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Research article

Nonlinear boundary value problems for fractional differential inclusions with Caputo-Hadamard derivatives on the half line

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Abstract: The authors establish sufficient conditions for the existence of solutions to a boundary value problem for fractional differential inclusions involving the Caputo-Hadamard type derivative of order $r \in (1,2]$ on infinite intervals. Both cases of convex and nonconvex valued right hand sides are considered. The technique of proof involves fixed point theorems combined with a diagonalization method.

Keywords: existence; fractional differential inclusions; Caputo-Hadamard type derivative; diagonalization method
Mathematics Subject Classification: 26A33, 34A08, 34A60, 34B15

1. Introduction

This paper deals with the existence of solutions to boundary value problems (BVP for short) for fractional differential inclusions. In particular, we consider the boundary value inclusion on an infinite interval

$$ {}^{H}_{C}D^{r}y(t) \in F(t,y(t)), \quad \text{for a.e } \ t \in J = [1,\infty),\ 1 < r \leq 2, \tag{1.1}$$

$$ y(1) = y_1,\ y \text{ bounded on } [1,\infty), \tag{1.2}$$

where ${}^{H}_{C}D^{r}$ is the Caputo-Hadamard fractional derivative, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, and $y_1 \in \mathbb{R}$.

Fractional order differential equations have proven to be effective models of various phenomena in engineering and the sciences such as viscoelasticity, electrochemistry, control theory, flows through porous media, electromagnetism, and others. Recently, they have been applied to problems in biological modeling and social interactions [14, 15]. The monographs of Abbas *et al.* [1–3], Hilfer [21], Kilbas *et al.* [23], Podlubny [26], Momani *et al.* [25] contain the mathematical background needed to understand the value of this modeling tool. For results on fractional order derivatives in general and Hadamard fractional derivatives in particular, we refer the reader to [5–7, 10, 17, 18, 20, 28].

The Caputo left-sided fractional derivative of order α is defined by

$$({}^cD^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $\alpha > 0$ and $n = [\alpha]+1$. This derivative is very useful in many applied problems because it satisfies its initial data which contains $y(0)$, $y'(0)$, etc., as well as the same data for boundary conditions.

The fractional derivative as presented by Hadamard in 1892 [19] differs from the well-known Caputo derivative in two significant ways. First, its kernel involves a logarithmic function with an arbitrary exponent, and secondly, the Hadamard derivative of a constant is not 0. The Caputo-Hadamard fractional derivative was introduced by Jarad et al. [22] is a modification of the Hadamard fractional derivative that maintains the property that the derivative of a constant is 0. In recent years there have been a number of papers examining problems involving the Caputo-Hadamard derivative, and as examples, we refer the reader to Adjabi et al. [4] and Shammack [27].

Here we present two results guaranteeing the existence of solutions to the problem (1.1)–(1.2); one is for the case where the right hand side is convex valued, and the other is for the nonconvex case. The nonlinear alternative of Leray-Schauder type is used in the proof for the convex case, and the Covitz-Nadler fixed point theorem for multivalued contraction maps is used in the nonconvex case. We should mention that each of approaches are then combined with the diagonalization method to obtain the results. It should be pointed out that this paper actually initiates the application of the diagonalization method to such classes of problems. The theorems in the present paper extend current results in the literature to the multivalued case.

2. Preliminaries

We begin by presenting some definitions and preliminary facts that are needed in the proofs of our results. We take $C(J,\mathbb{R})$ to be the space of all continuous functions from J into $\mathbb{R}$ and let $L^1(J,\mathbb{R})$ be the Banach space of Lebesgue integrable functions $y : J \longrightarrow \mathbb{R}$ with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

Also, we let $AC(J,\mathbb{R})$ denote the space of functions $y : J \to \mathbb{R}$ that are absolutely continuous.

Let $\delta = t\dfrac{d}{dt}$, $\delta^n = \delta(\delta^{n-1})$, set

$$AC^n_\delta(J,\mathbb{R}) = \{y : J \longrightarrow \mathbb{R} \,|\, \delta^{n-1}y(t) \in AC(J,\mathbb{R})\},$$

and let $AC^1(J,\mathbb{R})$ be the space of absolutely continuous functions $y : J \to \mathbb{R}$ with an absolutely continuous first derivative.

For any Banach space $(X, \|\cdot\|)$, we set:

$P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$,
$P_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$,
$P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$,
$P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

We say that a multivalued map $G : X \to \mathcal{P}(X)$ is *convex* (*closed*) valued if $G(X)$ is convex (closed) for all $x \in X$. A map G is *bounded on bounded sets* if $G(B) = \cup_{x\in B} G(x)$ is bounded in X for all $B \in P_b(X)$ (i.e., $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\} < \infty$). The mapping G is *upper semi-continuous (u.s.c)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subset N$. The mapping G is *completely continuous* if $G(B)$ is relatively compact for each $B \in P_b(X)$.

If G is a multivalued map that is completely continuous with nonempty compact values, then G is upper semi-continuous if and only if G has a closed graph (that is, if $x_n \to x_*$, $y_n \to y_*$, and $y_n \in G(x_n)$, then $y_* \in G(x_*)$). We say that $x \in X$ is a *fixed point* of G if $x \in G(x)$. The set of fixed points of the multivalued operator G will be denote by $FixG$. A multivalued map $G : J \to P_{cl}(\mathbb{R})$ is called *measurable* if for every $y \in \mathbb{R}$,

$$t \to d(y, G(t)) = inf\{|y - z| : z \in G(t)\}$$

is a measurable function.

Definition 2.1. *A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(1) $t \to F(t, u)$ is measurable for each $u \in \mathbb{R}$, and
(2) $u \to F(t, u)$ is upper semicontinuous for almost all $t \in J$.

Let (X, d) be a metric space induced from the normed space $(X, |\cdot|)$. the function $H_d : \mathcal{P}(X)\times\mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}$$

is known as the Hausdorff-Pompeiu metric.

Definition 2.2. *A multivalued operator $N : X \to P_{cl}(X)$ is called:*

(1) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \textit{for each} x,\ y \in X;$$

(2) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

For more details on multivalued maps see the books of Aubin and Cellina [8], Aubin and Frankowska [9], Castaing and Valadier [11], and Deimling [13].

Theorem 2.3. ([12, Covitz and Nadler]) *Let (X, d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Lemma 2.4. ([24]) *Let J be a compact real interval, F be a Carathéodory multivalued map, and let θ be a linear continuous map from $L^1(J,E) \mapsto C(J,E)$. Then the operator*

$$\theta \circ S_{F,y} : C(J,E) \mapsto P_{cp,c}(C(J,E)), \quad y \mapsto (\theta \circ S_{F,y})(y) = \theta(S_{F,y}),$$

is a closed graph operator in $L^1(J,E) \times C(J,E)$.

Definition 2.5. ([23]) *The Hadamard fractional integral of order r for a function $h : [1,+\infty) \to \mathbb{R}$ is defined as*

$$^{H}I^{r}h(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} \frac{h(s)}{s} ds, \ r > 0,$$

provided that the integral exists.

Definition 2.6. ([23]) *For a function h on the interval $[1,+\infty)$, the Hadamard fractional-order derivative of h of order r is defined by*

$$(^{H}D^{r}h)(t) \ = \ \frac{1}{\Gamma(n-r)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-r-1} \frac{h(s)}{s} ds, \ n-1 < r < n, \ n = [r]+1.$$

Here, $[r]$ denotes the integer part of r and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.7. ([22]) *For a function h belonging to $AC^n_\delta([a,b],\mathbb{R})$ with $a > 0$, we define the Caputo-type modification of the left-sided Hadamard fractional derivatives to be*

$$^{H}_{C}D^{r}y(t) = {}^{H}D^{r}\left[y(s) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{s}{a}\right)^k\right](t)$$

where $n = [\alpha]+1$.

Lemma 2.8. ([22]) *Let $y \in AC^n_\delta([a,b],\mathbb{R})$ or $C^n_\delta([a,b],\mathbb{R})$. Then*

$$^{H}I^{r}(^{H}_{C}D^{r})y(t) = y(t) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\log \frac{t}{a}\right)^k.$$

We next recall the nonlinear alternative of Leray-Schauder.

Theorem 2.9. *Let X be a Banach space and C a nonempty closed convex subset of X. Let U be a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to P_{cp,c}(C)$ be a upper semicontinuous compact map. Then either*

(1) T has fixed points in $\overline{U}$, or
(2) There exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda T(u)$.

3. Main results

We begin by defining what we mean by the problem (1.1)–(1.2) having a solution.

Definition 3.1. *A function $y \in AC^2_\delta(J,\mathbb{R})$ is said to be a solution of (1.1)–(1.2), if there exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,y(t))$ for a.e. $t \in J$ such that $^{H}_{C}D^{r}y(t) = v(t)$ and the function y satisfies the boundary condition (1.2).*

Lemma 3.2. *Let $h : [1,T] \to \mathbb{R}$ be continuous functions. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} h(s)\frac{ds}{s} \\ &- \frac{(r-1)(\log t)}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-2} h(s)\frac{ds}{s} + y_1. \end{aligned} \tag{3.1}$$

if and only if y is a solution of the nonlinear fractional problem

$${}^H_C D^r y(t) = h(t) \text{ for a.e. } t \in J = [1,T], \quad 1 < r \le 2, \tag{3.2}$$

$$y(1) = y_1, \ \ y'(T) = 0. \tag{3.3}$$

Proof. Applying the Hadamard fractional integral of order r to both sides of (3.2) and then using Lemma 2.8, we obtain

$$y(t) = c_1 + c_2 \log t + {}^H I^r h(t). \tag{3.4}$$

Applying (3.3) yields

$$c_1 = y_1$$

and

$$y'(t) = \frac{(r-1)}{t\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-2} h(s)\frac{ds}{s} + \frac{c_2}{t}.$$

Hence,

$$c_2 = \frac{-(r-1)}{\Gamma(r)}\int_1^T \left(\log\frac{T}{s}\right)^{r-2} h(s)\frac{ds}{s}.$$

Substituting into (3.4), we obtain (3.1).

Conversely, it is clear that if y satisfies equation (3.1), then (3.2) and (3.3) hold. □

Remark 3.3. *Notice that for $m \in \mathbb{N}$, there exists $J_m := [1, T_m] \subset J$ with*

$$1 < T_1 < T_2 < \cdots < T_m < \cdots$$

such that $T_m \to \infty$ as $m \to \infty$.

Definition 3.4. *For each $m \in \mathbb{N}$ and $y \in AC(J_m, \mathbb{R})$, define the set of selections of F by*

$$S_{F,y} = \{v \in L^1([1, T_m], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ a.e. } t \in [1, T_m]\}.$$

3.1. The convex case

Our first existence result is for the case where F is convex valued.

Theorem 3.5. *Assume that for each $m \in \mathbb{N}$:*

(H1) $F : J_m \times \mathbb{R} \to \mathcal{P}_{cp,p}(\mathbb{R})$ is a Carathéodory multi-valued map;

(H2) There exist $p \in C(J_m, \mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\|F(t,u)\|_{\mathcal{P}} \le p(t)\psi(|u|) \quad \text{for} \quad t \in J_m \text{ and } u \in \mathbb{R};$$

(H3) There exists $C > 0$ *such that*

$$\frac{C}{\frac{2(\log T_m)^r}{\Gamma(r+1)}\|p\|_\infty \psi(C) + |y_1|} > 1. \tag{3.5}$$

(H4) There exists $l \in L^1(J_m, \mathbb{R}^+)$ *with* ${}^H I^r l(t) < \infty$ *such that*

$$H_d(F(t,u), F(t,\bar{u})) \le l(t)|u - \bar{u}| \ \ \textit{for every} \ \ u, \bar{u} \in \mathbb{R}, \tag{3.6}$$

and

$$d(0, F(t,0)) \le l(t), \ \ \textit{a.e.} \ t \in J_m; \tag{3.7}$$

Then the problem (1.1)–(1.2) has at least one solution on J.

Proof. Fix $m \in \mathbb{N}$ and consider the related boundary value problem

$${}^H_C D^r y(t) \in F(t, y(t)), \quad \text{for a.e } \ t \in J_m, \ 1 < r \le 2, \tag{3.8}$$

$$y(1) = y_1, \ \ y'(T_m) = 0. \tag{3.9}$$

First, we shall show that the BVP (3.8)–(3.9) has a solution $y_m \in C(J_m, \mathbb{R})$ with

$$|y_m(t)| \le M \ \text{ for each } t \in J_m,$$

where $M > 0$ is a constant. To do this, consider the multivalued operator $N : C(J_m, \mathbb{R}) \to \mathcal{P}(C(J_m, \mathbb{R}))$ defined by

$$N(y) = \left\{ h \in C(J_m, \mathbb{R}) : \begin{array}{rcl} h(t) & = & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s} \\ & - & \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v(s) \frac{ds}{s} + y_1, \ v \in S_{F,y} \end{array} \right\}.$$

Clearly, from Lemma 3.2, the fixed points of N are solutions to (3.8)–(3.9). We shall show that N satisfies the hypotheses of the nonlinear Leray-Schauder alternative. We give the proof in steps.

Step 1: *$N(y)$ is convex for each $y \in C(J_m, E)$.* For $h_1, h_2 \in N(y)$, there exist $v_1, v_2 \in S_{F,y}$ such that

$$\begin{array}{rcl} h_i(t) & = & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_i(s) \frac{ds}{s} \\ & - & \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v_i(s) \frac{ds}{s} + y_1 \end{array}$$

for $t \in J_m$ and $i = 1, 2$. Letting $0 \le d \le 1$, we see that for each $t \in J_m$,

$$\begin{array}{rcl} (dh_1 + (1-d)h_2)(t) & = & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} [dv_1 + (1-d)v_2] \frac{ds}{s} \\ & - & \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} [dv_1 + (1-d)v_2] \frac{ds}{s}. \end{array}$$

Now F has convex values, so $S_{F,y}$ is convex; hence,

$$dh_1 + (1-d)h_2 \in N(y),$$

so $N(y)$ is convex.

Step 2: *N maps bounded sets into bounded sets in* $C(J_m,\mathbb{R})$. Let $B_{\mu_*} = \{y \in C(J_m,\mathbb{R}) : \|y\|_\infty \le \mu_*\}$ be a bounded set in $C(J_m,\mathbb{R})$ and $y \in B_{\mu_*}$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ &- \frac{(r-1)(\log t)}{\Gamma(r)}\int_1^{T_m}\left(\log\frac{T_m}{s}\right)^{r-2} v(s)\frac{ds}{s} + y_1. \end{aligned}$$

By (H2), we have, for each $t \in J_m$,

$$\begin{aligned} |h(t)| &\le \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1} |v(s)|\frac{ds}{s} \\ &+ \frac{(r-1)(\log t)}{\Gamma(r)}\int_1^{T_m}\left(\log\frac{T_m}{s}\right)^{r-2} |v(s)|\frac{ds}{s} + |y_1| \\ &\le \frac{(\log t)^r}{\Gamma(r+1)}\|p\|_\infty\psi(\mu_*) + \frac{(\log T_m)^{r-1}}{\Gamma(r+1)}\|p\|_\infty\psi(\mu_*) + |y_1|. \end{aligned}$$

Thus,

$$\|h\|_\infty \le \frac{2(\log T_m)^r}{\Gamma(r+1)}\|p\|_\infty\psi(\mu_*) + |y_1| := \ell.$$

Step 3: *N maps bounded sets into equicontinuous sets in* $C(J_m,\mathbb{R})$. Take t_1, $t_2 \in J_m$ with $t_1 < t_2$, and take B_{μ_*} to be a bounded set in $C(J_m,\mathbb{R})$ as we did in Step 2. Let $y \in B_{\mu_*}$ and $h \in N(y)$. Then,

$$\begin{aligned} |h(t_2) - h(t_1)| &= \Bigg|\frac{1}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1} - \left(\log\frac{t_1}{s}\right)^{r-1}\right] v(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1} v(s)\frac{ds}{s} \\ &+ \frac{(r-1)(\log t_2 - \log t_1)}{\Gamma(r)}\left[\int_1^{T_m}\left(\log\frac{T_m}{s}\right)^{r-2} v(s)\frac{ds}{s}\right]\Bigg| \\ &\le \frac{\|p\|_\infty\psi(\mu_*)}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1} - \left(\log\frac{t_1}{s}\right)^{r-1}\right]\frac{ds}{s} \\ &+ \frac{\|p\|_\infty\psi(\mu_*)}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}\frac{ds}{s} \\ &+ (\log t_2 - \log t_1)(r-1)\left|\frac{\|p\|_\infty\psi(\mu_*)}{\Gamma(r)}\int_1^{T_m}\left(\log\frac{T_m}{s}\right)^{r-2}\frac{ds}{s}\right|. \end{aligned}$$

As $t_1 \to t_2$, the right hand side of the inequality above approaches zero. Therefore, in view of Steps 1 to 3 and the Arzelà-Ascoli theorem, it follows that N is completely continuous.

Step 4: *N is upper semicontinuous.* We will show this by showing that N has a closed graph. Let $y_n \to y_*$, $h_n \in N(y_n)$, and $h_n \to h_*$. We need to prove that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ implies there exists $v_n \in S_{F,y_n}$ such that for $t \in J_m$,

$$\begin{aligned} h_n(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s) \frac{ds}{s} \\ &- \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v_n(s) \frac{ds}{s} + y_1. \end{aligned}$$

We need to show that there is a $v_* \in S_{F,y_*}$ such that, for each $t \in J_m$,

$$\begin{aligned} h_*(t) &= \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_*(s) \frac{ds}{s} \\ &- \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v_*(s) \frac{ds}{s} + y_1. \end{aligned}$$

Now $F(t, \cdot)$ is upper semi-continuous, so for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for every $n > N_\epsilon$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \epsilon B(0, 1), \quad a.e.\ t \in J_m.$$

Since F has compact values by (H1), there is a subsequence v_{n_k} of v_n such that

$$v_{n_k} \to v_* \quad \text{as} \quad k \to \infty$$

and

$$v_* \in F(t, y_*(t)), \quad a.e.\ \ t \in J_m.$$

For every $w \in F(t, y_*(t))$, we have

$$|v_{n_k}(t) - v_*(t)| \le |v_{n_k}(t) - w| + |w - v_*(t)|.$$

Then,

$$|v_{n_k}(t) - v_*(t)| \le d(v_{n_k}(t), F(t, y_*(t))).$$

We can obtain an analogous relation by interchanging the roles of v_{n_k} and v_*, so

$$|v_{n_k}(t) - v_*(t)| \le H_d(F(t, y_n(t)), F(t, y_*(t))) \le l(t)\|y_n - y_*\|_\infty$$

by (H4). It is easy to see that

$$\|h_{n_k} - h_*\|_\infty \to 0 \quad \text{as} \quad k \to \infty,$$

which is what we wished to show.

Step 5: *A priori bounds on solutions.* Let $y \in \lambda N(y)$ with $\lambda \in (0, 1]$. Then there is a $v \in S_{F,y}$ so that for each $t \in J_m$,

$$\begin{aligned} h(t) = &\frac{\lambda}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s} \\ &- \frac{\lambda(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v(s) \frac{ds}{s} + y_1. \end{aligned}$$

This implies by (H2) that, for each $t \in J_m$, we have

$$\begin{aligned}|h(t)| &\le \frac{1}{\Gamma(r)}\int_1^t \left(\log\frac{t}{s}\right)^{r-1}|v(s)|\frac{ds}{s}\\ &\quad + \frac{(r-1)(\log t)}{\Gamma(r)}\int_1^{T_m}\left(\log\frac{T_m}{s}\right)^{r-2}|v(s)|\frac{ds}{s} + |y_1|\\ &\le \frac{(\log t)^r}{\Gamma(r+1)}\|p\|_\infty\psi(\|y\|_\infty) + \frac{(\log T_m)^{r-1}}{\Gamma(r+1)}\|p\|_\infty\psi(\|y\|_\infty) + |y_1|\\ &\le 2\frac{(\log T_m)^r}{\Gamma(r+1)}\|p\|_\infty\psi(\|y\|_\infty) + |y_1|.\end{aligned}$$

Thus,

$$\frac{\|y\|_\infty}{2\frac{(\log T_m)^r}{\Gamma(r+1)}\|p\|_\infty\psi(\|y\|_\infty) + |y_1|} < 1.$$

Then by condition (3.5), there exists $C > 0$ such that $\|y\|_\infty \neq C$. Let $U = \{y \in C(J_m, \mathbb{R}) : \|y\|_\infty < C\}$. The operator $N : \overline{U} \to \mathcal{P}(C(J_m, \mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0, 1]$. It then follows from the Leray-Shauder nonlinear alternative that N has a fixed point $y \in \overline{U}$ that in turn is a solution of problem (3.8)–(3.9).

Step 6: *A diagonalization process.* First let $\mathbb{N}_m = \mathbb{N}^* - \{m\}$. For each $k \in \mathbb{N}$, let $y_k(t)$ be the solution of (3.8)–(3.9) whose existence is guaranteed by Steps 1–5 above, and set

$$u_k(t) = \begin{cases} y_k(t), & \text{for } t \in [1, T_m],\\ y_k(T_m), & \text{for } t \in [T_m, \infty).\end{cases}$$

For $m = 1$, there exists $v_k^1 \in S_{F,u}$ such that

$$\begin{aligned}u_k(t) &= \frac{1}{\Gamma(r)}\int_1^t\left(\log\frac{t}{s}\right)^{r-1} v_k^1(s)\frac{ds}{s}\\ &\quad - \frac{(r-1)(\log t)}{\Gamma(r)}\int_1^{T_m}\left(\log\frac{T_m}{s}\right)^{r-2} v_k^1(s)\frac{ds}{s} + y_1\end{aligned}$$

and

$$|u_k(t)| \le M \ \text{ for } \ t \in [1, T_1].$$

Now for $t_1, t_2 \in J_1$ with $t_1 < t_2$, we have

$$\begin{aligned}|u_k(t_2) - u_k(t_1)| &\le \frac{\|p\|_\infty\psi(M)}{\Gamma(r)}\int_1^{t_1}\left[\left(\log\frac{t_2}{s}\right)^{r-1} - \left(\log\frac{t_1}{s}\right)^{r-1}\right]\frac{ds}{s}\\ &\quad + \frac{\|p\|_\infty\psi(M)}{\Gamma(r)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{r-1}\frac{ds}{s}\\ &\quad + (\log t_2 - \log t_1)\left|\frac{r-1}{\Gamma(r)}\int_1^{T_1}\left(\log\frac{T_1}{s}\right)^{r-2} v_k^1(s)\frac{ds}{s}\right|.\end{aligned}$$

By the Arzelà-Ascoli Theorem, $\{u_k\}$ has a uniformly convergent subsequence, so there is a subset $\mathbb{N}_1$ of $\mathbb{N}$ and a function $z_1 \in C([1, T_1], \mathbb{R})$ such that

$$\{u_k\} \to z_1 \ \text{ as } \ k \to \infty \text{ through } \mathbb{N}_1.$$

Now for $k \in \mathbb{N}_1$ and $m = 2$, we have

$$|u_k(t)| \leq M \ \text{ for } \ t \in [1, T_2].$$

Also for $t_1, t_2 \in J_2$ with $t_1 < t_2$, there exists $v_k^2 \in S_{F,u}$ such that

$$\begin{aligned} |u_k(t_2) - u_k(t_1)| \leq & \frac{\|p\|_\infty \psi(M)}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1}\right] \frac{ds}{s} \\ & + \frac{\|p\|_\infty \psi(M)}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{ds}{s} \\ & + (\log t_2 - \log t_1) \left| \frac{r-1}{\Gamma(r)} \int_1^{T_2} \left(\log \frac{T_2}{s}\right)^{r-2} v_k^2(s) \frac{ds}{s} \right|. \end{aligned}$$

Again using the Arzelà-Ascoli Theorem, $\{u_k\}$ has a uniformly convergent subsequence, so there is a subset $\mathbb{N}_2$ of $\mathbb{N}_1$ and a function $z_2 \in C([1, T_1], \mathbb{R})$ such that

$$\{u_k\} \to z_2 \ \text{ as } \ k \to \infty \text{ through } \mathbb{N}_2$$

where $z_1 = z_2$ on $[1, T_1]$ since $\mathbb{N}_2 \subset \mathbb{N}_1$.

Proceeding inductively, we see that for $t_1, t_2 \in J_m$ with $t_1 < t_2$, there is $v_k^m \in S_{F,u}$, such that

$$\begin{aligned} |u_k(t_2) - u_k(t_1)| \leq & \frac{\|p\|_\infty \psi(M)}{\Gamma(r)} \int_1^{t_1} \left[\left(\log \frac{t_2}{s}\right)^{r-1} - \left(\log \frac{t_1}{s}\right)^{r-1}\right] \frac{ds}{s} \\ & + \frac{\|p\|_\infty \psi(M)}{\Gamma(r)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{r-1} \frac{ds}{s} \\ & + (\log t_2 - \log t_1) \left| \frac{r-1}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v_k^m(s) \frac{ds}{s} \right| \end{aligned}$$

and

$$\{u_k\} \to z_m \ \text{ as } \ k \to \infty \ \text{ through } \mathbb{N}_m.$$

Now, let $m \in \mathbb{N}$ with $s \leq T_m$, fix $t \in [1, \infty)$, and let $y(t) = z_m(t)$. Then $y \in C([1, \infty), \mathbb{R})$, $y(1) = y_1$, and $|y(t)| \leq M$ for $t \in [1, \infty)$.

Again for fixed $t \in [1, \infty)$ and $m \in \mathbb{N}$ with $s \leq T_m$, for $n \in \mathbb{N}_m$ there exists $v_n' \in S_{F,u}$ so that

$$\begin{aligned} u_n(t) = & \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n'(s) \frac{ds}{s} \\ & - \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v_n'(s) \frac{ds}{s} + y_1 \end{aligned}$$

as $n \to \infty$ through $\mathbb{N}_m$. Hence, there exists $v' \in S_{F,u}$, such that

$$z_m(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v'(s)\frac{ds}{s} - \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v'(s)\frac{ds}{s} + y_1,$$

that is, there exists $v \in S_{F,y}$ such that

$$y(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s)\frac{ds}{s} - \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v(s)\frac{ds}{s} + y_1.$$

We can apply this method for each $t \in [1, T_m]$ and each $m \in \mathbb{N}$. Thus,

$${}^H_C D^r y(t) \in F(t, y(t)) \quad \text{for a.e.} \quad t \in J = [1, T_m], \; 1 < r \leq 2, \tag{3.10}$$

for each $m \in \mathbb{N}$. This completes the proof of the theorem. □

3.2. The nonconvex case

We now consider the case where right hand side of problem (1.1)–(1.2) is nonconvex valued. In this case the proof relies on the fixed point result contained in Theorem 2.3.

Theorem 3.6. *In addition to condition (H4) assume that:*

(H5) $F : J_m \times \mathbb{R} \to P_{cp}(\mathbb{R})$ has the property that $F(\cdot, u) : J_m \to P_{cp}(\mathbb{R})$ is measurable for each $u \in \mathbb{R}$.

If

$$2\frac{(\log T_m)^r}{\Gamma(r+1)} \|l\|_{L^1(J_m,\mathbb{R})} < 1, \tag{3.11}$$

then the problem (1.1)–(1.2) has at least one solution on J.

Remark 3.7. By (H5), we can see that $S_{F,y}$ is nonempty for each $y \in C(J_m, \mathbb{R})$, so F has a measurable selection by [11, Theorem III.6].

Proof. We will show that N satisfies the conditions of Theorem 2.3. Once again our proof will be given in steps.

Step 1: $N(y) \in P_{cl}(C(J_m, \mathbb{R}))$ *for each* $y \in C(J_m, \mathbb{R})$. Let $(y_n)_{n\geq 0} \subset N(y)$ be such that $y_n \to \bar{y}$. Then, $\bar{y} \in C(J_m, \mathbb{R})$ and there exists $v_n \in S_{F,y}$, $n = 1, 2, \ldots$ such that, for each $t \in J_m$,

$$y_n(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_n(s)\frac{ds}{s} - \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v_n(s)\frac{ds}{s} + y_1.$$

From the fact that F has compact values and condition (H4), passing if necessary to a subsequence, we can conclude that v_n converges weakly to v in $L^1_w(J_m, \mathbb{R})$ (the space endowed with the weak topology).

Applying Mazur's theorem, this implies that v_n is strongly convergent to v and hence $v \in S_{F,y}$. Then, for $t \in J_m$,

$$y_n(t) \to \bar{y}(t) = \frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v(s) \frac{ds}{s} - \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v(s) \frac{ds}{s} + y_1.$$

Hence, $\bar{y} \in N(y)$.

Step 2: *There exists $\gamma < 1$ such that $H_d(N(y), N(\bar{y})) \leq \gamma \|y - \bar{y}\|_\infty$ for y, $\bar{y} \in C(J_m, \mathbb{R})$.* Let y, $\bar{y} \in C(J_m, \mathbb{R})$ and $h_1 \in N(y)$. Then, there exists $v_1 \in F(t, y(t))$ such that for each $t \in J_m$

$$\begin{aligned} h_1(t) = &\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_1(s) \frac{ds}{s} \\ &- \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v_1(s) \frac{ds}{s} + y_1. \end{aligned}$$

From (H4) it follows that

$$H_d(F(t, y(t)), F(t, \bar{y}(t)) \leq l(t)|y(t) - \bar{y}(t)|.$$

Hence, there exists $w \in F(t, \bar{y}(t))$ such that

$$|v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|, \ t \in J_m.$$

Consider $U : J_m \to \mathcal{P}(\mathbb{R})$ given by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq l(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \bar{y}(t))$ is measurable, there exists a function $v_2(t)$ that is a measurable selection for V. So, $v_2 \in F(t, \bar{y}(t))$, and for each $t \in J_m$,

$$|v_1(t) - v_2(t)| \leq l(t)|y(t) - \bar{y}(t)|, \ t \in J_m.$$

Let us define, for $v_2 \in S_{F,\bar{y}}$,

$$\begin{aligned} h_2(t) = &\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} v_2(s) \frac{ds}{s} \\ &- \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-2} v_2(s) \frac{ds}{s} + y_1. \end{aligned}$$

Then, for each $t \in J_m$,

$$\begin{aligned} |h_1(t) - h_2(t)| \leq &\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |v_1(s) - v_2(s)| \frac{ds}{s} \\ &+ \frac{(r-1)(\log t)}{\Gamma(r)} \int_1^{T_m} \left(\log \frac{T_m}{s}\right)^{r-1} |v_1(s) - v_2(s)| \frac{ds}{s} \\ \leq &\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{r-1} |y(s) - \bar{y}(s)| l(s) \frac{ds}{s} \end{aligned}$$

$$+\frac{(r-1)(\log t)}{\Gamma(r)}\int_1^{T_m}\left(\log\frac{T_m}{s}\right)^{r-1}|y(s)-\bar{y}(s)|l(s)\frac{ds}{s}$$
$$\leq\left[2\frac{(\log T_m)^r}{\Gamma(r+1)}\int_1^{T_m}l(s)ds\right]\|y-\bar{y}\|_\infty.$$

Thus,

$$\|h_1-h_2\|_\infty\leq\left[2\frac{(\log T_m)^r}{\Gamma(r+1)}\|l\|_{L^1(J_m,\mathbb{R})}\right]\|y-\bar{y}\|_\infty.$$

An analogous relation obtained by interchanging the roles of y and $\bar{y}$ gives

$$H_d(N(y),N(\bar{y}))\leq\left[2\frac{(\log T_m)^r}{\Gamma(r+1)}\|l\|_{L^1(J_m,\mathbb{R})}\right]\|y-\bar{y}\|_\infty.$$

Hence, by (3.11), N is a contraction, so by Theorem 2.3, N has a fixed point y that is a solution to (1.1)–(1.2).

Step 3: *The diagonalization process.* We can use a similar argument to the one used in Step 6 in the proof of Theorem 3.5. Thus

$${}^H_CD^r y(t)\in F(t,y(t)) \ \text{ for a.e. } \ t\in J=[1,T_m], \quad 1<r\leq 2, \tag{3.12}$$

for each $m\in\mathbb{N}$. This proves the theorem. □

4. An example

We conclude this paper with an example illustrating our main result. We apply Theorem 3.5 to the fractional differential inclusion

$${}^H_CD^r y(t)\in F(t,y(t)), \quad \text{for a.e } \ t\in J=[1,\infty), \ 1<r\leq 2, \tag{4.1}$$

$$y(1)=y_1, \ y \text{ is bounded on } [1,\infty). \tag{4.2}$$

We set

$$F(t,y)=\{v\in\mathbb{R}: f_1(t,y)\leq v\leq f_2(t,y)\}$$

where $f_1,f_2:[1,T_m]\times\mathbb{R}\mapsto\mathbb{R}$. We assume that for each $t\in[1,T_m]$, $f_1(t,\cdot)$ is lower semi-continuous (i.e., the set $\{y\in\mathbb{R}: f_1(t,y)>\mu\}$ is open for each $\mu\in\mathbb{R}$), and assume that for each $t\in[1,T_m]$, $f_2(t,\cdot)$ is upper semi-continuous (i.e., the set $\{y\in\mathbb{R}: f_2(t,y)<\mu\}$ is open for each $\mu\in\mathbb{R}$). Assume that there exist $p\in C([1,T_m],\mathbb{R}^+)$ and a continuous and nondecreasing function $\psi:[0,\infty)\mapsto(0,\infty)$ such that

$$\begin{aligned}\|F(t,y)\|_{\mathcal{P}}&=\sup\{|v|,v(t)\in F(t,y)\}\\&=\max(|f_1(t,y)|,|f_2(t,y)|)\leq p(t)\psi(|y|),\quad\text{for each } t\in[1,T_m],\ y\in\mathbb{R}.\end{aligned}$$

It is clear that F is compact and convex-valued and is upper semi-continuous. Finally, we assume that there exists a number $C>0$ such that

$$\frac{C}{\frac{2(\log T_m)^r}{\Gamma(r+1)}\|p\|_\infty\psi(C)+|y_1|}>1. \tag{4.3}$$

Since all the conditions of Theorem 3.5 are satisfied, problem (4.1)–(4.2) has at least one bounded solution y on $[1,\infty)$.

5. Conclusions

In this paper we consider a boundary value problem for a fractional differential inclusion involving the Caputo-Hadamard type derivative of order $r \in (1, 2]$ on the infinite interval $[1, \infty)$. We give sufficient conditions for the existence of solutions in case the right hand side of the inclusion is convex valued and where it is not. In the convex valued case, the nonlinear alternative of Leray-Schauder type is used in the proof, and in the nonconvex case, the Covitz-Nadler fixed point theorem for multivalued contractions is applied. Due to the fact that our problem is on an infinite interval, a diagonalization method was needed to complete the proofs. This was the first time the diagonalization method has been applied to such problems.

Conflicts of interest

All authors declare no conflicts of interest in this paper.

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