{X_{n:n+1}}(x)$ is decreasing in x. Then the proof is completed. □

The results of Theorem 4 and Theorem 5 are based on Theorem 4(iii) and (iv) in Navarro [17], we present a sufficient condition for the reversed hazard rate and likelihood ratio orders between 2-out-of-$(n+1)$ and 2-out-of-n system. Theorem 4 and Theorem 5 state that 2-out-of-$(n+1)$ system is more reliable than 2-out-of-n system in the sense of the reversed hazard rate and likelihood ratio orders. Now, we present the following example to illustrate the above results.

Example 6. Consider the Clayton copula with generator $\psi(t) = (t+1)^{-1}$, we have

$$\left(\frac{t\psi'(t)}{\psi(t)}\right)'' = \frac{2}{(1+t)^3} \geq 0, \quad \left(\frac{t\psi''(t)}{\psi'(t)}\right)' = \frac{-2}{(1+t)^2} \leq 0, \quad \left(\frac{t\psi'''(t)}{\psi''(t)}\right)' = -\frac{3}{(1+t)^3} \leq 0.$$

Thus, the generator satisfies all conditions of Theorem 4 and Theorem 5. Assume that X_i has common exponential distribution function e^{-x}, $i = 1, 2, 3, 4$. To display the whole of survival curves of $X_{2:3}$ and $Y_{3:4}$ on $[0, \infty)$, we perform the transformation $(x+1)^{-1} : [0, \infty) \longmapsto [0, 1]$. Then, as seen in Figure 1(a), $X_{2:3} \leq_{rh} X_{3:4}$, and Figure 1(b) confirms that $X_{2:3} \leq_{lr} X_{3:4}$.



(a) The curve of $\bar{F}_{(X_{2:3}+1)^{-1}}(x)/\bar{F}_{(X_{3:4}+1)^{-1}}(x)$ on $(0, 1)$. (b) The curve of $f_{(X_{2:3}+1)^{-1}}(x)/f_{(X_{3:4}+1)^{-1}}(x)$ on $(0, 1)$.

Figure 1. The curves of $\bar{F}_{(X_{2:3}+1)^{-1}}(x)/\bar{F}_{(X_{3:4}+1)^{-1}}(x)$ and $f_{(X_{2:3}+1)^{-1}}(x)/f_{(X_{3:4}+1)^{-1}}(x)$.

In the following, in the context of system consist of dependent and homogeneous components, we build the comparison results for the relative ageing between parallel system and 2-out-of-n/2-out-of-$(n+1)$ system. First, we present the relative ageing between parallel and 2-out-of-n systems with respect to the increasing reversed hazard ratio.

Theorem 7. *Suppose homogeneous random variables $X_1, X_2, \ldots, X_n$ having an Archimedean copula with generator ψ. If both $t(\psi''(t)/\psi'(t) - \psi'(t)/\psi(t))$ and $t\psi'(t)/\psi(t)$ are decreasing in $t \geq 0$, then*

$$X_{n-1:n} \prec_b X_{n:n}.$$

Proof. The reversed hazard rate functions of $X_{n-1:n}$ and $X_{n:n}$ can be expressed as

$$r_{X_{n-1:n}}(x) = n(n-1)\phi'(F(x))f(x)\frac{\psi'((n-1)\phi(F(x))) - \psi'(n\phi(F(x)))}{n\psi((n-1)\phi(F(x))) - (n-1)\psi(n\phi(F(x)))}$$

and

$$r_{X_{n:n}}(x) = n\phi'(F(x))f(x)\frac{\psi'(n\phi(F(x)))}{\psi(n\phi(F(x)))}.$$

Let $u = F(x)$, it is sufficient to show that

$$A_3(u) = \frac{r_{X_{n-1:n}}(x)}{r_{X_{n:n}}(x)} \overset{\text{sgn}}{=} \frac{\frac{\psi'\left((n-1)\phi(u)\right)}{\psi'\left(n\phi(u)\right)} - 1}{n\frac{\psi\left((n-1)\phi(u)\right)}{\psi\left(n\phi(u)\right)} - (n-1)}$$

is decreasing in $u \in [0, 1]$. Taking the derivative of $A_3(u)$, we have

$$\begin{aligned}
&A_3'(u)\\
\overset{\text{sgn}}{=}\ & \phi'(u)\left[(n-1)\frac{\psi''((n-1)\phi(u))}{\psi'((n-1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right]\left(n\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))} - (n-1)\right)\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))}\\
& -\phi'(u)\left[(n-1)\frac{\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right]\left(n\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))}\\
=\ & \phi'(u)\left[\left((n-1)\frac{\psi''((n-1)\phi(u))}{\psi'((n-1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right)\left(n\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))} - (n-1)\right)\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))}\right.\\
& \left.-\left((n-1)\frac{\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right)\left(n\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))}\right]\\
=\ & \phi'(u)\Delta_3(u),
\end{aligned}$$

where

$$\begin{aligned}
&\Delta_3(u)\\
=\ & \left((n-1)\frac{\psi''((n-1)\phi(u))}{\psi'((n-1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right)\left(n\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))} - (n-1)\right)\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))}\\
& -\left((n-1)\frac{\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right)\left(n\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))}.
\end{aligned}$$

As $\phi'(u) \le 0$, we just need to show that $\Delta_3(u)$ is nonnegative. Note that

$$\begin{aligned}
& \left(n\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))} - (n-1)\right)\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))} - \left(n\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))}\\
=\ & n\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))} - (n-1)\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))}\\
=\ & \left[n\phi(u)\frac{\psi'(n\phi(u))}{\psi(n\phi(u))} - (n-1)\phi(u)\frac{\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))}\right]\frac{\psi((n-1)\phi(u))}{\phi(u)\psi'(n\phi(u))} \ge 0,
\end{aligned}$$

where the last inequality is due to the assumption that $t\psi'(t)/\psi(t)$ is decreasing in $t \ge 0$ and $\psi'(x) \le 0$. Thus

$$\Delta_3(u)$$

$$
\begin{aligned}
&\geq \left[\left((n-1)\frac{\psi''((n-1)\phi(u))}{\psi'((n-1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right) - \left((n-1)\frac{\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right)\right] \\
&\quad \times\left(n\frac{\psi'((n-1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n-1)\phi(u))}{\psi(n\phi(u))} \\
&\overset{\text{sgn}}{=} \left((n-1)\frac{\psi''((n-1)\phi(u))}{\psi'((n-1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right) - \left((n-1)\frac{\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right) \\
&= (n-1)\phi(u)\left(\frac{\psi''((n-1)\phi(u))}{\psi'((n-1)\phi(u))} - \frac{\psi'((n-1)\phi(u))}{\psi((n-1)\phi(u))}\right) - n\phi(u)\left(\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))} - \frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right) \geq 0,
\end{aligned}
$$

where the last inequality is by the assumption that $t(\psi''(t)/\psi'(t) - \psi'(t)/\psi(t))$ is decreasing in $t \geq 0$. Hence, we complete the proof. □

Next, we establish the relative ageing between parallel system and 2-out-of-$(n+1)$ system in the sense of increasing reversed hazard ratio.

Theorem 8. *Suppose homogeneous random variables $X_1, X_2, \ldots, X_{n+1}$ having an Archimedean copula with generator ψ. If both $t(\psi''(t)/\psi'(t) - \psi'(t)/\psi(t))$ and $t\psi'(t)/\psi(t)$ are decreasing in $t \geq 0$, then*

$$X_{n:n+1} \prec_b X_{n:n}.$$

Proof. The reversed hazard rate functions of $X_{n:n+1}$ and $X_{n:n}$ can be expressed as

$$r_{X_{n:n+1}}(x) = n(n+1)\phi'(F(x))f(x)\frac{\psi'(n\phi(F(x))) - \psi'((n+1)\phi(F(x)))}{(n+1)\psi(n\phi(F(x))) - n\psi((n+1)\phi(F(x)))}$$

and

$$r_{X_{n:n}}(x) = n\phi'(F(x))f(x)\frac{\psi'(n\phi(F(x)))}{\psi(n\phi(F(x)))},$$

respectively. Let $u = F(x)$, it just need to show that

$$A_4(u) = \frac{r_{X_{n:n+1}}(x)}{r_{X_{n:n}}(x)} \overset{\text{sgn}}{=} \frac{\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))} - 1}{n\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))} - (n+1)}$$

is decreasing in $u \in [0,1]$. Taking the derivative of $A_4(u)$, we have

$$
\begin{aligned}
&A_4'(u) \\
&\overset{\text{sgn}}{=} \phi'(u)\left[\left((n+1)\frac{\psi''((n+1)\phi(u))}{\psi'((n+1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right)\left(n\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))} - (n+1)\right)\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))}\right. \\
&\quad \left.-\left((n+1)\frac{\psi'((n+1)\phi(u))}{\psi((n-1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right)\left(n\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))}\right] \\
&= \phi'(u)\Delta_4(u),
\end{aligned}
$$

where

$$\Delta_4(u)$$

$$
\begin{aligned}
&= \left((n+1)\frac{\psi''((n+1)\phi(u))}{\psi'((n+1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right)\left(n\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))} - (n+1)\right)\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))} \\
&\quad -\left((n+1)\frac{\psi'((n+1)\phi(u))}{\psi((n+1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right)\left(n\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))}.
\end{aligned}
$$

As $\phi'(u) \le 0$, it only need to show that $\Delta_4(u)$ is nonnegative. Note that

$$
\begin{aligned}
&\left(n\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))} - (n+1)\right)\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))} - \left(n\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))} \\
&= n\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))} - (n+1)\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))} \\
&= \left[n\phi(u)\frac{\psi'(n\phi(u))}{\psi(n\phi(u))} - (n+1)\phi(u)\frac{\psi'((n+1)\phi(u))}{\psi((n+1)\phi(u))}\right]\frac{\psi((n+1)\phi(u))}{\phi(u)\psi'(n\phi(u))} \le 0,
\end{aligned}
$$

where the last inequality is due to the assumption that $t\psi'(t)/\psi(t)$ is decreasing in $t \ge 0$ and $\psi'(x) \le 0$. Thus

$$
\begin{aligned}
&\Delta_4(u) \\
\ge\ & \left[\left((n+1)\frac{\psi''((n+1)\phi(u))}{\psi'((n+1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right) - \left((n+1)\frac{\psi'((n+1)\phi(u))}{\psi((n+1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right)\right] \\
&\times\left(n\frac{\psi'((n+1)\phi(u))}{\psi'(n\phi(u))} - n\right)\frac{\psi((n+1)\phi(u))}{\psi(n\phi(u))} \\
\overset{\text{sgn}}{=}\ & -\left((n+1)\frac{\psi''((n+1)\phi(u))}{\psi'((n-1)\phi(u))} - n\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))}\right) + \left((n+1)\frac{\psi'((n+1)\phi(u))}{\psi((n+1)\phi(u))} - n\frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right) \\
\overset{\text{sgn}}{=}\ & n\phi(u)\left(\frac{\psi''(n\phi(u))}{\psi'(n\phi(u))} - \frac{\psi'(n\phi(u))}{\psi(n\phi(u))}\right) - (n+1)\phi(u)\left(\frac{\psi''((n+1)\phi(u))}{\psi'((n+1)\phi(u))} - \frac{\psi'((n+1)\phi(u))}{\psi((n+1)\phi(u))}\right) \ge 0,
\end{aligned}
$$

where the last inequality is by the assumption that $t(\psi''(t)/\psi'(t) - \psi'(t)/\psi(t))$ is decreasing in $t \ge 0$. Hence, we complete the proof. □

Theorem 7 and Theorem 8 state that 2-out-of-n system and 2-out-of-$(n+1)$ system ageing faster than parallel system consisting of n components in the sense of the increasing reversed hazard ratio, respectively. The next example demonstrates the above results.

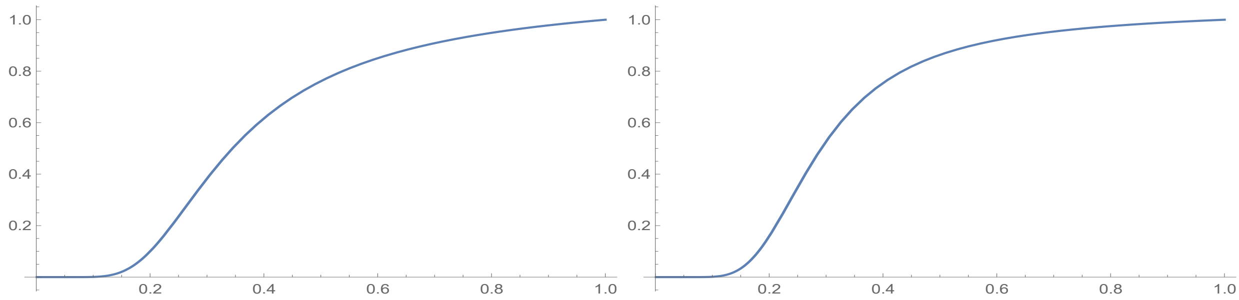
Example 9. *Consider the Clayton copula with generator $\psi(t) = (t+1)^{-1}$, we have*

$$
t\left(\frac{\psi''(t)}{\psi'(t)} - \frac{\psi'(t)}{\psi(t)}\right) = -1 + \frac{1}{1+t}, \quad \left(\frac{t\psi'(t)}{\psi(t)}\right)' = \frac{-1}{(1+t)^2}.
$$

Thus, the generator satisfies the conditions of Theorem 7 and Theorem 8. Assume that X_i has common exponential distribution function e^{-x}, $i = 1,2,3,4,5$. The curves of $h_{(X_{3:4}+1)^{-1}}(x)/h_{(X_{4:4}+1)^{-1}}(x)$ and $h_{(X_{4:5}+1)^{-1}}(x)/h_{(X_{4:4}+1)^{-1}}(x)$ are plotted in Figure 2(a) and Figure 2(b), respectively, from which we can see that $X_{3:4} \prec_b X_{4:4}$ and $X_{4:5} \prec_b X_{4:4}$.

4. Comparison results for the case of random sample size

In this section, we study the ordering results of the second largest order statistics from random number of homogeneous samples coupled by Archimedean copula. First, we present the comparison result in the sense of the reversed hazard rate order between $X_{N_1-1:N_1}$ and $X_{N_2-1:N_2}$.



(a) The curve of $h_{(X_{3:4}+1)^{-1}}(x)/h_{(X_{4:4}+1)^{-1}}(x)$ on $(0, 1)$. (b) The curve of $h_{(X_{4:5}+1)^{-1}}(x)/h_{(X_{4:4}+1)^{-1}}(x)$ on $(0, 1)$.

Figure 2. The curves of $h_{(X_{3:4}+1)^{-1}}(x)/h_{(X_{4:4}+1)^{-1}}(x)$ and $h_{(X_{4:5}+1)^{-1}}(x)/h_{(X_{4:4}+1)^{-1}}(x)$.

Theorem 10. *Suppose homogeneous random variables $X_1, X_2, \ldots$ having an Archimedean copula with generator ψ, and let $N_1(\geq 2)$ and $N_2(\geq 2)$ be positive integer-valued random variables which are independent of X_i, $i = 1, 2, \ldots$. If $t\psi'(t)/\psi(t)$ is convex and $t\psi''(t)/\psi'(t)$ is decreasing, and $N_1 \leq_{Lt-r} N_2$, then*

$$X_{N_1-1:N_1} \leq_{rh} X_{N_2-1:N_2}.$$

Proof. The distribution functions of $X_{N_j-1:N_j}$ can be expressed as

$$\begin{aligned} F_{X_{N_j-1:N_j}}(x) &= P(X_{N_j-1:N_j} \leq x) \\ &= P(X_{N_j-1:N_j} \leq x|N_j = n)P(N_j = n) \\ &= P(X_{n-1:n} \leq x)P(N_j = n) \\ &= \sum_{n=2}^{\infty} F_{X_{n-1:n}}(x)P(N_j = n) \\ &= L_{N_j}(-\log F_{X_{n-1:n}}(x)), \quad j = 1, 2. \end{aligned}$$

As $N_1 \leq_{Lt-r} N_2$ implies that $L_{N_2}(x)/L_{N_1}(x)$ is decreasing in $x > 0$. Note that $-\log F_{X_{n-1:n}}(x)$ is decreasing in $x > 0$, thus

$$\frac{L_{N_2}(-\log F_{X_{n-1:n}}(x))}{L_{N_1}(-\log F_{X_{n-1:n}}(x))}$$

is increasing in $x > 0$. Then we have $X_{N_1-1:N_1} \leq_{rh} X_{N_2-1:N_2}$. Which finishes the proof. □

Shaked and Wong [34] have also shown that $N_1 \leq_{rh} N_2$ implies $N_1 \leq_{Lt-r} N_2$, thus we have the following corollary.

Corollary 11. *Suppose homogeneous random variables $X_1, X_2, \ldots$ having an Archimedean copula with generator ψ, and let $N_1(\geq 2)$ and $N_2(\geq 2)$ be positive integer-valued random variable which are independent of X_i, $i = 1, 2, \ldots$. If $t\psi'(t)/\psi(t)$ is convex and $t\psi''(t)/\psi'(t)$ is decreasing, and $N_1 \leq_{rh} N_2$, then*

$$X_{N_1-1:N_1} \leq_{rh} X_{N_2-1:N_2}.$$

The following theorem establishes the likelihood ratio order between $X_{N_1-1:N_1}$ and $X_{N_2-1:N_2}$.

Theorem 12. *Suppose homogeneous random variables $X_1, X_2, \dots$ having an Archimedean copula with generator ψ, and let $N_1(\geq 2)$ and $N_2(\geq 2)$ be positive integer-valued random variables which are independent of X_i, $i = 1, 2, \dots$. If $t\psi'''(t)/\psi''(t)$ is decreasing, and $N_1 \leq_{lr} N_2$, then*

$$X_{N_1-1:N_1} \leq_{lr} X_{N_2-1:N_2}.$$

Proof. The density function of $X_{N_j-1:N_j}$ can be expressed as

$$f_{X_{N_j-1:N_j}}(x) \quad = \quad \sum_{n=2}^{\infty} f_{X_{n-1:n}}(x) P(N_j = n), \quad j = 1, 2.$$

As $N_1 \leq_{lr} N_2$ implies that

$$\frac{P(N_1 = n)}{P(N_2 = n)} \leq \frac{P(N_1 = n+1)}{P(N_2 = n+1)},$$

thus $P(N_j = n)$ is TP_2 in $n \geq 2$ and $j(j = 1, 2)$. According to Theorem 5, for $x_1 \leq x_2$, we have

$$\frac{f_{X_{n-1:n}}(x_2)}{f_{X_{n:n+1}}(x_2)} \leq \frac{f_{X_{n-1:n}}(x_1)}{f_{X_{n:n+1}}(x_1)},$$

that is, $f_{X_{n-1:n}}(x)$ is TP_2 in $n \geq 2$ and x. Then, by the Theorem 5.1 of Karlin [42], $f_{X_{N_j-1:N_j}}(x)$ is TP_2 in x and $j(j = 1, 2)$. Therefore, $X_{N_1-1:N_1} \leq_{lr} X_{N_2-1:N_2}$. Thus we complete the proof. □

From Corollary 11 and Theorem 12, we obtain the following two corollaries.

Corollary 13. *Suppose homogeneous random variables $X_1, X_2, \dots$ having an Archimedean copula with generator ψ, and let $N(\geq 2)$ be positive integer-valued random variable which is independent of X_i, $i = 1, 2, \dots$. If $t\psi'(t)/\psi(t)$ is convex and $t\psi''(t)/\psi'(t)$ is decreasing, and $\sum_{k=2}^{n} P(N \leq n)/\sum_{k=2}^{n-1} P(N \leq n-1)$ is decreasing in $n \geq 2$, then*

$$X_{N-1:N} \leq_{rh} X_{N:N+1}.$$

Corollary 14. *Suppose homogeneous random variables $X_1, X_2, \dots$ having an Archimedean copula with generator ψ, and let $N(\geq 2)$ be positive integer-valued random variable which is independent of X_i, $i = 1, 2, \dots$. If $t\psi'''(t)/\psi''(t)$ is decreasing, and $P(N = n)/P(N = n-1)$ is decreasing in $n \geq 2$, then*

$$X_{N-1:N} \leq_{lr} X_{N:N+1}.$$

In the following, we present an example which satisfies the condition $P(N = n)/P(N = n-1)$ is decreasing in $n \geq 2$ in Corollary 14.

Example 15. *Suppose positive integer-valued random variable N follows distribution with density function*

$$P(N = k) = \frac{\lambda^{k-2}}{(k-2)!} e^{-\lambda}, \quad k = 2, 3, \dots,$$

where $\lambda > 0$. It is easy to check that

$$\frac{P(N = n)}{P(N = n-1)} = \frac{\lambda}{n-2}$$

is decreasing in $n \geq 2$.

5. Some applications

5.1. Reliability theory

In reliability theory, the k-out-of-n system as the popular fault tolerant system has been widely applied in industrial engineering and system security. Specifically, $X_{1:n}$ and $X_{n:n}$ denote the lifetimes of series and parallel systems, $X_{2:n}$ and $X_{n-1:n}$ characterize the lifetime of the fail-safe system and 2-out-of-n system, respectively. Theorem 4 states that adding a more component to 2-out-of-n system as a redundancy will lead to a more reliable system in the sense of the reversed hazard rate order.

5.2. Auction theory

The second-price sealed-bid auction is of important theoretical and practical interest in auction theory. There are several bidders competing to buy a good, bidders hand in their bids to the auctioneers simultaneously without the knowledge of their rivals' bids. The bidder with the highest bid wins the object and pays the second highest bid in the English auction. Theorem 4 states that attracting one more bidder makes the final price of second-price sealed-bid auction stochastically higher in terms of the reversed hazard rate order.

6. Concluding remarks

In this paper, in the context of system consisting of dependent and homogeneous components, we investigate the problem of stochastic comparisons of the second largest order statistics, and we build the reversed hazard rate and likelihood ratio orders, and we further generalize the corresponding results to the case of random sample size. We also derive some results for relative ageing between parallel systems and 2-out-of-n/2-out-of-$(n + 1)$ systems in terms of the increasing reversed hazard rate order. And we present two applications of the main results. The hazard rate and likelihood ratio orders for the second smallest order statistics can be obtained in a similar method, also, relative ageing between series systems and $(n-1)$-out-of-n/n-out-of-$(n+1)$ systems in terms of the increasing hazard rate order.

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Conflict of interest

The authors declare no conflict of interest.

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