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Research article

On Hermite-Hadamard type inequalities for n-polynomial convex stochastic processes

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Abstract: In this note, our purpose is to introduce the concept of n-polynomial convex stochastic processes and study some of their algebraic properties. We establish new refinements for integral version of Hölder and power mean inequality. Also, we are concerned to extend several Hermite-Hadamard type inequalities for n-polynomial convex stochastic processes by using Hölder, Hölder-İşcan, power mean and improved power mean integral inequalities. Moreover, we give comparison of obtained results.

Keywords: convex stochastic process; n-polynomial; Hölder-İşcan integral inequality; Hermite-Hadamard inequality
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1. Introduction and Preliminaries

In 1980, Nikodem [1] proposed convex stochastic processes and examined the regularity properties of these processes. Skowroński in 1992, obtained some further results on Jensen-convex and Wright-convex stochastic processes [2, 3]. Some interesting properties of convex and Jensen-convex processes are also presented in [4]. Moreover, Kotrys [5] in 2011 developed Hermite-Hadamard type inequalities for convex stochastic processes. In recent years, many studies have been done in the literature on some types of convexity for stochastic processes see [9–11] and Hermite-Hadamard inequalities for related convex stochastic processes [12–18].

A stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be convex if we have

$$X(tc + (1-t)d, .) \leq tX(c, .) + (1-t)X(d, .) \qquad (a.e.),$$

for all $c, d \in I$, $t \in [0, 1]$. X is said to be concave on $I \neq \emptyset$, if above inequality reverses.

The notion of stochastic processes for convexity is of great importance in optimization and also useful for numerical approximations when there exist probabilistic quantities in the literature [19]. In [20], the authors studied stochastic optimization problem under constraints in a general framework including fnancial models with constrained portfolios, labor income and large investor models and reinsurance models (see [21–27] for more details). A constrained stochastic successive convex approximation (CSSCA) algorithm is proposed in [28] to find a stationary point for a general non-convex stochastic optimization problem, whose objective and constraint functions are non-convex and involve expectations over random states. The algorithm solved a sequence of convex objective/feasibility optimization problems obtained by replacing the objective/constraint functions in the original problems with some convex surrogate functions. For more on applications of convex stochastic processes see [29–36].

Let $X : I \times \Omega \to \mathbb{R}$ be the Jensen convex and mean-square continuous in the interval $I \times \Omega$ then

$$X\left(\frac{c+d}{2}, .\right) \leq \frac{1}{d-c} \int_c^d X(x, .)dx \leq \frac{X(c, .) + X(d, .)}{2} \qquad (a.e.), \tag{1.1}$$

for any $c, d \in I$, $c < d$. If the stochastic process X is concave then both inequalities hold in the reversed direction . This double inequality is well known in the literature as Hermite-Hadamard inequality [5]. In this article, we establish new refinements of Hölder and power mean integral inequality. Also, we present the counterpart of the research made by T. Toplu *et al.* in [6] for stochastic processes.
Let us present some important and useful definitions for this research.
Let $(\Omega, \mathcal{A}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a random variable if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is a stochastic process if for every $x \in I$, the function $X(x, .)$ is a random variable. Readers can look at [7, 8] for the defnitions and basic properties.

Definition 1.1. [8] Consider a stochastic process $X(t, .)$ such that the expectation squared is bounded, i.e. $\mathbb{E}[X(t, .)]^2 < \infty$ for all $t \in I$. The stochastic process X is defined:
(i) continuous in probability on I, if

$$\mu - \lim_{c \to c_\circ} X(c, .) = X(c_\circ, .),$$

for all $c_\circ \in I$, where $\mu - \lim$ represents limit in the probability;
(ii) mean-square continuous in I, if

$$\lim_{c \to c_\circ} \mathbb{E}(X(c, .) - X(c_\circ, .))^2 = 0,$$

for all $c_\circ \in I$;
(iii) increasing (decreasing) if for all $c, d \in I$, $c < v$

$$X(c, .) \leq X(d, .); \qquad X(c, .) \geq X(d, .) \qquad (a.e.);$$

(iv) mean-square differentiable at a point $c \in I$, if there is a stochastic process X' (derivative of X) such that

$$X'(c,.) = P - \lim_{c \to c_\circ} \frac{X(c,.) - X(c_\circ,.)}{c - c_\circ}.$$

The stochastic process $X : I \times \Omega \to \mathbb{R}$ is continuous (differentiable) if it is continuous (differentiable) at every point of interval I.

Definition 1.2. [8] Assume a stochastic process $X : I \times \Omega \to \mathbb{R}$ with $\mathbb{E}\left[X(a)^2\right] < \infty$, where $a \in I$. A random variable $Y : \Omega \to \mathbb{R}$ is said to be the mean-square integral of the process X on $[c, d]$ if for every normal sequence of partitions of $[c, d] \subseteq I$, $c = t_\circ < t_1 < ... < t_n = d$ and for all $\Theta_k \in [t_{k-1}, t_k]$, k=1, ..., n, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{k=1}^{n} X(\Theta_k,.)(t_k - t_{k-1}) - Y(.)\right)^2\right] = 0.$$

Then we write

$$\int_c^d X(s,.)ds = Y(.) \qquad (a.e).$$

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process X.

The paper is organized as follow: In section 2, we derive Hölder-İşcan and improved power mean integral inequality, whereas in section 3, we introduce the notion of n-polynomial convex stochastic process and present some interesting properties for these processes. Furthermore, section 4 and section 5 are devoted to state and prove Hermite-Hadamard inequality and some new type of Hermite-Hadamard inequalities respectively for n-polynomial convex stochastic processes.

2. Hölder-İşcan and improved power mean integral inequality

In this section, we establish new refinements of Hölder and power mean integral inequality.

Theorem 2.1. ***(Hölder-İşcan integral inequality)*** *Let* $X, Y : [c, d]\times\Omega \to \mathbb{R}$ *be real stochastic processes and* $|X'|^p$, $|Y'|^p$ *be mean square integrable on* $[c, d]$*. If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then we have almost everywhere:*

$$\int_c^d |X(x,.)Y(x,.)|dx \leq \frac{1}{d-c}\left\{\left(\int_c^d (d-x)|X'(x,.)|^p dx\right)^{\frac{1}{p}} \left(\int_c^d (d-x)|Y'(x,.)|^q dx\right)^{\frac{1}{q}} + \left(\int_c^d (x-c)|X'(x,.)|^p dx\right)^{\frac{1}{p}} \left(\int_c^d (x-c)|Y'(x,.)|^q dx\right)^{\frac{1}{q}}\right\}. \tag{2.1}$$

Proof. Using the Hölder integral inequality, we obtain

$$\int_c^d |X(x,.)Y(x,.)|dx \leq \frac{1}{d-c}\left\{ \int_c^d |(d-x)^{\frac{1}{p}} X(x,.)(d-x)^{\frac{1}{q}} Y(x,.)|dx\right.$$

$$+\int_c^d |(x-c)^{\frac{1}{p}} X(x,.)(x-c)^{\frac{1}{q}} Y(x,.)|dx\Big\}$$

$$\leq \frac{1}{d-c}\Big\{\left(\int_c^d (d-x)|X^{'}(x,.)|^{P} dx\right)^{\frac{1}{p}} \left(\int_c^d (d-x)|Y^{'}(x,.)|^{q} dx\right)^{\frac{1}{q}}$$

$$+\left(\int_c^d (x-c)|X^{'}(x,.)|^{p} dx\right)^{\frac{1}{p}} \left(\int_c^d (x-c)|Y^{'}(x,.)|^{q} dx\right)^{\frac{1}{q}}\Big\}.$$

□

Theorem 2.2. ***(Improved power mean integral inequality)*** *Let $X, Y : [c, d]\times\Omega \to \mathbb{R}$ be real stochastic processes and $|X|$, $|X||Y|^q$ be mean square integrable on $[c, d]$. If $q \geq 1$, then the following inequality holds almost everywhere:*

$$\int_c^d |X(x,.)Y(x,.)|dx \leq \frac{1}{d-c}\Big\{\left(\int_c^d (d-x)|X(x,.)|dx\right)^{1-\frac{1}{q}} \left(\int_c^d (d-x)||X(x,.)|Y^{'}(x,.)|^{q} dx\right)^{\frac{1}{q}}$$

$$+\left(\int_c^d (x-c)|X(x,.)|dx\right)^{1-\frac{1}{q}} \left(\int_c^d (x-c)||X(x,.)|Y(x,.)|^{q} dx\right)^{\frac{1}{q}}\Big\}. \tag{2.2}$$

Proof. Firstly, let $q > 1$. Using Hölder inequality , we easily see that

$$\int_c^d |X(x,.)Y(x,.)|dx = \frac{1}{d-c}\Big\{\int_c^d |(d-x)^{\frac{1}{p}} X^{\frac{1}{p}}(x,.)(d-x)^{\frac{1}{q}} X^{\frac{1}{q}}(x,.)Y(x,.)|dx$$

$$+\int_c^d |(x-c)^{\frac{1}{p}} X^{\frac{1}{p}}(x,.)(x-c)^{\frac{1}{q}} X^{\frac{1}{q}} Y(x,.)|dx\Big\}$$

$$\leq \frac{1}{d-c}\Big\{\left(\int_c^d (d-x)|X(x,.)|dx\right)^{1-\frac{1}{q}} \left(\int_c^d (d-x)|X(x,.)||Y(x,.)|^{q} dx\right)^{\frac{1}{q}}$$

$$+\left(\int_c^d (x-c)|X(x,.)|dx\right)^{1-\frac{1}{q}} \left(\int_c^d (x-c)|X(x,.)||Y(x,.)|^{q} dx\right)^{\frac{1}{q}}\Big\}.$$

For $q = 1$, the inequality (2.2) holds trivially. □

3. Main definition and basic properties

In this section, we introduce our main definition and present some results concerning to the basic properties of n-polynomial convex stochastic processes.

Definition 3.1. Let $X : I \times \Omega \to \mathbb{R}$ be a non-negative stochastic process and $n \in \mathbb{N}$. Then X is said to be n-polynomial convex if

$$X(tc+(1-t)d,.) \leq \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]X(c,.) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k]X(d,.) \quad (a.e.), \tag{3.1}$$

for every $c, d \in I$ and $t \in [0, 1]$.

Remark 3.2. It is worth pointing out that for

1. $n = 1$ in (3.1), the 1-polynomial convexity implies classical convexity.
2. $h(t) = \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]$ in (3.1), we obtain an h-convex stochastic process (see [10]).

Remark 3.3. Let $X : I \times \Omega \to \mathbb{R}$ be a 2-polynomial convex stochastic process, then

$$X(tc+(1-t)d,.) \leq \frac{3t-t^2}{2}X(c,.) + \frac{2-t-t^2}{2}X(d,.) \qquad (a.e.),$$

for every $c, d \in I$ and $t \in [0,1]$. It is clear that for all $t \in [0,1]$

$$t \leq \frac{3t-t^2}{2} \quad and \quad 1-t \leq \frac{2-t-t^2}{2}.$$

This shows that every non-negative convex stochastic process is also a 2-polynomial convex stochastic process. More generally, every non-negative convex stochastic process is also an n-polynomial convex stochastic process.

Proposition 3.4. *Let $X, Y : I \times \Omega \to \mathbb{R}$ be n-polynomial convex stochastic processes and $\gamma \in \mathbb{R}$ ($\gamma \geq 0$), then $X+Y$ and γX are also n-polynomial convex stochastic processes.*

Proof. Consider $c, d \in I$ and $t \in (0,1)$ arbitrary.

$$\begin{aligned}(X+Y)(tc+(1-t)d,.) &= X(tc+(1-t)d,.) + Y(tc+(1-t)d,.)\\ &\leq \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k](X(c,.)+Y(c,.)) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k](X(d,.)+Y(d,.))\\ &= \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k](X+Y)(c,.) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k](X+Y)(d,.)) \qquad (a.e.).\end{aligned}$$

Now consider $\gamma \geq 0$. Then,

$$\begin{aligned}\gamma X(tc+(1-t)d,.) &\leq \gamma\frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]X(c,.) + \gamma\frac{1}{n}\sum_{k=1}^{n}[1-t^k]X(d,.)\\ &= \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]\gamma X(c,.) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k]\gamma X(d,.) \qquad (a.e.).\end{aligned}$$

□

Proposition 3.5. *If $X : I \times \Omega \to \mathbb{R}$ is convex and $Y : I \times \Omega \to \mathbb{R}$ is an n-polynomial convex and non-decreasing stochastic process, then $Y \circ X : I \times \Omega \to \mathbb{R}$ is n-polynomial convex stochastic process.*

Proof. By definition of composition, we have

$$\begin{aligned}(Y \circ X)(tc+(1-t)d,.) &= Y(X(tc+(1-t)d,.))\\ &\leq Y(tX(c,.)+(1-t)X(d,.))\end{aligned}$$

$$\leq \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]Y(X(c,.)) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k]Y(X(d,.))$$
$$= \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k](Y\circ X)(c,.) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k](Y\circ X)(d,.) \qquad (a.e.),$$

which completes the prove. □

Theorem 3.6. *Let $X_\alpha : [a,b]\times\Omega \to \mathbb{R}$, $b > 0$, be an arbitrary collection of n-polynomial convex stochastic processes and define a stochastic process $Y(c,.) = \sup_\alpha X_\alpha(c,.)$. If $J = \{v \in [a,b] : Y(v,.) < \infty\}$is non-empty, then J is an interval and Y is an n-polynomial convex stochastic process on $J\times\Omega$.*

Proof. Let $t\in[0,1]$ and $c,d\in J$ be arbitrary. Then

$$\begin{aligned}
Y(tc+(1-t)d,.) &= \sup_\alpha X_\alpha(tc+(1-t)d,.)\\
&\leq \sup_\alpha\left[\frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]X_\alpha(c,.) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k]X_\alpha(d,.)\right]\\
&\leq \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]\sup_\alpha X_\alpha(c,.) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k]\sup_\alpha X_\alpha(d,.)\\
&= \frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]Y(c,.) + \frac{1}{n}\sum_{k=1}^{n}[1-t^k]Y(d,.)\\
&< \infty.
\end{aligned}$$

This implies that J is an interval and Y is an n-polynomial convex stochastic process on $J\times\Omega$. □

4. Hermite-Hadamard inequality

In this section, we will establish Hermite-Hadamard inequality for n-polynomial convex, mean square integrable stochastic process.

Theorem 4.1. *Let $X : I\times\Omega \to \mathbb{R}$ be an n-polynomial convex, mean square integrable stochastic process. Then for every $c,d\in I$, $c<d$, we have almost everywhere*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)X\left(\frac{c+d}{2},.\right) \leq \frac{1}{d-c}\int_c^d X(x,.)dx \leq \left(\frac{X(c,.)+X(d,.)}{n}\right)\sum_{k=1}^{n}\frac{k}{k+1}. \tag{4.1}$$

Proof. By using inequality (3.1), we have

$$\begin{aligned}
X\left(\frac{c+d}{2},.\right) &= X\left(\frac{[tc+(1-t)d]+[(1-t)c+td]}{2},.\right)\\
&\leq \frac{1}{n}\sum_{k=1}^{n}\left[1-\left(1-\frac{1}{2}\right)^k\right]X(tc+(1-t)d,.) + \frac{1}{n}\sum_{k=1}^{n}\left[1-\left(\frac{1}{2}\right)^k\right]X((1-t)c+td,.)
\end{aligned}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\left[1-\left(\frac{1}{2}\right)^{k}\right][X(tc+(1-t)d,.)+X((1-t)c+td,.)].$$

By integrating last inequality with respect to $t \in [0,1]$, we get

$$\begin{aligned} X\left(\frac{c+d}{2},.\right) &\leq \frac{1}{n}\sum_{k=1}^{n}\left[1-\left(\frac{1}{2}\right)^{k}\right]\left[\int_0^1 X(tc+(1-t)d,.)dt+\int_0^1 X((1-t)c+td,.)dt\right] \\ &= \frac{2}{d-c}\left(\frac{n+2^{-n}-1}{n}\right)\int_c^d X(x,.)dx, \end{aligned}$$

implies

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)X\left(\frac{c+d}{2},.\right) \leq \frac{1}{d-c}\int_c^d X(x,.)dx. \tag{4.2}$$

Now, changing the variable as $x = tc + (1-t)d$, and using the inequality (3.1), we have

$$\begin{aligned} \frac{1}{d-c}\int_c^d X(x,.)dx &= \int_0^1 X(tc+(1-t)d,.)dt \\ &\leq \int_0^1 \left[\frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]X(c,.)+\frac{1}{n}\sum_{k=1}^{n}[1-t^k]X(d,.)\right]dt \\ &= \frac{X(c,.)}{n}\int_0^1 \sum_{k=1}^{n}[1-(1-t)^k]dt+\frac{X(d,.)}{n}\int_0^1 \sum_{k=1}^{n}[1-t^k]dt \\ &= \frac{X(c,.)}{n}\sum_{k=1}^{n}\int_0^1 [1-(1-t)^k]dt+\frac{X(d,.)}{n}\sum_{k=1}^{n}\int_0^1 [1-t^k]dt \\ &= \frac{X(c,.)}{n}\sum_{k=1}^{n}\frac{k}{k+1}+\frac{X(d,.)}{n}\sum_{k=1}^{n}\frac{k}{k+1} \\ &= \frac{X(c,.)+X(d,.)}{n}\sum_{k=1}^{n}\frac{k}{k+1}, \end{aligned} \tag{4.3}$$

where

$$\int_0^1 [1-(1-t)^k]dt = \int_0^1 [1-t^k]dt = \frac{k}{k+1}.$$

Combining (4.2) and (4.3) yields (4.1). □

Remark 4.2. For $n = 1$, the inequality (4.1) coincides with the inequality (1.1).

5. New inequalities of Hermite-Hadamard type

In this section, we purpose to develop new Hermite-Hadamard type inequalities for stochastic processes whose first derivative in absolute value , raised to a certain power is n-polynomial convex stochastic process. In order to prove these inequalities it is necessary to use the following lemma:

Lemma 5.1. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square differentiable stochastic process on I° and X' is mean square integrable on $[c, d]$, where $c, d \in I$, $c < d$. Then we have almost everywhere*

$$\frac{X(c,.) + X(d,.)}{2} - \frac{1}{d-c}\int_c^d X(x,.)dx = \frac{d-c}{2}\int_0^1 (1-2t)X'(tc + (1-t)d,.)dt. \tag{5.1}$$

Proof. Using integration by parts, we get

$$\begin{aligned}\int_0^1 (1-2t)X'(tc+(1-t)d,.)dt &= \frac{X(tc+(1-t)d,.)}{c-d}(1-2t)\Big|_0^1 + 2\int_0^1 \frac{X(tc+(1-t)d,.)}{c-d}dt \\ &= \frac{X(c,.)+X(d,.)}{d-c} - \frac{2}{d-c}.\frac{1}{d-c}\int_c^d X(x,.)dx.\end{aligned}$$

This implies that

$$\frac{d-c}{2}\int_0^1 (1-2t)X'(tc+(1-t)d,.)dt = \frac{X(c,.)+X(d,.)}{2} - \frac{1}{d-c}\int_c^d X(x,.)dx.$$

□

Theorem 5.2. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square differentiable stochastic process on I° and X' be mean square integrable on $[c, d]$. If $|X'|$ is an n-polynomial convex stochastic process on $[c, d]$, then we have almost everywhere:*

$$\left|\frac{X(c,.)+X(d,.)}{2} - \frac{1}{d-c}\int_c^d X(x,.)dx\right| \le \frac{d-c}{n}\sum_{k=1}^{n}\left[\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\right]A(|X'(c,.)|, |X'(d,.)|), \tag{5.2}$$

where A is the arithmetic mean.

Proof. Using Lemma 5.1 and the property of n-polynomial convexity of stochastic process $|X'|$, we have

$$\begin{aligned}\left|\frac{X(c,.)+X(d,.)}{2} - \frac{1}{d-c}\int_c^d X(x,.)dx\right| &= \left|\frac{d-c}{2}\int_0^1 (1-2t)X'(tc+(1-t)d,.)dt\right| \\ &\le \frac{d-c}{2}\int_0^1 |1-2t|\left(\frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]|X'(c,.)| + \frac{1}{n}\sum_{k=1}^{n}[1-t^k]|X'(d,.)|\right)dt \\ &\le \frac{d-c}{2n}\left(|X'(c,.)|\int_0^1 |1-2t|\sum_{k=1}^{n}[1-(1-t)^k]dt + |X'(d,.)|\int_0^1 |1-2t|\sum_{k=1}^{n}[1-t^k]dt\right) \\ &= \frac{d-c}{2n}\left(|X'(c,.)|\sum_{k=1}^{n}\int_0^1 |1-2t|[1-(1-t)^k]dt + |X'(d,.)|\sum_{k=1}^{n}\int_0^1 |1-2t|[1-t^k]dt\right) \\ &= \frac{d-c}{2n}\left(|X'(c,.)|\sum_{k=1}^{n}\left[\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\right] + |X'(d,.)|\sum_{k=1}^{n}\left[\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\right]\right) \\ &= \frac{d-c}{n}\sum_{k=1}^{n}\left[\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\right]\left(\frac{|X'(c,.)|+|X'(d,.)|}{2}\right)\end{aligned}$$

$$= \frac{d-c}{n} \sum_{k=1}^{n} \left[\frac{(k^2+k+2)2^k - 2}{(k+1)(k+2)2^{k+1}} \right] A(|X^{'}(c,.)|, |X^{'}(d,.)|),$$

where A is the arithmetic mean and

$$\int_0^1 |1-2t|[1-(1-t)^k]dt = \int_0^1 |1-2t|[1-t^k]dt = \frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}.$$

□

Corollary 5.3. *For $n = 1$ in (5.2), we obtain the following inequality for convex stochastic processes:*

$$\left| \frac{X(c,.)+X(d,.)}{2} - \frac{1}{d-c} \int_c^d X(x,.)dx \right| \leq \frac{d-c}{4} A(|X^{'}(c,.)|, |X^{'}(d,.)|) \qquad (a.e.).$$

Theorem 5.4. *Let $X : I \times \Omega \to \mathbb{R}$ be a mean square differentiable stochastic process on I° with $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and assume that $X^{'}$ be mean square integrable on $[c, d]$. If $|X^{'}|^q$ is an n-polynomial convex stochastic process on $[c, d]$, then the following inequality holds almost everywhere for $t \in [0, 1]$:*

$$\left| \frac{X(c,.)+X(d,.)}{2} - \frac{1}{d-c} \int_c^d X(x,.)dx \right| \leq \frac{d-c}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{n} \sum_{k=1}^{n} \frac{k}{k+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}}(|X^{'}(c,.)|^q, |X^{'}(d,.)|^q). \tag{5.3}$$

Proof. By using Lemma 5.1, Hölder's integral inequality and the definition of n-polynomial convexity of stochastic process $|X^{'}|^q$, we get

$$\begin{aligned}
\left| \frac{X(c,.)+X(d,.)}{2} - \frac{1}{d-c} \int_c^d X(x,.)dx \right| &\leq \frac{d-c}{2} \int_0^1 |1-2t|X^{'}(tc+(1-t)d,.)dt \\
&\leq \frac{d-c}{2} \left(\int_0^1 |1-2t|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |X^{'}(tc+(1-t)d,.)|^q dt \right)^{\frac{1}{q}} \\
&\leq \frac{d-c}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{|X^{'}(c,.)|^q}{n} \sum_{k=1}^{n} \int_0^1 [1-(1-t)^k]dt + \frac{|X^{'}(d,.)|^q}{n} \sum_{k=1}^{n} \int_0^1 [1-t^k]dt \right)^{\frac{1}{q}} \\
&= \frac{d-c}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(|X^{'}(c,.)|^q \frac{1}{n} \sum_{k=1}^{n} \frac{k}{k+1} + |X^{'}(d,.)|^q \frac{1}{n} \sum_{k=1}^{n} \frac{k}{k+1} \right)^{\frac{1}{q}} \\
&= \frac{d-c}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{2}{n} \sum_{k=1}^{n} \frac{k}{k+1} \right)^{\frac{1}{q}} A^{\frac{1}{q}}(|X^{'}(c,.)|^q, |X^{'}(d,.)|^q),
\end{aligned}$$

where A is the arithmetic mean and

$$\int_0^1 |1-2t|^p dt = \frac{1}{p+1},$$

$$\int_0^1 [1-(1-t)^k]dt = \int_0^1 [1-t^k]dt = \frac{k}{k+1}.$$

□

Corollary 5.5. *For $n = 1$ in (5.3), we obtain the following inequality for convex stochastic processes:*

$$\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right| \le \frac{d-c}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}(|X'(c,.)|^q, |X'(d,.)|^q) \qquad (a.e.).$$

Theorem 5.6. *Let $X : I\times\Omega \to \mathbb{R}$ be a mean square differentiable stochastic process on I° with $q \ge 1$ and X' be mean square integrable on $[c, d]$. If $|X'|^q$ is an n-polynomial convex stochastic process on $[c, d]$, then the following inequality holds almost everywhere for $t \in [0, 1]$:*

$$\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right| \le \frac{d-c}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|X'(c,.)|^q, |X'(d,.)|^q). \tag{5.4}$$

Proof. Assume $q > 1$. By using Lemma 5.1, Hölder's integral inequality and the property of n-polynomial convexity of stochastic process $|X'|^q$, we get

$$\begin{aligned}
\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right| &\le \frac{d-c}{2}\int_0^1 |1-2t|X'(tc+(1-t)d,.)dt\\
&\le\frac{d-c}{2}\left(\int_0^1|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2t||X'(tc+(1-t)d,.)|^q dt\right)^{\frac{1}{q}}\\
&\le\frac{d-c}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg(\int_0^1|1-2t|\Bigg[\frac{1}{n}\sum_{k=1}^{n}[1-(1-t)^k]|X'(c,.)|^q\\
&\quad+\frac{1}{n}\sum_{k=1}^{n}[1-t^k]|X'(d,.)|^q\Bigg]dt\Bigg)^{\frac{1}{q}}\\
&=\frac{d-c}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg[\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\int_0^1|1-2t|[1-(1-t)^k]dt\\
&\quad+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\int_0^1|1-2t|[1-t^k]dt\Bigg]^{\frac{1}{q}}\\
&=\frac{d-c}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\Bigg[\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\\
&\quad+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\Bigg]^{\frac{1}{q}}\\
&=\frac{d-c}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|X'(c,.)|^q, |X'(d,.)|^q),
\end{aligned}$$

where

$$\int_0^1|1-2t|dt=\frac{1}{2},$$

$$\int_0^1|1-2t|[1-(1-t)^k]dt=\int_0^1|1-2t|[1-t^k]dt=\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}.$$

For $q = 1$, we use the estimates from the proof of Theorem 5.2. Thus, the proof of Theorem 5.6 is completed. □

Remark 5.7. If we take $q = 1$ in Theorem 5.6, then we get the conclusion of Theorem 5.2.

Corollary 5.8. *For $n = 1$ in* (5.4)*, we obtain the following inequality for convex stochastic processes:*

$$\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right| \leq \frac{d-c}{4}A^{\frac{1}{q}}(|X^{'}(c,.)|^q, |X^{'}(d,.)|^q) \qquad (a.e.).$$

Now, we will prove the Theorem 5.4 by using Hölder-İşcan integral inequality. Then we will compare the results obtained in this Theorem and Theorem 5.4.

Theorem 5.9. *Let $X : I \times \Omega \to \mathbb{R}$ be mean square differentiable on I° with $c < d$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $X^{'}$ be mean square integrable on $[c, d]$ and $|X^{'}|^q$ be an n-polynomial convex stochastic process on $[c, d]$, then the following inequality holds almost everywhere for $t \in [0, 1]$:*

$$\begin{aligned}&\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right| \qquad (5.5)\\ &\leq \frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X^{'}(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{2(k+2)}+\frac{|X^{'}(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k(k+3)}{2(k+1)(k+2)}\right)^{\frac{1}{q}}\\ &\quad+\frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X^{'}(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k(k+3)}{2(k+1)(k+2)}+\frac{|X^{'}(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{2(k+2)}\right)^{\frac{1}{q}}.\end{aligned}$$

Proof. By using Lemma 5.1, Hölder-İşcan integral inequality and the property of n-polynomial convexity of stochastic process $|X^{'}|^q$, we obtain

$$\begin{aligned}\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right| &\leq \frac{d-c}{2}\int_0^1 |1-2t|X^{'}(tc+(1-t)d,.)dt\\ &\leq \frac{d-c}{2}\left(\int_0^1(1-t)|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)|X^{'}(tc+(1-t)d,.)|^q dt\right)^{\frac{1}{q}}\\ &\quad+\frac{d-c}{2}\left(\int_0^1 t|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t|X^{'}(tc+(1-t)d,.)|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

implies

$$\begin{aligned}&\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right|\\ &\leq \frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X^{'}(c,.)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)[1-(1-t)^k]dt+\frac{|X^{'}(d,.)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)[1-t^k]dt\right)^{\frac{1}{q}}\\ &\quad+\frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X^{'}(c,.)|^q}{n}\sum_{k=1}^{n}\int_0^1 t[1-(1-t)^k]dt+\frac{|X^{'}(d,.)|^q}{n}\sum_{k=1}^{n}\int_0^1 t[1-t^k]dt\right)^{\frac{1}{q}},\end{aligned}$$

which gives

$$\left|\frac{X(c,.)+X(d,.)}{2} - \frac{1}{d-c}\int_c^d X(x,.)dx\right|$$
$$\leq \frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{2(k+2)} + \frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k(k+3)}{2(k+1)(k+2)}\right)^{\frac{1}{q}}$$
$$+ \frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k(k+3)}{2(k+1)(k+2)} + \frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{2(k+2)}\right)^{\frac{1}{q}}$$

where

$$\int_0^1 (1-t)|1-2t|^p dt = \int_0^1 t|1-2t|^p dt = \frac{1}{2(p+1)},$$
$$\int_0^1 (1-t)[1-(1-t)^k]dt = \int_0^1 t[1-t^k]dt = \frac{k}{2(k+2)},$$

and

$$\int_0^1 (1-t)[1-t^k]dt = \int_0^1 t[1-(1-t)^k] = \frac{k(k+3)}{2(k+1)(k+2)}.$$

Hence, the proof of Theorem 5.9 is completed.. □

Corollary 5.10. *For $n = 1$ in* (5.5)*, we obtain the following inequality for convex stochastic processes:*

$$\left|\frac{X(c,.)+X(d,.)}{2} - \frac{1}{d-c}\int_c^d X(x,.)dx\right|$$
$$\leq \frac{d-c}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|X'(c,.)|^q+2|X'(d,.)|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{2|X'(c,.)|^q+|X'(d,.)|^q}{3}\right)^{\frac{1}{q}}\right] \qquad (a.e.).$$

Remark 5.11. The inequality (5.5) gives better result than (5.3). We will prove it by showing that

$$\frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{2(k+2)} + \frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k(k+3)}{2(k+1)(k+2)}\right)^{\frac{1}{q}}$$
$$+ \frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k(k+3)}{2(k+1)(k+2)} + \frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{2(k+2)}\right)^{\frac{1}{q}}$$
$$\leq \frac{d-c}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}(|X'(c,.)|^q, |X'(d,.)|^q).$$

If we use the concavity of the stochastic process $Y : [0,\infty)\times\Omega \to \mathbb{R}$, defined by $Y(u,.) = u^{\lambda}$, $0 < \lambda \leq 1$, we get

$$\frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{2(k+2)} + \frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k(k+3)}{2(k+1)(k+2)}\right)^{\frac{1}{q}}$$

$$+\frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k(k+3)}{2(k+1)(k+2)}+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{2(k+2)}\right)^{\frac{1}{q}}$$
$$\leq \frac{d-c}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}2\left[\frac{1}{2}\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{k+1}+\frac{1}{2}\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right]$$
$$=\frac{d-c}{2}2^{\frac{1}{q}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{k}{k+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|X'(c,.)|^q,|X'(d,.)|^q)$$

which is required.

Theorem 5.12. *Let $X : I\times\Omega\to\mathbb{R}$ be mean square differentiable on I° with $c<d$, $q\geq 1$. Assume that X' be mean square integrable on $[c,d]$ and $|X'|^q$ be an n-polynomial convex stochastic process on $[c,d]$, then the following inequality holds almost everywhere for $t\in[0,1]$:*

$$\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right| \tag{5.6}$$
$$\leq \frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}S_1(k)+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}S_2(k)\right)^{\frac{1}{q}}$$
$$+\frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}S_2(k)+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}S_1(k)\right)^{\frac{1}{q}},$$

where

$$S_1(k)=\int_0^1(1-t)|1-2t|[1-(1-t)^k]dt=\int_0^1 t|1-2t|[1-t^k]dt$$
$$=\frac{(k^2+k+2)2^k-2}{2^{k+2}(k+2)(k+3)},$$
$$S_2(k)=\int_0^1 t|1-2t|[1-(1-t)^k]dt=\int_0^1(1-t)|1-2t|[1-t^k]dt$$
$$=\frac{(k+5)\left[(k^2+k+2)2^k-2\right]}{2^{k+2}(k+1)(k+2)(k+3)}.$$

Proof. Assume first that $q>1$. From Lemma 5.1, improved power-mean integral inequality and the definition of n-polynomial convexity of stochastic process $|X'|^q$, we have

$$\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right|\leq\frac{d-c}{2}\int_0^1|1-2t|X'(tc+(1-t)d,.)dt$$
$$\leq\frac{d-c}{2}\left(\int_0^1(1-t)|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)|1-2t||X'(tc+(1-t)d,.)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{d-c}{2}\left(\int_0^1 t|1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t|1-2t||X'(tc+(1-t)d,.)|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{d-c}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\Big(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)|1-2t|[1-(1-t)^k]dt$$

$$+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\int_0^1(1-t)|1-2t|[1-t^k]dt\Big)^{\frac{1}{q}}$$

$$+\frac{d-c}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\Big(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}\int_0^1 t|1-2t|[1-(1-t)^k]dt$$

$$+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}\int_0^1 t|1-2t|[1-t^k]dt\Big)^{\frac{1}{q}}$$

$$=\frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}S_1(k)+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}S_2(k)\right)^{\frac{1}{q}}$$

$$+\frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}S_2(k)+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}S_1(k)\right)^{\frac{1}{q}}$$

where

$$\int_0^1(1-t)|1-2t|dt=\int_0^1 t|1-2t|dt=\frac{1}{4}.$$

For $q=1$, we use the estimates from the proof of Theorem 5.2. Thus, the proof of Theorem 5.12 is completed. □

Corollary 5.13. *For $n=1$ in* (5.6)*, we obtain the following inequality for convex stochastic processes:*

$$\left|\frac{X(c,.)+X(d,.)}{2}-\frac{1}{d-c}\int_c^d X(x,.)dx\right|$$

$$\leq\frac{d-c}{8}\left[\left(\frac{|X'(c,.)|^q}{4}+\frac{3|X'(d,.)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|X'(c,.)|^q}{4}+\frac{|X'(d,.)|^q}{4}\right)^{\frac{1}{q}}\right]\qquad(a.e.).$$

Remark 5.14. The inequality (5.6) gives better estimate than (5.4). Let us prove it by showing

$$\frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}S_1(k)+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}S_2(k)\right)^{\frac{1}{q}}$$

$$+\frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}S_2(k)+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}S_1(k)\right)^{\frac{1}{q}}$$

$$\leq\frac{d-c}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}}\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|X'(c,.)|^q,|X'(d,.)|^q).$$

By using concavity of the stochastic process $Y:[0,\infty)\times\Omega\to\mathbb{R}$, defined by $Y(u,.)=u^\lambda$, $0<\lambda\leq1$, we obtain

$$\frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}S_1(k)+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}S_2(k)\right)^{\frac{1}{q}}$$

$$+\frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|X'(c,.)|^q}{n}\sum_{k=1}^{n}S_2(k)+\frac{|X'(d,.)|^q}{n}\sum_{k=1}^{n}S_1(k)\right)^{\frac{1}{q}}$$

$$\leq\frac{d-c}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}[S_1(k)+S_2(k)]\right)^{\frac{1}{q}}A^{\frac{1}{q}}(|X'(c,.)|^q,|X'(d,.)|^q),$$

where

$$S_1(k)+S_2(k)=\frac{(k^2+k+2)2^k-2}{(k+1)(k+2)2^{k+1}},$$

which completes the proof of remark.

6. Applications for special means

For two positive numbers c, d with $c < d$, define

$$\begin{aligned}
A(c,d) &= \frac{c+d}{2},\\
G(c,d) &= \sqrt{cd},\\
H(c,d) &= \frac{2cd}{c+d},\\
I(c,d) &= \begin{cases} \frac{1}{e}\left(\frac{d^d}{c^c}\right)^{\frac{1}{d-c}}, & c\neq d\\ c, & c=d\end{cases}
\end{aligned}$$

$$L(c,d)=\begin{cases}\frac{d-c}{lnd-lnc}, & c\neq d\\ c, & c=d,\end{cases}$$

$$L_p(c,d)=\begin{cases}\left(\frac{d^{p+1}-c^{p+1}}{(p+1)(d-c)}\right)^{\frac{1}{p}}, & c\neq d, p\in\mathbb{R}\setminus\{-1,0\}\\ c, & c=d\end{cases}$$

These means are respectively called the arithmetic, geometric, harmonic, identric, logarithmic and p-logarithmic means of two positive numbers c and d.

Applying Theorem 4.1 to $X(u,.)=-lnu$, $u\in(0,1]$ results in the following inequality for means:

Corollary 6.1. *Assume $c, d \in (0,1]$ with $c < d$, then*

$$\frac{1}{2}\left(\frac{m}{m+2^{-m}-1}\right)lnG\leq lnI\leq\frac{lnA}{m}\sum_{k=1}^{m}\frac{k}{k+1}\qquad(a.e.).$$

Applying Theorem 4.1 to $X(u,.)=u^m$, $u\in[0,\infty)$ results in the following inequality for means.

Corollary 6.2. *Assume $m\in(-\infty,0)\cup[1,\infty)\setminus\{-1\}$ and $c,d\in[0,\infty)$ with $c<d$, then*

$$\frac{1}{2}\left(\frac{m}{m+2^{-m}-1}\right)A^n(c,d)\leq L_m^m(c,d)\leq\frac{2}{m}A(c^m,d^m)\sum_{k=1}^{m}\frac{k}{k+1}\qquad(a.e.).$$

Taking $X(u,.) = u^{-1}$, $u \in (0,\infty)$ in Theorem 4.1 results in the inequality for means as follows.

Corollary 6.3. *Assume $c, d \in (0,\infty)$ with $c < d$, then*

$$\frac{1}{2}\left(\frac{m}{m+2^{-m}-1}\right)A^{-1}(c,d) \leq L^{-1}(c,d) \leq \frac{2}{m}H^{-1}(c,d)\sum_{k=1}^{m}\frac{k}{k+1} \qquad (a.e.).$$

7. Conclusions

The more generalized class of convex stochastic processes named as n-polynomial convex stochastic process is introduced in the present note. Hermite-Hadamard inequality and some of its refined forms hold for this new generalization. Some existing results in literature became the particular cases of these results as mentioned in remarks. All the results and inequalities derived here are new, interesting and important in the field of integral inequalities.

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Conflict of interest

The authors declare that they do not have any competing interests.

References

1. K. Nikodem, On convex stochastic processes, *Aequationes Math.*, **20** (1980), 184–197.
2. A. Skowronski, On some properties of j-convex stochastic processes, *Aequationes Math.*, **44** (1992), 249–258.
3. A. Skowronski, On Wright-convex stochastic processes, *Ann. Math. Sil*, **9** (1995), 29–32.
4. Z. Pales, Nonconvex functions and separation by power means, *Math. Inequal. Appl.*, **3** (2000), 169–176.
5. D. Kotrys, Hermite-Hadamard inequality for convex stochastic processes, *Aequationes Math.*, **83** (2012), 143–151.
6. T. Toplu, M. kadakal, I. İşcan, On n-polynomial convexity and some related inequalities, *AIMS Math.*, **5** (2020), 1304–1318.
7. Z. Brzezniak, T. Zastawniak, *Basic stochastic processes: a course through exercises*, Springer Science & Business Media, 2000.
8. K. Sobczyk, *Stochastic differential equations with applications to physics and engineering*, Springer Science & Business Media, 2013.
9. D. Kotrys, Remarks on strongly convex stochastic processes, *Aequationes Math.*, **86** (2013), 91–98.
10. D. Barrez, L. Gonzlez, N. Merentes, A. Moros, On h-convex stochastic processes, *Mathematica Aeterna*, **5** (2015), 571–581.

11. M. Shoaib Saleem, M. Ghafoor, H. Zhou, J. Li, Generalization of h-convex stochastic processes and some classical inequalities, *Math. Probl. Eng.,* **2020** (2020), 1–9.
12. G. Farid, W. Nazeer, M. S. Saleem, S. Mehmood, S. M. Kang, Bounds of Riemann-Liouville fractional integrals in general form via convex functions and their applications, *Mathematics*, **6** (2018), 248.
13. S. Zhao, S. I. Butt, W. Nazeer, J. Nasir, M. Umar, Y. Liu, Some Hermiteensenercer type inequalities for k-Caputo-fractional derivatives and related results, *Adv. Differ. Equ-NY*, **2020** (2020), 1–17.
14. H. Bai, M. S. Saleem, W. Nazeer, M. S. Zahoor, T. Zhao, Hermite-Hadamard-and Jensen-type inequalities for interval nonconvex function, *J. Math.*, **2020** (2020), 1–6.
15. S. M. Kang, G. Farid, W. Nazeer, B. Tariq, Hadamard and Fejadamard inequalities for extended generalized fractional integrals involving special functions, *J. Inequal. Appl.*, **2018** (2018), 1–11.
16. Y. C. Kwun, G. Farid, S. Ullah, W. Nazeer, K. Mahreen, S. M. Kang, Inequalities for a unified integral operator and associated results in fractional calculus, *IEEE Access*, **7** (2019), 126283–126292.
17. X. Z. Yang, G. Farid, W. Nazeer, Y. M. Chu, C. F. Dong, Fractional generalized Hadamard and Fej-Hadamard inequalities for m-convex function, *AIMS Math.*, **5** (2020), 6325–6340.
18. Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah, S. M. Kang, Generalized riemann-liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of hadamard inequalities, *IEEE access*, **6** (2018), 64946–64953.
19. P. Chen, A. Quarteroni, G. Rozza, Stochastic optimal Robin boundary control problems of advection-dominated elliptic equations, *SIAM J. Numer. Anal.*, **51** (2013), 2700–2722.
20. M. Mnif, H. Pham, Stochastic optimization under constraints, *Stoch. Proc. Appl.,* **93** (2001), 149–180.
21. P. Bank, F. Riedel, Optimal consumption choice under uncertainty with intertemporal substitution, *Ann. Appl. Probab.*, **11** (1999), 788.
22. D. Cuoco, Optimal consumption and equilibrium prices with portfolio constraints and stochastic income, *J. Econom. Theory*, **72** (1997), 33–73.
23. D. Cuoco, J. CvitaniCc, Optimal consumption choice for a large investor, *J. Econom. Dyn. Control* , **22** (1998), 401–436.
24. J. Cvitanić, I. Karatzas, Convex duality in convex portfolio optimization, *Ann. Appl. Probab.*, **2** (1992), 767–818.
25. N. El Karoui, M. Jeanblanc, Optimization of consumption with labor income, *Financ. Stoch.*, **4** (1998), 409–440.
26. H. He, H. Pagès, Labor income, borrowing constraints and equilibrium asset prices, *Economic Theory*, **3** (1993), 663–696.
27. R. Merton, Optimum consumption and portfolio rules in a continuous-time model, *J. Econom. Theory*, **3** (1971), 373–413.
28. A. Liu, V. K. Lau, B. Kananian, Stochastic successive convex approximation for non-convex constrained stochastic optimization, *IEEE T. Signal Proces.,* **67** (2019), 4189–4203.
29. A. Nemirovski, A. Shapiro, Convex approximations of chance constrained programs, *SIAM J. Optimiz.*, **17** (2007), 969–996.

30. A. Rakhlin, O. Shamir, K. Sridharan, Making gradient descent optimal for strongly convex stochastic optimization, *arXiv preprint arXiv:1109.5647*, 2011.
31. S. Ghadimi, G. Lan, Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization i: A generic algorithmic framework, *SIAM J. Optimiz.*, **22** (2012), 1469–1492.
32. H. Yu, M. Neely, X. Wei, Online convex optimization with stochastic constraints, *Advances in Neural Information Processing Systems*, **30** (2017), 1428–1438.
33. A. Defazio, F. Bach, S. Lacoste-Julien, SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives, *Advances in neural information processing systems*, **27** (2014), 1646–1654.
34. Y. Xu, W. Yin, Block stochastic gradient iteration for convex and nonconvex optimization, *SIAM J. Optimiz.*, **25** (2015), 1686–1716.
35. A. K. Sahu, D. Jakovetic, D. Bajovic, S. Kar, Communication-efficient distributed strongly convex stochastic optimization: Non-asymptotic rates, *arXiv preprint arXiv:1809.02920*, 2018.
36. M. Mahdavi, T. Yang, R. Jin, Stochastic convex optimization with multiple objectives, *Advances in Neural Information Processing Systems*, **26** (2013), 1115–1123.