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Research article

Some aspects of generalized von Neumann-Jordan type constant

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Abstract: In recent times, Takahashi has introduced von Neumann-Jordan type constants $C_{-\infty}(X)$. In the present manuscript, we establish a novel geometric constant $C_{-\infty}(a, X)$ in a Banach space X. Next, it is shown that $\frac{1}{2} + \frac{2a}{4+a^2} \leqslant C_{-\infty}(a, X) \leqslant 1$ for all $a \geqslant 0$. Further, between the generalized James constant $J(a, X)$ and $C_{-\infty}(a, X)$, a relationship is investigated. For uniform normal structure, a few sufficient conditions were established. Finally, we investigate some relations between the two constants $N(X)$ and $C_{-\infty}(a, X)$.

Keywords: Banach spaces; geometric constants; normal structure
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1. Introduction

In the current decade, numerous geometric constants have been investigated for a Banach space X. Particular attention was given to the two constants; the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$ and $J(X)$ (the James constant), where the results are rigorously investigated and analyzed. For a Banach space X, several studies on the James constant $J(X)$ and also on the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$ have been conducted by Gao [10, 11], Yang and Wang [22], and Kato, Maligranda, and Takahashi [16, 17]. Interested readers in this field are advised to see the work presented in [3, 9, 12, 13, 18] and the references mentioned therein.

The remaining work is organized as follows. To provide a base for the study, some definitions and related results are presented in Section 2. In the subsequent section, the association of $C_{-\infty}(a, X)$ with other constants as well as its equivalent forms are considered. Further, by applying the ultraproduct techniques and using coefficients in $C_{-\infty}(a, X)$, we obtained sufficient criteria for uniform normal-structure of Banach spaces. Finally, in the last Section 4, a relation between the two constants $N(X)$ and $C_{-\infty}(a, X)$ is presented.

2. Preliminaries

A Banach space X is said to be uniformly non-square whenever there exists a constant $\sigma \in (0,1)$ in such a way that for each $x, y \in S_X$ either $\frac{\|x+y\|}{2} \leqslant 1-\sigma$ or $\frac{\|x-y\|}{2} \leqslant 1-\sigma$. Similarly, the constant

$$J(X) = \sup\{\min\{\|x+y\|,\ \|x-y\|\} : x, y \in S_X\},$$

is known as the James constant of the space X or the non-square constant.

In 1937, Clarkson [5] defined the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$ of the form

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\left(\|x\|^2 + \|y\|^2\right)} : x, y \in X,\ \text{not both zero}\right\}.$$

In the following, we will present some properties about von Neumann-Jordan constant (see [15, 16]):

(i) The constant $C_{\mathrm{NJ}}(X)$ lies between 1 and 2 inclusively and $C_{\mathrm{NJ}}(X) = 1$ if and only if X is a Hilbert space;
(ii) X is uniformly non-square if and only if $C_{\mathrm{NJ}}(X) < 2$;
(iii) $C_{\mathrm{NJ}}(X^*) = C_{\mathrm{NJ}}(X)$.

The generalisation of the von Neumann-Jordan constant of X can be written as (see [8])

$$C_{\mathrm{NJ}}(a, X) = \sup\left\{\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in X \text{ not all zero}, \right.$$
$$\left. \text{and } a\|x\| \geqslant \|y-z\|\right\}.$$

Some of its properties are:

(i) $2 \geqslant C_{\mathrm{NJ}}(a, X) \geqslant 1 + \frac{4a}{4+a^2}$ whenever $a \geqslant 0$ and for $a \geqslant 2$, $C_{\mathrm{NJ}}(a, X) = 2$;
(ii) For a Hilbert space X, $C_{\mathrm{NJ}}(a, X) = \frac{4a}{4+a^2} + 1$;
(iii) $C_{\mathrm{NJ}}(a, X)$ is a continuous function of the argument a.

The constant $J_{X,t}(\tau)$ of James type was introduced by Takahashi in [21] as a generalization of the James constant $J(X)$. For $\tau \geqslant 0$ and $-\infty \leqslant t < \infty$, the constant $J_{X,t}(\tau)$ can be defined as

$$J_{X,t}(\tau) = \sup\left\{\mathcal{M}_t(\|x - \tau y\|, \|x + \tau y\|) : x, y \in S_X\right\}.$$

It is interesting to notice that

$$\mathcal{M}_t(a, b) := \left(\frac{a^t + b^t}{2}\right)^{1/t} \quad (-\infty < t < \infty,\ \text{and } t \neq 0),$$
$$\mathcal{M}_{-\infty}(a, b) := \lim_{t \to -\infty} \mathcal{M}_t(a, b) = \min\{a, b\},$$
$$M_0(a, b) := \lim_{t \to 0} \mathcal{M}_t(a, b) = \sqrt{ab},$$

where a and b are positive numbers.

Let $-\infty \leqslant t < \infty$, then the modulus $C_t(X)$ of von Neumann-Jordan type could be defined as follows:

$$C_t(X) = \sup\left\{\frac{J_{X,t}^2(\tau)}{1 + \tau^2} : 0 \leqslant \tau \leqslant 1\right\}.$$

We refer the readers to see [21] and [23] for background on $C_t(X)$.

Definition 2.1. [21] Let us assume a Banach space X, then the constant $C_{-\infty}(X)$ can be expressed as

$$C_{-\infty}(X) = \sup\left\{ \min\left\{ \frac{\|x+y\|^2}{\|x\|^2+\|y\|^2}, \frac{\|x-y\|^2}{\|x\|^2+\|y\|^2} \right\} : x, y \in X, (x, y) \neq (0, 0) \right\}.$$

To introduce the geometric theory, Brodskii and Milman [4] in 1948 has presented the following important concept:

Definition 2.2. If for each closed convex and bounded non-singleton set K (being a subset of a Banach space X) $r(K) < \operatorname{diam}(K)$, then the space is said to have normal structure. Here, $r(K) := \inf\{\sup\{\|x - y\| : y \in K\} : x \in K\}$ and $\operatorname{diam}(K) := \sup\{\|x - y\| : x, y \in K\}$ are the Chebyshev radius and the usual diameter of K, respectively.

Similarly, if every weakly convex and compact set $K \subset X$ (consist of at least two members) has normal structure, then the space X is believed to have weak normal structure. Generally, normal structure playing a key role in finding the fixed points of nonexpansive mappings, for an instant see [20]. To be more specific, for reflexive Banach spaces having normal structure, the fixed point theorem for nonexpansive mappings holds.

Let us consider a Banach space X and a filter $\mathcal{F}$ on $\mathbb{N}$. A sequence $\{x_n\}$ of the space X will converge to x with respect to the filter, that is $\lim_{\mathcal{F}} x_i = x$, if for every U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathcal{F}$. If a filter is maximal with respect to sets inclusion then such filter $\mathcal{U}$ on $\mathbb{N}$ is said to be an ultrafilter. If an ultrafilter is of the type $\{A \subset \mathbb{N} : i_0 \in A\}$ for a fixed i in $\mathbb{N}$ then it is called trivial and nontrivial in other cases. Assume that $l_\infty(X)$ stand for the product space $\Pi_{n\in\mathbb{N}}\|X\|$ and having the norm $\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty$. It is to be noted that whenever $\mathcal{U}$ is nontrivial, then one can embed isometrically X into $\widetilde{X}$. It is also worthy to notice that for a super-reflexive space X (i.e., $\widetilde{X^*} = (\widetilde{X})^*$), X has normal structure of uniform type if $\widetilde{X}$ has normal structure and conversely (see [14]). The subsequent definition is presented for our future needs.

Definition 2.3. [2] Let us assume that $\mathcal{P}$ is a Banach space property, then one can say that a Banach space X has the property super-$\mathcal{P}$ if every Banach space finitely representable in X has the property $\mathcal{P}$.

In the following assertion, we will state a stronger result about the heredity (properties of spaces which transfer to their subspaces naturally) of a property $\mathcal{P}$.

Corollary 2.0.1. [2] Assume that $\mathcal{P}$ is a Banach space heredity property. Then a Banach space X has super-$\mathcal{P}$ if and only if every ultrapower $\widetilde{X}$ of X has $\mathcal{P}$.

Theorem 2.1. [2] Every Banach space X with super-normal structure has uniform normal structure.

3. The $C_{-\infty}(a, X)$ constant

We introduce a new constant based on the James constant and $C_{-\infty}(X)$. In the rest of the work, we will take into account Banach spaces having at least dimension of 2. The results will become more clear with the help of the following concept:

Definition 3.1. Consider a Banach space X, then one can express $C_{-\infty}(a, X)$ as: for $a \geqslant 0$,

$$C_{-\infty}(a, X) = \sup\left\{ \min\left\{ \frac{\|x+y\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}, \frac{\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2} \right\} : x, y, z \in X, \right.$$
$$\left. 0 < \|x\| + \|y\| + \|z\| \text{ and } a\|x\| \geqslant \|y - z\| \right\}.$$

In the following, we will collect some important properties being hold by this novel constant.

Remark. In a Banach space X, we have

(i) $C_{-\infty}(0, X) = \frac{1}{2}C_{-\infty}(X)$;
(ii) $C_{-\infty}(a, X)$ is an increasing function for $a \geqslant 0$;
(iii) The function $C_{-\infty}(a, X)$ is continuous for $a \geqslant 0$.

The following proposition establishes another form of $C_{-\infty}(a, X)$:

Proposition 1. For a Banach space X and $a \geqslant 0$, we have

$$
\begin{aligned}
C_{-\infty}(a, X) = \sup\Bigg\{ &\min\left\{\frac{\|x+y\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}, \frac{\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}\right\} : x, y, z \in B_X, \\
&0 < \|x\| + \|y\| + \|z\|, \text{ and } a\|x\| \geqslant \|y-z\|\Bigg\} \\
= \sup\Bigg\{ &\min\left\{\frac{\|x+y\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}, \frac{\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}\right\} : x, y, z \in B_X, \\
&\text{of which at least one must be a member of } S_X \text{ and } a\|x\| \geqslant \|y-z\|\Bigg\}.
\end{aligned}
$$

Proposition 2. If X is a Banach space, then

(i) $\frac{1}{2} + \frac{2a}{4+a^2} \leqslant C_{-\infty}(a, X) \leqslant 1,\ \forall\, a \geqslant 0$;
(ii) $C_{-\infty}(a, X) = 1$ whenever $a \geqslant 2$.

Proof. (i) First of all, we need to prove the inequality lying to the left. For this, letting $x \in S_X$ and assume $y = -z = \frac{a}{2}x$, clearly we have $y - z = ax$, we get

$$
\begin{aligned}
C_{-\infty}(a, X) &\geqslant \min\left\{\frac{\|x+y\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}, \frac{\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}\right\} \\
&= \min\left\{\frac{(1+\frac{a}{2})^2\|x\|^2}{2\|x\|^2+2\cdot(\frac{a^2}{4})\|x\|^2}, \frac{(1+\frac{a}{2})^2\|x\|^2}{2\|x\|^2+2\cdot(\frac{a^2}{4})\|x\|^2}\right\} \\
&= \min\left\{\frac{(1+\frac{a}{2})^2}{2(1+\frac{a^2}{4})}, \frac{(1+\frac{a}{2})^2}{2(1+\frac{a^2}{4})}\right\} \\
&= \min\left\{\frac{1}{2} + \frac{2a}{4+a^2}, \frac{1}{2} + \frac{2a}{4+a^2}\right\}.
\end{aligned}
$$

Next, we show that $C_{-\infty}(a, X) \leqslant 1$. Applying the usual inequality of triangles, we have

$$
\begin{aligned}
\min\left\{\frac{\|x+y\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}, \frac{\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}\right\} &\leqslant \frac{\|x+y\|^2+\|x-z\|^2}{4\|x\|^2+2\|y\|^2+2\|z\|^2} \\
&\leqslant \frac{2\|x\|^2+2\|y\|^2+2\|x\|^2+2\|z\|^2}{4\|x\|^2+2\|y\|^2+2\|z\|^2} \\
&\leqslant \frac{4\|x\|^2+2\|y\|^2+2\|z\|^2}{4\|x\|^2+2\|y\|^2+2\|z\|^2} = 1,
\end{aligned}
$$

whence $C_{-\infty}(a, X) \leqslant 1$.

(ii) It is to be noted that the developed function $a \mapsto \frac{1}{2} + \frac{2a}{4+a^2}$ is increasing on $[0, 2]$ strictly and the function attain 2 (being its maximum) at $a = 2$. This fact suggest that $C_{-\infty}(a, X) = 1$ for all $a \geqslant 2$. ■

Example 3.1. (Day-James $l_\infty - l_1$ space) Consider $X = \mathbb{R}^2$ be equipped with the norm defined by

$$||x|| = \begin{cases} ||x||_\infty & \text{if } 0 \leqslant x_1x_2, \\ ||x||_1 & \text{if } 0 \geqslant x_1x_2. \end{cases}$$

By taking x = (1, 1), y = (0, 1) and z = (−1, 0), we have ||x + y|| = 2, ||x − z|| = 2, ||z|| = 1. Thus

$$1 = \min\left\{\frac{||x+y||^2}{2||x||^2 + ||y||^2 + ||z||^2}, \frac{||x-z||^2}{2||x||^2 + ||y||^2 + ||z||^2}\right\} \leqslant C_{-\infty}(a, X) \leqslant 1,$$

whence $C_{-\infty}(a, X) = 1$, where $a \geqslant 1$.

Example 3.2. (Day-James $l_2 - l_1$ space) Assume that $X = \mathbb{R}^2$ and the norm is defined of the form

$$||x|| = \begin{cases} ||x||_2 & \text{if } 0 \leqslant x_1x_2, \\ ||x||_1 & \text{if } 0 \geqslant x_1x_2. \end{cases}$$

By choosing $x = (\frac{1}{2}, -\frac{1}{2})$, $y = (0, 1)$ and $z = (1, 0)$. Hence, ||x + y|| = 2, ||x − z|| = 2, ||z|| = 1. Thus

$$1 = \min\left\{\frac{||x+y||^2}{2||x||^2 + ||y||^2 + ||z||^2}, \frac{||x-z||^2}{2||x||^2 + ||y||^2 + ||z||^2}\right\} \leqslant C_{-\infty}(a, X) \leqslant 1,$$

whence $C_{-\infty}(a, X) = 1$, where $a \geqslant 2$.

Theorem 3.3. Whenever H is a Hilbert space, $C_{-\infty}(a, H) = \frac{1}{2} + \frac{2a}{4+a^2}$, $\forall\, a \in [0, 2]$.

Proof. Let a is from the interval $[0, 2]$ and $x(\neq 0), y, z \in H$ and $||y - z|| = t||x||$ for arbitrary $t \in [0, a]$. Then

$$\begin{aligned} \min\left\{\frac{||x+y||^2}{2||x||^2 + ||y||^2 + ||z||^2}, \frac{||x-z||^2}{2||x||^2 + ||y||^2 + ||z||^2}\right\} &\leqslant \frac{||x+y||^2 + ||x-z||^2}{4||x||^2 + 2||y||^2 + 2||z||^2} \\ &\leqslant \frac{2||x||^2 + ||y||^2 + ||z||^2 + 2||y-z||||x||}{4||x||^2 + 2||y||^2 + 2||z||^2} \\ &\leqslant \frac{1}{2} + \frac{2t||x||^2}{4||x||^2 + ||y+z||^2 + ||y-z||^2} \\ &\leqslant \frac{1}{2} + \frac{2t||x||^2}{4||x||^2 + ||y-z||^2} \\ &= \frac{1}{2} + \frac{2t}{4+t^2} \\ &\leqslant \frac{1}{2} + \frac{2a}{4+a^2}. \end{aligned}$$

Hence, by Proposition 2, $C_{-\infty}(a, H) = \frac{1}{2} + \frac{2a}{4+a^2}$. ■

Definition 3.2. [6] The modulus of convexity of a Banach space X is defined as

$$\sigma_X(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X,\ ||x-y|| = \epsilon\right\}, \quad 0 \leqslant \epsilon \leqslant 2.$$

In the following, we intend to present a relationship between the two constants; the modulus of convexity $\sigma_X(\cdot)$ and $C_{-\infty}(\cdot, X)$.

Theorem 3.4. Assume a Banach space X, $\epsilon \in [0,2]$, and $0 \leqslant \beta$. If $C_{-\infty}(\beta, X) < \frac{|\epsilon-\beta|^2}{3+(1+\beta)^2}$, then $\sigma_X(\epsilon) > 0$.

Proof. Contrary to the conclusion, we supposed that $\sigma_X(\epsilon) = 0$. Thus, there exist $x_n, y_n \in S_X$ in such a way that $\|x_n - y_n\| = \epsilon \ \forall\ n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. Set $z_n = y_n - \beta x_n$. So for all $n \in \mathbb{N}$, we obtain $\beta x_n = y_n - z_n$, $\|y_n - \beta x_n\| = \|z_n\| \leqslant 1 + \beta$ and $\|x_n - z_n\| \geqslant |\|x_n - y_n\| - \|\beta x_n\|| = |\epsilon - \beta|$. Therefore, we have

$$\begin{aligned}\frac{|\epsilon-\beta|^2}{3+(1+\beta)^2} &= \min\left\{\frac{4}{3+(1+\beta)^2}, \frac{|\epsilon-\beta|^2}{3+(1+\beta)^2}\right\} \\ &\leqslant \liminf_{n\to\infty} \min\left\{\frac{\left\|x_n+y_n\right\|^2}{2\|x_n\|^2+\left\|y_n\right\|^2+\|z_n\|^2}, \frac{\|x_n-z_n\|^2}{2\|x_n\|^2+\left\|y_n\right\|^2+\|z_n\|^2}\right\} \\ &\leqslant C_{-\infty}(\beta, X) < \frac{|\epsilon-\beta|^2}{3+(1+\beta)^2}.\end{aligned}$$

■

Corollary 3.4.1. If $C_{-\infty}(0, X) < \frac{\epsilon^2}{4}$ for all $\epsilon \in [0,2]$, then $\sigma_X(\epsilon) > 0$ whenever X is a Banach space.

In the below Proposition, we will develop a relationship between the constant $C_{\mathrm{NJ}}(a, X)$ and $C_{-\infty}(a, X)$.

Proposition 3. In a Banach space X, the following holds

$$2C_{-\infty}(a, X) \leqslant C_{\mathrm{NJ}}(a, X).$$

Proof. Consider the general points x, y, z from the space X, then we have

$$\begin{aligned}\min\left\{\frac{\|x+y\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}, \frac{\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2}\right\} &\leqslant \frac{1}{2}\frac{\|x+y\|^2+\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2} \\ &\leqslant \frac{1}{2}C_{\mathrm{NJ}}(a, X),\end{aligned}$$

which guarantees that the inequality holds true. ■

In article [7], the authors consider the constant $J(a, X)$ defined as follows

$$J(a, X) = \sup\left\{\|x-z\| \wedge \|x+y\| : a\|x\| \geqslant \|y-z\| \text{ and } x, y, z \in B_X\right\}.$$

The following result is rather well-known.

Lemma 3.5. [7] For a Banach space X and $0 \leqslant a < 2$, if $C_{\mathrm{NJ}}(a, X) = 2$, then there exist sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ in B_X which satisfies

(i) All of the sequences $\|x_n\|, \left\|y_n\right\|, \|z_n\|$ converge to 1;
(ii) $\|x_n - z_n\|$ and $\left\|x_n + y_n\right\|$ tend to 2;
(iii) $\left\|y_n - z_n\right\| \leqslant a\|x_n\|, \ \forall\ n$.

Proposition 4. If X is a Banach space and a lies in the interval $[0,2]$, then we have the following equivalent statements:

(i) $C_{-\infty}(a, X) = 1$;
(ii) $C_{\mathrm{NJ}}(a, X) = 2$;

(iii) $J(a,X)=2$.

Proof. (i) ⇔ (ii): If $C_{\mathrm{NJ}}(a,X)=2$, then Lemma 3.5 suggest that there exist sequences $\{\mathrm{x}_n\},\{\mathrm{y}_n\},\{\mathrm{z}_n\}$ in S_X holding $\|x_n-z_n\|,\|x_n+y_n\|\to 2$ as well as $\|y_n-z_n\|\leqslant a\|x_n\|$ for each n. Thus, $C_{-\infty}(a,X)=1$. The converse part can be easily proved with the help of Proposition 3.

(ii) ⇔ (iii): From Corollary 3.4 in [7]. ∎

Next, we prove the following theorem.

Theorem 3.6. In a Banach space X, $C_{-\infty}(a,X)\geqslant\frac{(1+a)^2}{2+2a^2}$ for all $a\in(0,1]$ if and only if $J(1,X)=2$.

Proof. As the function $C_{-\infty}(a,X)$ is a continuous one, so

$$C_{-\infty}(1,X)=\lim_{a\to 1}C_{-\infty}(a,X)\geqslant\lim_{a\to 1}\frac{(1+a)^2}{2+2a^2}=1.$$

Thus we have $C_{-\infty}(1,X)=1$. From Proposition 4, and hence $J(1,X)=2$.

For the second part of the proof, assume $J(1,X)=2$. By Proposition 4 and Lemma 3.5, there must exist sequences $\{\mathrm{x}_n\},\{\mathrm{y}_n\},\{\mathrm{z}_n\}$ in S_X which obey $\|\mathrm{x}_n-\mathrm{z}_n\|,\|\mathrm{x}_n+\mathrm{y}_n\|\to 2$ and $\|\mathrm{y}_n-\mathrm{z}_n\|\leqslant 1$ for each n. Thus, $\|a\mathrm{y}_n-a\mathrm{z}_n\|\leqslant a,\ \forall\, n$. Next, take into account the inequalities

$$\left\|\mathrm{x}_n+\mathrm{y}_n\right\|-\left\|\mathrm{y}_n-a\mathrm{y}_n\right\|\leqslant\left\|a\mathrm{y}_n+\mathrm{x}_n\right\|\leqslant 1+a$$

and

$$-\|\mathrm{z}_n-a\mathrm{z}_n\|+\|\mathrm{x}_n-\mathrm{z}_n\|\leqslant\|\mathrm{x}_n-a\mathrm{z}_n\|\leqslant 1+a.$$

Hence

$$\lim_{n\to\infty}\left\|a\mathrm{y}_n+\mathrm{x}_n\right\|=1+a,\quad \lim_{n\to\infty}\|\mathrm{x}_n-a\mathrm{z}_n\|=1+a.$$

Combining all of the above concerned relations

$$C_{-\infty}(a,X)\geqslant\lim_{n\to\infty}\min\left\{\frac{\|a\mathrm{y}_n+\mathrm{x}_n\|^2}{2\|\mathrm{x}_n\|^2+\|a\mathrm{y}_n\|^2+\|a\mathrm{z}_n\|^2},\ \frac{\|\mathrm{x}_n-a\mathrm{z}_n\|^2}{2\|\mathrm{x}_n\|^2+\|a\mathrm{y}_n\|^2+\|a\mathrm{z}_n\|^2}\right\}=\frac{(1+a)^2}{2+2a^2}.$$

∎

For proving our main result, we need to prove the following most important proposition.

Proposition 5. $C_{-\infty}(a,X)=C_{-\infty}(a,\widetilde{X})$.

Proof. Obviously, $C_{-\infty}(a,\widetilde{X})\geqslant C_{-\infty}(a,X)$. We only have to prove that $C_{-\infty}(a,\widetilde{X})\leqslant C_{-\infty}(a,X)$. For this purpose, consider $\sigma>0$, α from the interval $[0,a]$ and the points $\tilde{x},\tilde{y},\tilde{z}$ (not zeros collectively) from the space X for which the relation $\|\tilde{y}-\tilde{z}\|=\alpha\|\tilde{x}\|$ holds. Whenever $\tilde{x}=0$, then

$$\min\left\{\frac{\|\tilde{x}+\tilde{y}\|^2}{2\|\tilde{x}\|^2+\|\tilde{y}\|^2+\|\tilde{z}\|^2},\ \frac{\|\tilde{x}-\tilde{z}\|^2}{2\|\tilde{x}\|^2+\|\tilde{y}\|^2+\|\tilde{z}\|^2}\right\}\leqslant\frac{1}{2}\frac{\|\tilde{x}+\tilde{y}\|^2+\|\tilde{x}-\tilde{z}\|^2}{2\|\tilde{x}\|^2+\|\tilde{y}\|^2+\|\tilde{z}\|^2}=\frac{1}{2}\leqslant C_{-\infty}(a,X).$$

For the case of $\tilde{x}\neq 0$, we will choose $\epsilon>0$ in such a way that $\epsilon<\sigma\|\tilde{x}\|$. As $\|\tilde{x}\|=\lim_{\mathcal{U}}\|x_n\|$ and

$$\begin{aligned}c:=\min&\left\{\frac{\|\tilde{x}+\tilde{y}\|^2}{2\|\tilde{x}\|^2+\|\tilde{y}\|^2+\|\tilde{z}\|^2},\ \frac{\|\tilde{x}-\tilde{z}\|^2}{2\|\tilde{x}\|^2+\|\tilde{y}\|^2+\|\tilde{z}\|^2}\right\}\\ &=\lim_{\mathcal{U}}\min\left\{\frac{\|\mathrm{x}+\mathrm{y}\|^2}{2\|\mathrm{x}\|^2+\|\mathrm{y}\|^2+\|\mathrm{z}\|^2},\ \frac{\|\mathrm{x}-\mathrm{z}\|^2}{2\|\mathrm{x}\|^2+\|\mathrm{y}\|^2+\|\mathrm{z}\|^2}\right\}:=\lim_{\mathcal{U}}c_n,\end{aligned}$$

which suggest that $E := \{n \in \mathbb{N} : |c_n - c| < \sigma$ and $\|y_n - z_n\| \leqslant \alpha\|x_n\| + \epsilon < (\alpha + \sigma)\|x_n\|\}$ belongs to $\mathcal{U}$. Particularly, noticing that $x_n \neq 0$ for every $n \in E$, there exists n such that

$$\begin{aligned} c &< \min\left\{\frac{\|x_n + y_n\|^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2}, \quad \frac{\|x_n - z_n\|^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2}\right\} + \sigma \\ &\leqslant C_{-\infty}(a + \sigma, X) + \sigma. \end{aligned}$$

Hence, the conclusion follows from the arbitrariness of σ and from the continuity of $C_{-\infty}(\cdot, X)$. This completes the proof. ■

We begin by starting a lemma which will be our main tool.

Lemma 3.7. [8] Assume a Banach space X without weak normal structure, then for any $0 < \epsilon < 1$ and every $\frac{1}{2} < r \leqslant 1$, then there exists $x_1 \in S_X$ as well as x_2, x_3 in rS_X which satisfies

(i) $x_2 - x_3 = ax_1$ with $|a - r| < \varepsilon$,

(ii) $\|x_1 - x_2\| > 1 - \varepsilon$,

(iii) $\|x_1 + x_2\| > (1 + r) - \varepsilon, \|x_3 + (-x_1)\| > (3r - 1) - \varepsilon$.

Theorem 3.8. If

$$C_{-\infty}(r, X) < \frac{(3r - 1)^2}{2(1 + r^2)}, \quad for\ some\ r \in (\frac{1}{2}, 1],$$

then the Banach space X has uniform normal structure.

Proof. The theorem will be proved if we show that the underlying conditions ensuring that X has normal structure. Because, Proposition 5 will guarantees that $\widetilde{X}$ has also normal structure and ultimately X has super-normal structure by virtue of Corollary 2.0.1, and finally with the help of Theorem 2.1, we can say that X has uniform normal structure.

Whenever $C_{-\infty}(r, X) < \frac{(3r-1)^2}{2(1+r^2)}$, we could observe that X is uniformly non-square and as a result, the space is reflexive. Generally in such cases, both the weak normal and normal structures coincide. Thus, we only need to prove that X has weak normal structure.

As $C_{-\infty}(\cdot, X)$ is continuous, so $C_{-\infty}(r', X) < \frac{(3r-1)^2}{2(1+r^2)}$ for some $r' > r$. Next, we choose a natural number m such that $r + (\frac{1}{m}) \leqslant r'$. Contrary to the conclusion, we supposed that X does not have weak normal structure. Then, by Lemma 3.7 there exists sequences $\{x_n\}$ in S_X and $\{y_n\}, \{z_n\}$ from rS_X in such a way that for every $n \in \mathbb{N}$,

$$a_n x_n = y_n - z_n \text{ with } |a_n - r| < \frac{1}{n + m},$$

$$\|x_n - y_n\|^2 > \left(1 - \frac{1}{n + m}\right)^2, \quad \|x_n + y_n\|^2 > \left(1 + r - \frac{1}{n + m}\right)^2,$$

and

$$\|x_n - z_n\|^2 > \left((3r - 1) - \frac{1}{n + m}\right)^2.$$

It is worthy to notice that $a_n = \|y_n - z_n\| < \frac{1}{n+m} + r < r + \frac{1}{m} < r'$, and

$$\liminf_{n\to\infty} \left\|x_n + y_n\right\|^2 \geqslant (r + 1)^2 \text{ and } (3r - 1)^2 \leqslant \liminf_{n\to\infty} \|x_n - z_n\|^2 .$$

Thus

$$\begin{aligned}\frac{(3r-1)^2}{2\left(1+r^2\right)} &\leqslant \liminf_{n\to\infty} \min\left\{\frac{\left\|x_n+y_n\right\|^2}{2\left\|x_n\right\|^2+\left\|y_n\right\|^2+\left\|z_n\right\|^2}, \frac{\left\|x_n-z_n\right\|^2}{2\left\|x_n\right\|^2+\left\|y_n\right\|^2+\left\|z_n\right\|^2}\right\}\\ &\leqslant C_{-\infty}(r', X)\\ &< \frac{(3r-1)^2}{2\left(1+r^2\right)}.\end{aligned}$$

A contradiction was raised due to our wrong supposition and hence X has weak normal structure as desired. ■

4. The coefficients $N(X)$ and $C_{-\infty}(a, X)$ and normal structure

If A is a non-empty bounded subset of a Banach space X, then the number

$$r(A) = \inf\{\sup \|x-y\| : x \in A\}$$

is said to be the Chebyshev radius of A or Chebyshev radius with respect to A. Similarly, by $Z(A)$ we mean the set of all those elements x of A for which the infimum could be attained and in certain cases, it may be empty as well. The set $Z(A)$ is known as Chebyshev center with respect to A. If

$$r(A) \leqslant \operatorname{diam} A$$

holds true for each convex bounded set $A(\subset X)$ having $\operatorname{diam} A$ greater than zero, then the Banach space X will have normal structure. The normal structure coefficient (denoted by $N(X)$) for a space X can be expressed as

$$N(X) = \inf\left\{\frac{\operatorname{diam} A}{r(A)}\right\},$$

where this infimum is taken all over the convex bounded closed sets $A \subset X$ with $\operatorname{diam} A > 0$. Clearly, $2 \geqslant N(X) \geqslant 1$ and the Banach space X is said to have uniform normal structure whenever $N(X) > 1$. We refer the reader to [1] and [19] for background on $N(X)$.

We begin by stating a theorem from [19] which will be our main tool.

Theorem 4.1. Let us assume a non-empty convex and compact subset C of a finite-dimensional space X and x_0 be a member of C. If $x_0 \in Z(C)$, then there must exist points $x_1, \cdots, x_n \in C$, functionals $x_1^*, \cdots, x_n^* \in S_{X^*}$, and non-negative scalars $\lambda_1, \cdots, \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, and

$$x_i^*(x_0 - x_i) = r(C) = \|x_0 - x_i\|$$

for $i = 1, 2, 3, \cdots, n$ and

$$0 \leqslant \sum_{i=1}^{n} \lambda_i x_i^*(x - x_0)$$

for each $x \in C$.

By following the ideas in [19] and [22], it is handy to prove Theorem 4.2.

Theorem 4.2. If X is a nontrivial Banach space with the constant $N(X)$, then for every a, we have

$$\sqrt{\frac{\max_r f(r)}{C_{-\infty}(a,X)}} \leqslant N(X),$$

with

$$f(r) = \frac{(a+1)^2}{2(r^2+1)},$$

where r belongs to the interval $[a,1]$ and $1 \geqslant a \geqslant 0$.

Proof. Case 1. If $C_{-\infty}(a,X) = 1$, then one can notice that

$$\max_{a\leqslant r\leqslant 1} f(r) = \max_{a\leqslant r\leqslant 1} \frac{(a+1)^2}{2\left(r^2+1\right)} \leqslant \max_{a\leqslant r\leqslant 1} \frac{(r+1)^2}{2\left(r^2+1\right)} \leqslant 1.$$

Hence $1 \geqslant \sqrt{\frac{\max_r f(r)}{C_{-\infty}(a,X)}}$, and by using $1 \leqslant N(X)$, we can write $N(X) \geqslant \sqrt{\frac{\max_r f(r)}{C_{-\infty}(a,X)}}$ with $r \in [a,1]$.

Case 2. If $C_{-\infty}(a,X) < 1$, then the underlying space is uniformly non-square and thus reflexive. Further, we shall assume that C is a convex hull such that $\operatorname{diam} C = d$ and $r(C) = 1$. Also, we can write $\sup\{\|x\| : \mathrm{x} \in C\} = 1$ and so by Proposition 4.1 we could have points $\mathrm{x}_1, \cdots, \mathrm{x}_n$, norm-1 functionals $\mathrm{x}_1^*, \mathrm{x}_2^*, \cdots, \mathrm{x}_n^*$ and non-negative real numbers $\lambda_1, \cdots, \lambda_n$ in such a way that $\sum_{i=1}^n \lambda_i = 1$, $\|\mathrm{x}_i\| = \mathrm{x}_i^*(-\mathrm{x}_i) = 1$ for $i = 1,2,\cdots,n$ and $0 \leqslant \sum_{i=1}^n \lambda_i \mathrm{x}_i^*(\mathrm{x}_j)$ for $j = 1,2,\cdots,n$. Further, for any $r \in [a,1]$, we can set

$$\mathrm{x}_{i,j} = \frac{1}{d}(\mathrm{x}_i - \mathrm{x}_j), \quad \mathrm{y}_{i,j} = \frac{r\mathrm{x}_i}{d}, \quad \mathrm{z}_{i,j} = \frac{(r-a)\mathrm{x}_i + a\mathrm{x}_j}{d} \quad \text{for} \quad i,j = 1,2,\ldots,n.$$

Clearly, $1 \geqslant \|\mathrm{x}_{i,j}\|, r \geqslant \|\mathrm{y}_{i,j}\|, r \geqslant \|\mathrm{z}_{i,j}\|$, and $a\|\mathrm{x}_{i,j}\| = \|\mathrm{y}_{i,j} - \mathrm{z}_{i,j}\|$. Next, we obtain the following estimates:

$$\begin{aligned}
&\sum_{i,j=1}^{n} \lambda_i\lambda_j \left\|\mathrm{x}_{i,j} + \mathrm{y}_{i,j}\right\|^2 \\
&\geqslant \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \left(\mathrm{x}_i^*\left(\mathrm{x}_{i,j} + \mathrm{y}_{i,j}\right)\right)^2 \\
&= \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \left(\frac{1+r}{d} + \frac{1}{d}\mathrm{x}_i^*\left(\mathrm{x}_j\right)\right)^2 \\
&= \frac{(r+1)^2}{d^2} + \frac{2(r+1)}{d^2} \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \mathrm{x}_i^*\left(\mathrm{x}_j\right) \\
&\quad + \left(\frac{1}{d^2}\right) \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \left(\mathrm{x}_i^*\left(\mathrm{x}_j\right)\right)^2 \\
&\geqslant \frac{(r+1)^2}{d^2} \qquad \text{for any } r \in [a,1]
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{i,j=1}^{n} \lambda_i \lambda_j \left\| \mathrm{x}_{i,j} - \mathrm{z}_{i,j} \right\|^2 \\
&\geqslant \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j \left(\mathrm{x}_j^* \left(\mathrm{x}_{i,j} - \mathrm{z}_{i,j} \right) \right)^2 \\
&= \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \left(\frac{1+a}{d} + \frac{1+a-r}{d} \mathrm{x}_j^* \left(\mathrm{x}_i \right) \right)^2 \\
&= \frac{(1+a)^2}{d^2} + \frac{2(1+a)(1+a-r)}{d^2} \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j \mathrm{x}_j^* \left(\mathrm{x}_i \right) \\
&\quad + \frac{(1+a-r)^2}{d^2} \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j \left(\mathrm{x}_j^* \left(\mathrm{x}_i \right) \right)^2 \\
&\geqslant \frac{(a+1)^2}{d^2} \quad \text{for any } r \in [a, 1].
\end{aligned}
$$

Ultimately, there exists i and j such that

$$
\min\{ \left\| \mathrm{x}_{i,j} + \mathrm{y}_{i,j} \right\|^2, \left\| \mathrm{x}_{i,j} - \mathrm{z}_{i,j} \right\|^2 \} \geqslant \frac{(a+1)^2}{d^2}.
$$

From the definition of the $C_{-\infty}(a, X)$ constant, we could say

$$
C_{-\infty}(a, X) \geqslant \min \left\{ \frac{\left\| \mathrm{x}_{i,j} + \mathrm{y}_{i,j} \right\|^2, \quad \left\| \mathrm{x}_{i,j} - \mathrm{z}_{i,j} \right\|^2}{2 \left\| \mathrm{x}_{i,j} \right\|^2 + \left\| \mathrm{y}_{i,j} \right\|^2 + \left\| \mathrm{z}_{i,j} \right\|^2} \right\} \geqslant \frac{(a+1)^2}{2d^2 \left(r^2 + 1 \right)},
$$

which implies

$$
d \geqslant \sqrt{\frac{\max_r f(r)}{C_{-\infty}(a, X)}},
$$

and r is again from the interval $[a, 1]$.

As set C is arbitrary, thus we have the required result. ■

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Conflict of interest

The authors declare that there is no conflict of interest regarding this manuscript.

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