Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(6): 6087–6106.
DOI:10.3934/math.2021357
Received: 05 December 2020
Accepted: 17 March 2021
Published: 01 April 2021

Research article

The Fekete-Szegö type inequalities for certain subclasses analytic functions associated with petal shaped region

Pinhong Long[1,*], **Xing Li**[1], **Gangadharan Murugusundaramoorthy**[2,*]**and Wenshuai Wang**[1]

[1] School of Mathematics and Statistics, Ningxia University, Yinchuan, Ningxia 750021, China
[2] Department of Mathematics, VIT University, Vellore, Tamil Nadu 632014, India

* **Correspondence:** Emails: longph@nxu.edu.cn, gmsmoorthy@yahoo.com.

Abstract: In the article we introduce several new subclasses of analytic functions associated with pedal shaped functions. By using differential subordination and convolution operator theory, we obtain the bound estimations of the coefficients a_2 and a_3, and the logarithmic coefficients d_1 and d_2 as well as Fekete-Szegö type functional inequalities for these subclasses.

Keywords: Fekete-Szegö problem; univalent function; subordination; pedal shaped function
Mathematics Subject Classification: 30C45, 30C50

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + a_2z^2 + a_3z^3 + a_4z^4 + \cdots , \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z :| z |< 1\}$ and normalized by $f(0) = 0$ and $f'(0) = 1$. Recall that, $\mathcal{S} \subset \mathcal{A}$ is the univalent function in $\mathbb{D} = \{z :| z |< 1\}$ and has the star-like and convex function classes as its sub-classes which their geometric conditions satisfy

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \tag{1.2}$$

respectively. The two well-known sub-classes have been used to define different subclasses of analytical functions in different direction with different perspective and their results are too voluminous in literature.

For two functions f and g, f is said to be subordinate to g, written as $f \prec g$, if there exists a Schwartz function $w(z)$ such that

$$f(z) = g(w(z)), \quad z \in \mathbb{D}, \tag{1.3}$$

where $w(0) = 0$ and $| w(z) |< 1$ for $z \in \mathbb{D}$.

Goodman [2] proposed the concept of conic domain to generalize convex function which generated the first parabolic region as an image domain of analytic function. Besides, he also introduced and studied the class $\mathcal{UCV}$ of uniformly convex functions which satisfy

$$\Re\left\{1 + (z-\psi)\frac{f''(z)}{f'(z)}\right\} > 0, (z, \psi \in \mathbb{D}).$$

Lately, Ma and Minda [8] and Rønning[15] independently studied the underneath characterization

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \mathbb{D}. \tag{1.4}$$

Rønning [15] also defined a class $\mathcal{ST}$ as below

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \mathbb{D}. \tag{1.5}$$

Further, we say that f of the form (1.1) is in $\mathcal{USD}$, if

$$\Re\{f'(z)\} \geq |f'(z) - 1|, z \in \mathbb{D}. \tag{1.6}$$

The above characterization given in (1.4), resulted in the first parabolic region of the form

$$\Omega = \{w; \Re(w) >| w - 1 |\}, \tag{1.7}$$

which was later generalized by Kanas and Wisniowska [3, 4] to

$$\Omega_k = \{w; \Re(w) > k \mid w - 1 \mid\}, k \geq 0. \tag{1.8}$$

We note that Ω_k represents the right half plane for $k = 0$, hyperbolic region for $0 < k < 1$, parabolic region for $k = 1$ and elliptic region for $k > 1$.

From then on, the generalized conic region (1.8) has been studied by many researchers (see [10, 12] and also references cited therein). Moreover, the conic domain Ω was generalized to domain $\Omega[A, B]$, $-1 \leq B < A \leq 1$, by Noor and Malik [13] via

$$\begin{aligned}\Omega[A, B] = \{u + iv : [(B^2 - 1)(u^2 + v^2) - 2(AB - 1)u + (A^2 - 1)]^2 \\ > [-2(B + 1)(u^2 + v^2) + 2(A + B + 2)u - 2(A + 1)]^2 + 4(A - B)^2v^2\},\end{aligned}$$

which is called petal shaped region (also see [11]).

A function $p(z) \in \mathcal{UP}[A, B]$, if and only if

$$p(z) \prec \frac{(A + 1)\tilde{p}(z) - (A - 1)}{(B + 1)\tilde{p}(z) - (B - 1)}, \tag{1.9}$$

where "$\prec$" denotes subordination, and $\tilde{p} = 1 + \frac{2}{\pi^2}\left(log\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right)^2$.

Fixing $A = 1$ and $B = -1$ in (1.9), the usual classes of functions studied by Goodman [1] and Kanas [3, 4] can be obtained.

Furthermore, the classes $\mathcal{UCV}[A, B]$ and $\mathcal{ST}[A, B]$ are uniformly Janowski convex and starlike defined below:
A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}[A, B]$ $-1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\left(1+\frac{zf''(z)}{f'(z)}\right)-(A-1)}{(B+1)\left(1+\frac{zf''(z)}{f'(z)}\right)-(A+1)}\right) > \left|\frac{(B-1)\left(1+\frac{zf''(z)}{f'(z)}\right)-(A-1)}{(B+1)\left(1+\frac{zf''(z)}{f'(z)}\right)-(A+1)}-1\right|, \tag{1.10}$$

or equivalently

$$1+\frac{zf''(z)}{f'(z)} \in \mathcal{UP}[A, B].$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{ST}[A, B]$, $-1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}\right) > \left|\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}-1\right|, \tag{1.11}$$

or equivalently

$$\frac{zf'(z)}{f(z)} \in \mathcal{UP}[A, B].$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{USD}[A, B]$, $-1 \leq B < A \leq 1$, if and only if

$$\Re\left(\frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)}\right) > \left|\frac{(B-1)f'(z)-(A-1)}{(B+1)f'(z)-(A+1)}-1\right|, \tag{1.12}$$

or equivalently

$$f'(z) \in \mathcal{UP}[A, B].$$

It can easily be seen that $f(z) \in \mathcal{UCV}[A, B] \Leftrightarrow zf'(z) \in \mathcal{ST}[A, B]$. Setting $A = 1$ and $B = -1$ in (1.10) and (1.11), we obtained the classes of functions investigated by Goodman [2] and Rønning [15].

The relevant connection to Fekete-Szegö problem is a way of maximizing the non-linear functional $\left|a_3 - \mu a_2^2\right|$ for various subclasses of univalent functions. To know much more of history, we refer the readers to [5, 7, 16]. Inspired by earlier work in [14], in this paper we study the coefficient inequalities for certain subclasses of analytical functions related to petal type region. The first few coefficient bounds and the relevant connection to Fekete-Szegö inequalities were obtained for the classes of functions defined. Also note that, the results obtained here have not been in literature and the varying of parameters involved can give rise to new or known results. For the purpose of the main results, the following lemmas and definitions are needed.

Lemma 1.1. *[1] Let $\mathcal{P}$ be the class of all analytic functions $h(z)$ of the following form*

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (z \in \mathbb{D}) \tag{1.13}$$

satisfying $\Re[h(z)] > 0$ and $h(0) = 1$. Then $| c_n | \leq 2 (n \in \mathbb{N})$ and the result is best possible for $h(z) = \frac{1+\rho z}{1-\rho z}, |\rho| = 1$.

The next lemmas give us a majorant for the coefficients of the functions of the class $\mathcal{P}$, and more details may be found in [9]:

Lemma 1.2. *[9] If $h(z) \in \mathcal{P}$ is given by (1.13), then, for any complex μ,*

$$|c_2 - \mu c_1^2| \leq 2 \max \{1, |2\mu - 1|\}$$

and the result is sharp for the functions

$$h_0(z) = \frac{1+z}{1-z} \quad or \quad h(z) = \frac{1+z^2}{1-z^2}, \quad (z \in \mathbb{D}).$$

Lemma 1.3. *[9] (Lemma 1 and Remark, pp. 162-163) Let $h(z) \in \mathcal{P}$ be given by (1.13). Then*

$$| c_2 - \mu c_1^2 | \leq \begin{cases} -4\mu + 2, & \text{if } \mu \leq 0, \\ 2, & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 2, & \text{if } \mu \geq 1. \end{cases}$$

When $\mu < 0$ or $\mu > 1$, the equality holds if and only if $h_1 = \dfrac{1+z}{1-z}$ or one of its rotations. If $0 < \mu < 1$, then equality holds if and only if $h_2 = \dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $\mu = 0$, the equality holds if and only if

$$h_3(z) = \left(\frac{1}{2} + \frac{\eta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\eta}{2}\right)\frac{1-z}{1+z}, \ (0 \leq \eta \leq 1)$$

or one of its rotations. If $\mu = 1$, then the sharp result holds for the following function

$$\frac{1}{h_3(z)} = \left(\frac{1}{2} + \frac{\eta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\eta}{2}\right)\frac{1-z}{1+z}, \ (0 \leq \eta \leq 1)$$

or one of its rotations. Although the above upper bound is sharp, when $0 < \mu < 1$, it can be improved as follows:

$$|c_2 - \mu c_1^2| + \mu |c_1|^2 \leq 2, \quad (0 < \mu \leq \frac{1}{2}) \tag{1.14}$$

and

$$|c_2 - \mu c_1^2| + (1-\mu)|c_1|^2 \leq 2, \quad (\frac{1}{2} < \mu < 1). \tag{1.15}$$

Definition 1.4. For $0 \leq \lambda \leq 1$, and $-1 \leq B < A \leq 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}[\lambda, A, B]$, if and only if

$$\Re\left(\frac{(B-1)\mathcal{F}(z) - (A-1)}{(B+1)\mathcal{F}(z) - (A+1)}\right) > \left|\frac{(B-1)\mathcal{F}(z) - (A-1)}{(B+1)\mathcal{F}(z) - (A+1)} - 1\right|, \tag{1.16}$$

or equivalently $(f'(z))^{\lambda} \left(\frac{zf'(z)}{f(z)}\right)^{1-\lambda} = \mathcal{F}(z) \in \mathcal{UP}[A, B]$.

Remark 1.5. We note that by fixing $\lambda = 0$, then $\mathcal{N}[\lambda, A, B] \equiv \mathcal{ST}[A, B]$ and $\lambda = 1$, then $\mathcal{N}[\lambda, A, B] \equiv \mathcal{USD}[A, B]$

Definition 1.6. For $0 \leq \lambda \leq 1$, and $-1 \leq B < A \leq 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}[\lambda, A, B]$, if and only if

$$\Re\left(\frac{(B-1)\mathcal{G}(z)-(A-1)}{(B+1)\mathcal{G}(z)-(A+1)}\right) > \left|\frac{(B-1)\mathcal{G}(z)-(A-1)}{(B+1)\mathcal{G}(z)-(A+1)} - 1\right|, \tag{1.17}$$

or equivalently $(f'(z))^{\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\lambda} = \mathcal{G}(z) \in \mathcal{UP}[A, B]$.

Remark 1.7. We note that by taking $\lambda = 0$, then $\mathcal{M}[\lambda, A, B] \equiv \mathcal{UCV}[A, B]$ and $\lambda = 1$, then $\mathcal{M}[\lambda, A, B] \equiv \mathcal{USD}[A, B]$

Definition 1.8. For $0 \leq \lambda \leq 1$, and $-1 \leq B < A \leq 1$, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{W}[\lambda, A, B]$, if and only if

$$\Re\left(\frac{(B-1)\mathcal{Q}(z)-(A-1)}{(B+1)\mathcal{Q}(z)-(A+1)}\right) > \left|\frac{(B-1)\mathcal{Q}(z)-(A-1)}{(B+1)\mathcal{Q}(z)-(A+1)} - 1\right|, \tag{1.18}$$

or equivalently $\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\lambda} = \mathcal{Q}(z) \in \mathcal{UP}[A, B]$.

Remark 1.9. Assuming $\lambda = 0$, we note that $\mathcal{W}[\lambda, A, B] \equiv \mathcal{UCV}[A, B]$ and $\lambda = 1$, then $\mathcal{W}[\lambda, A, B] \equiv \mathcal{ST}[A, B]$

2. Coefficient bounds and Fekete-Szegö inequality

In this section, we let $-1 \leq B < A \leq 1$, $0 \leq \lambda \leq 1$, unless otherwise stated. To prove our main results we recall the following:
Let $h \in \mathcal{P}$ of the form (1.13). Consider

$$h(z) = \frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. Then, it follows easily that

$$w(z) = \frac{h(z)-1}{h(z)+1} = \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_2 c_1}{2} + \frac{c_1^3}{8}\right)z^3 + \cdots. \tag{2.1}$$

Now, if $\tilde{p}(z) = 1 + R_1 z + R_2 z^2 + \cdots$, where $R_1 = \frac{8}{\pi^2}$, $R_2 = \frac{16}{3\pi^2}$ and $R_3 = \frac{184}{45\pi^2}$ (see [6]), then we have

$$\tilde{p}(w(z)) = 1 + R_1 w(z) + R_2 (w(z))^2 + R_3 (w(z))^3 + \cdots. \tag{2.2}$$

Hence, from (2.1) and (2.2) we get

$$\tilde{p}(w(z)) = 1 + \frac{4c_1}{\pi^2}z + \frac{4}{\pi^2}\left(c_2 - \frac{c_1^2}{6}\right)z^2 + \frac{4}{\pi^2}\left(c_3 - \frac{c_1c_2}{3} + \frac{2c_1^3}{45}\right)z^3 + \cdots. \tag{2.3}$$

If $p \in \mathcal{UP}[A, B]$, from the relation (2.3), we may derive

$$\begin{aligned} p(z) &= \frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B-1)} \\ &= \frac{2 + (A+1)\frac{4}{\pi^2}c_1z + (A+1)\frac{4}{\pi^2}(c_2 - \frac{c_1^2}{6})z^2 + \cdots}{2 + (B+1)\frac{4}{\pi^2}c_1z + (B+1)\frac{4}{\pi^2}(c_2 - \frac{c_1^2}{6})z^2 + \cdots}. \end{aligned}$$

Thereby, it implies that

$$\begin{aligned} p(z) = 1 &+ \frac{2(A-B)c_1}{\pi^2}z + \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)c_1^2}{\pi^2}\right)z^2 + \frac{8(A-B)}{\pi^2} \\ &\times \left[\left(\frac{(B+1)^2}{\pi^4} + \frac{B+1}{6\pi^2} + \frac{1}{90}\right)c_1^3 - \left(\frac{B+1}{\pi^2} + \frac{1}{12}\right)c_1c_2 + \frac{c_3}{4}\right]z^3 + \cdots. \end{aligned} \tag{2.4}$$

Theorem 2.1. *Let $f \in \mathcal{N}[\lambda, A, B]$ and f be of the form* (1.1)*. Then, we have*

$$|a_2| \leq \frac{4(A-B)}{(1+\lambda)\pi^2}, \tag{2.5}$$

$$|a_3| \leq \frac{4(A-B)}{(2+\lambda)\pi^2} \max\left\{1, 2|\widetilde{\Theta}|\right\} \tag{2.6}$$

for

$$\widetilde{\Theta} = \frac{2(B+1)}{\pi^2} - \frac{(2+\lambda)(1-\lambda)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3}.$$

Furthermore, for a complex number μ, we get

$$|a_3 - \mu a_2^2| \leq \frac{4(A-B)}{(2+\lambda)\pi^2} \max\{1, 2|\widetilde{\Lambda}|\}, \tag{2.7}$$

where

$$\widetilde{\Lambda} = \frac{2(B+1)}{\pi^2} - \frac{(2+\lambda)(1-\lambda-2\mu)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3}.$$

Proof. Assume that $f \in \mathcal{N}[\lambda, A, B]$. Then, it follows from the relation (1.9) and Definition 1.4 that

$$(f'(z))^{\lambda}\left(\frac{zf'(z)}{f(z)}\right)^{1-\lambda} = \frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B-1)},$$

where $w(z)$ is such that $w(0) = 0$ and $|\, w(z) \,| < 1$. From (2.4) we get

$$\frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B-1)} = 1 + \frac{2(A-B)c_1}{\pi^2}z + \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)c_1^2}{\pi^2}\right)z^2$$

$$+\frac{8(A-B)}{\pi^2}\left[\left(\frac{(B+1)^2}{\pi^4}+\frac{B+1}{6\pi^2}+\frac{1}{90}\right)c_1^3-\left(\frac{B+1}{\pi^2}+\frac{1}{12}\right)c_1c_2+\frac{c_3}{4}\right]z^3+\cdots. \tag{2.8}$$

If $f(z)$ is given by (1.1), then we have

$$\begin{aligned}(f'(z))^\lambda\left(\frac{zf'(z)}{f(z)}\right)^{1-\lambda} &= 1+(1+\lambda)a_2z\\ &+\frac{1}{2}(2+\lambda)[2a_3-(1-\lambda)a_2^2]z^2+\cdots.\end{aligned} \tag{2.9}$$

From the comparison of coefficients of z and z^2 in (2.8) and (2.9), we obtain

$$a_2=\frac{2(A-B)c_1}{\pi^2(1+\lambda)} \tag{2.10}$$

and

$$\frac{1}{2}(2+\lambda)[2a_3-(1-\lambda)a_2^2]=\frac{2(A-B)}{\pi^2}\left(c_2-\frac{c_1^2}{6}-\frac{2(B+1)c_1^2}{\pi^2}\right). \tag{2.11}$$

By using (2.10) in (2.11), we get

$$\begin{aligned}a_3 &= \frac{2(A-B)}{(2+\lambda)\pi^2}\left[c_2-\frac{1}{6}c_1^2-\frac{2(B+1)}{\pi^2}c_1^2+\frac{(2+\lambda)(1-\lambda)(A-B)}{(1+\lambda)^2\pi^2}c_1^2\right]\\ &= \frac{2(A-B)}{(2+\lambda)\pi^2}(c_2-\vartheta c_1^2),\end{aligned} \tag{2.12}$$

where

$$\vartheta=\frac{1}{6}+\frac{2(B+1)}{\pi^2}-\frac{(2+\lambda)(1-\lambda)(A-B)}{(1+\lambda)^2\pi^2}.$$

In view of Lemma 1.1, from (2.10) we get

$$|a_2|\leq\frac{4(A-B)}{(1+\lambda)\pi^2},$$

and by applying Lemma 1.2 to (2.12), we get

$$\begin{aligned}|a_3| &\leq \frac{4(A-B)}{(2+\lambda)\pi^2}\max\{1,|2\vartheta-1|\}\\ &= \frac{4(A-B)}{(2+\lambda)\pi^2}\max\left\{1,2\left|\frac{2(B+1)}{\pi^2}-\frac{(2+\lambda)(1-\lambda)(A-B)}{(1+\lambda)^2\pi^2}-\frac{1}{3}\right|\right\}\end{aligned}$$

so that we get the desired result in (2.5) and (2.6).

Now, for $\mu\in\mathbb{C}$, we note that

$$\begin{aligned}&|a_3-\mu a_2^2|\\ &= \frac{2(A-B)}{(2+\lambda)\pi^2}\left|c_2-c_1^2\left(\frac{1}{6}+\frac{2(B+1)}{\pi^2}-\frac{(2+\lambda)(1-\lambda-2\mu)(A-B)}{(1+\lambda)^2\pi^2}\right)\right|\end{aligned}$$

$$= \frac{2(A-B)}{(2+\lambda)\pi^2}\left|c_2 - \hbar c_1^2\right|,$$

where

$$\hbar = \frac{1}{6} + \frac{2(B+1)}{\pi^2} - \frac{(2+\lambda)(1-\lambda-2\mu)(A-B)}{(1+\lambda)^2\pi^2}.$$

By Lemma 1.2, we get

$$|a_3 - \mu a_2^2| \leq \frac{4(A-B)}{(2+\lambda)\pi^2}\max\left\{1, 2\left|\frac{2(B+1)}{\pi^2} - \frac{(2+\lambda)(1-\lambda-2\mu)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3}\right|\right\},$$

which yields the desired result in (2.7). □

Theorem 2.2. *Let* $f \in \mathcal{M}[\lambda, A, B]$ *be of the form* (1.1). *Then*

$$|a_2| \leq \frac{2(A-B)}{\pi^2}, \tag{2.13}$$

$$|a_3| \leq \frac{4(A-B)}{3(2-\lambda)\pi^2}\max\left\{1, 2|\widetilde{\Xi}|\right\} \tag{2.14}$$

for

$$\widetilde{\Xi} = \frac{2(B+1)}{\pi^2} - \frac{2(A-B)(1-\lambda)}{\pi^2} - \frac{1}{3}.$$

Further, for a complex number μ*, we have*

$$|a_3 - \mu a_2^2| \leq \frac{4(A-B)}{3(2-\lambda)\pi^2}\max\{1, 2|\widetilde{\Pi}|\} \tag{2.15}$$

for

$$\widetilde{\Pi} = \frac{2(B+1)}{\pi^2} - \frac{(A-B)}{2\pi^2}[4(1-\lambda) - 3\mu(2-\lambda)] - \frac{1}{3}.$$

Proof. Suppose that $f \in \mathcal{M}[\lambda, A, B]$ and of the form (1.1). Then, it follows from the relation (1.9) and Definition 1.6 that

$$(f'(z))^\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\lambda} = \frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B-1)},$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. From (2.4) we assert that

$$\begin{aligned}\frac{(A+1)\tilde{p}(w(z)) - (A-1)}{(B+1)\tilde{p}(w(z)) - (B-1)} &= 1 + \frac{2(A-B)c_1}{\pi^2}z \\ &+ \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)}{\pi^2}c_1^2\right)z^2 + \frac{8(A-B)}{\pi^2} \\ &\times\left[\left(\frac{(B+1)^2}{\pi^4} + \frac{B+1}{6\pi^2} + \frac{1}{90}\right)c_1^3 - \left(\frac{B+1}{\pi^2} + \frac{1}{12}\right)c_1c_2 + \frac{c_3}{4}\right]z^3 + \cdots.\end{aligned} \tag{2.16}$$

Since f is given by (1.1), we know

$$(f'(z))^\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\lambda} = 1 + 2a_2z$$

$$+[3(2-\lambda)a_3-4(1-\lambda)a_2^2]z^2+\cdots. \tag{2.17}$$

By the comparison of coefficients of z and z^2 in (2.16) and (2.17), we get

$$a_2=\frac{(A-B)c_1}{\pi^2} \tag{2.18}$$

of which by Lemma 1.1 gives

$$|a_2|\le\frac{2(A-B)}{\pi^2},$$

and

$$\begin{aligned} a_3 &= \frac{2(A-B)}{3(2-\lambda)\pi^2}\left[c_2-\left(\frac{1}{6}+\frac{2(B+1)}{\pi^2}-\frac{2(A-B)(1-\lambda)}{\pi^2}\right)c_1^2\right] \\ &= \frac{2(A-B)}{3(2-\lambda)\pi^2}(c_2-\vartheta c_1^2), \end{aligned} \tag{2.19}$$

where

$$\vartheta=\frac{1}{6}+\frac{2(B+1)}{\pi^2}-\frac{2(A-B)(1-\lambda)}{\pi^2}.$$

By Lemma 1.2, we can deduce that

$$\begin{aligned} |a_3| &\le \frac{4(A-B)}{3(2-\lambda)\pi^2}\max\{1,|2\vartheta-1|\} \\ &= \frac{4(A-B)}{3(2-\lambda)\pi^2}\max\left\{1,2\left|\frac{2(B+1)}{\pi^2}-\frac{2(A-B)(1-\lambda)}{\pi^2}-\frac{1}{3}\right|\right\}, \end{aligned}$$

which yields the desired result in (2.14).

Now, for a complex number μ,

$$\begin{aligned} &\left|a_3-\mu a_2^2\right| \\ =& \frac{2(A-B)}{3(2-\lambda)\pi^2}\left|c_2-c_1^2\left(\frac{1}{6}+\frac{2(B+1)}{\pi^2}-\frac{2(1-\lambda)(A-B)}{\pi^2}+\frac{3\mu(A-B)(2-\lambda)}{2\pi^2}\right)\right| \\ =& \frac{2(A-B)}{3(2-\lambda)\pi^2}\left|c_2-\varrho c_1^2\right|, \end{aligned}$$

where

$$\varrho=\frac{1}{6}+\frac{2(B+1)}{\pi^2}-\frac{(A-B)}{2\pi^2}\left[4(1-\lambda)-3\mu(2-\lambda)\right].$$

Hence, by means of Lemma 1.2 we get the desired result (2.15). □

Theorem 2.3. *If the function $f\in\mathcal{W}[\lambda,A,B]$ is of the form* (1.1)*,then*

$$|a_2| \le \frac{4(A-B)}{(2-\lambda)\pi^2}, \tag{2.20}$$

$$|a_3| \le \frac{2(A-B)}{(3-2\lambda)\pi^2}\max\{1,2|\widetilde{\Phi}|\} \tag{2.21}$$

for

$$\widetilde{\Phi} = \frac{2(B+1)}{\pi^2} + \frac{(A-B)(\lambda^2+5\lambda-8)}{(2-\lambda)^2\pi^2} - \frac{1}{3}.$$

Moreover, for a complex number μ, we obtain

$$|a_3 - \mu a_2^2| \leq \frac{2(A-B)}{(3-2\lambda)\pi^2} \max\{1, 2|\widetilde{\Psi}|\} \tag{2.22}$$

for

$$\widetilde{\Psi} = \frac{2(B+1)}{\pi^2} + \frac{(A-B)[\lambda^2+5\lambda-8+4(3-2\lambda)\mu]}{(2-\lambda)^2\pi^2} - \frac{1}{3}.$$

Proof. Let $f \in \mathcal{W}[\lambda, A, B]$. Then, applying the relation (1.9) and Definition 1.8 we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\lambda} = \frac{(A+1)\tilde{p}(w(z))-(A-1)}{(B+1)\tilde{p}(w(z))-(B-1)}, \tag{2.23}$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. From (2.4) we get

$$\begin{aligned}\frac{(A+1)\tilde{p}(w(z))-(A-1)}{(B+1)\tilde{p}(w(z))-(B-1)} &= 1 + \frac{2(A-B)c_1}{\pi^2}z \\ &+ \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)c_1^2}{\pi^2}\right)z^2 + \cdots .\end{aligned} \tag{2.24}$$

In view of (1.1), we obtain

$$\begin{aligned}\left(\frac{zf'(z)}{f(z)}\right)^{\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\lambda} &= 1 + (2-\lambda)a_2 z \\ &+ \left[2(3-2\lambda)a_3 + \frac{1}{2}(\lambda^2+5\lambda-8)a_2^2\right]z^2 + \cdots .\end{aligned} \tag{2.25}$$

By the comparison of coefficients of z and z^2 in (2.24) and (2.25), we get

$$a_2 = \frac{2(A-B)c_1}{(2-\lambda)\pi^2} \tag{2.26}$$

and

$$2(3-2\lambda)a_3 + \frac{1}{2}(\lambda^2+5\lambda-8)a_2^2 = \frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)}{\pi^2}c_1^2\right).$$

Then, by (2.26),

$$\begin{aligned}a_3 &= \frac{1}{2(3-2\lambda)}\left[\frac{2(A-B)}{\pi^2}\left(c_2 - \frac{c_1^2}{6} - \frac{2(B+1)}{\pi^2}c_1^2\right) - \frac{2(A-B)^2[\lambda^2+5\lambda-8]}{(2-\lambda)^2\pi^4}c_1^2\right] \\ &= \frac{(A-B)}{(3-2\lambda)\pi^2}\left[c_2 - c_1^2\left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{(A-B)(\lambda^2+5\lambda-8)}{(2-\lambda)^2\pi^2}\right)\right].\end{aligned} \tag{2.27}$$

Following the procedure as in the above theorems, we can get the desired results given by (2.20) and (2.21).

Now, for a complex number μ,

$$
\begin{aligned}
&|a_3 - \mu a_2^2| \\
= &\left|\frac{(A-B)}{(3-2\lambda)\pi^2}\Big[c_2 - c_1^2\Big(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{(A-B)[\lambda^2+5\lambda-8+4\mu(3-2\lambda)]}{(2-\lambda)^2\pi^2}\Big)\Big]\right| \\
= &\frac{(A-B)}{(3-2\lambda)\pi^2}|c_2 - \xi c_1^2|,
\end{aligned}
$$

where

$$\xi = \frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{(A-B)[\lambda^2+5\lambda-8+4\mu(3-2\lambda)]}{(2-\lambda)^2\pi^2}.$$

Therefore, in light of Lemmas 1.1 and 1.2 we get the desired result (2.22). □

If we choose real μ, then by Lemma 1.3 we derive the next results for Fekete-Szegö problem for these classes above.

Theorem 2.4. *If the function $f \in \mathcal{N}[\lambda, A, B]$ be of the form* (1.1)*, and $\mu \in \mathbb{R}$ then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{8(A-B)}{(2+\lambda)\pi^2}\left\{-\frac{2(B+1)}{\pi^2} + \frac{(2+\lambda)(1-\lambda-2\mu)(A-B)}{(1+\lambda)^2\pi^2} + \frac{1}{3}\right\}, \ (\mu \leq \Upsilon_1); \\ \frac{4(A-B)}{(2+\lambda)\pi^2}, \ (\Upsilon_1 \leq \mu \leq \Upsilon_2); \\ \frac{8(A-B)}{(2+\lambda)\pi^2}\left\{\frac{2(B+1)}{\pi^2} - \frac{(2+\lambda)(1-\lambda-2\mu)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3}\right\}, \ (\mu \geq \Upsilon_2), \end{cases}$$

where

$$\Upsilon_1 = \frac{1-\lambda}{2} - \frac{[\frac{\pi^2}{6} + 2(B+1)](1+\lambda)^2}{2(2+\lambda)(A-B)}$$

and

$$\Upsilon_2 = \frac{1-\lambda}{2} - \frac{[-\frac{5\pi^2}{6} + 2(B+1)](1+\lambda)^2}{2(2+\lambda)(A-B)}.$$

Moreover, we set

$$\Upsilon_3 = \frac{1-\lambda}{2} - \frac{[-\frac{\pi^2}{3} + 2(B+1)](1+\lambda)^2}{2(2+\lambda)(A-B)}.$$

Then, each of the following results holds:
(A) For $\mu \in [\Upsilon_1, \Upsilon_3]$,

$$|a_3 - \mu a_2^2| + \frac{(1+\lambda)^2\pi^2(\widetilde{\Lambda} + \frac{1}{2})}{2(2+\lambda)(A-B)}|a_2|^2 \leq \frac{4(A-B)}{(2+\lambda)\pi^2};$$

(B) For $\mu \in [\Upsilon_3, \Upsilon_2]$,

$$|a_3 - \mu a_2^2| + \frac{(1+\lambda)^2\pi^2(-\widetilde{\Lambda} + \frac{1}{2})}{2(2+\lambda)(A-B)}|a_2|^2 \leq \frac{4(A-B)}{(2+\lambda)\pi^2},$$

where

$$\widetilde{\Lambda} = \frac{2(B+1)}{\pi^2} - \frac{(2+\lambda)(1-\lambda-2\mu)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3}.$$

Proof. If $f \in \mathcal{N}[\lambda, A, B]$ is given by (1.1), based on the proof of Theorem 2.1 we see

$$a_3 - \mu a_2^2 = \frac{2(A-B)}{(2+\lambda)\pi^2}[c_2 - \hbar c_1^2], \tag{2.28}$$

where

$$\hbar = \frac{1}{6} + \frac{2(B+1)}{\pi^2} - \frac{(2+\lambda)(1-\lambda-2\mu)(A-B)}{(1+\lambda)^2\pi^2}.$$

For $\hbar \in \mathbb{R}$, we know that $\hbar \geq 1$ is equivalent to $\mu \geq \Upsilon_2$, and $\hbar \leq 0$ is equivalent to $\mu \leq \Upsilon_1$. Therefore, taking the modulus on both sides of the above equality, with the aid of the inequality in Lemma 1.3 we obtain the first estimates of Theorem 2.4.

For the proof of the second part, note that $0 < \hbar \leq 1/2$ is equivalent to $\Upsilon_1 < \mu \leq \Upsilon_3$. By using the relations (2.28) and (2.10) , and then by applying the inequality (1.14) of Lemma 1.3, we get

$$\begin{aligned}\frac{2(A-B)}{(2+\lambda)\pi^2}\left[|c_2 - \hbar c_1^2| + \hbar|c_1^2|\right] &= |a_3 - \mu a_2^2| + \frac{2(A-B)\hbar}{(2+\lambda)\pi^2}|c_1^2| \\ = \quad |a_3 - \mu a_2^2| + \frac{(1+\lambda)^2\pi^2(\widetilde{\Lambda} + \frac{1}{2})}{2(2+\lambda)(A-B)}|a_2^2| &\leq \frac{4(A-B)}{(2+\lambda)\pi^2}\end{aligned}$$

such that the required inequality (A) holds.

Similarly, we can easily check that $1/2 \leq \hbar < 1$ is equivalent to $\Upsilon_3 \leq \mu < \Upsilon_2$. From the relations (2.28) and (2.10), and the inequality (1.15) of Lemma 1.3 we obtain

$$\begin{aligned}\frac{2(A-B)}{(2+\lambda)\pi^2}\left[|c_2 - \hbar c_1^2| + (1-\hbar)|c_1^2|\right] &= |a_3 - \mu a_2^2| + \frac{2(A-B)(1-\hbar)}{(2+\lambda)\pi^2}|c_1^2| \\ = \quad |a_3 - \mu a_2^2| + \frac{(1+\lambda)^2\pi^2(-\widetilde{\Lambda} + \frac{1}{2})}{2(2+\lambda)(A-B)}|a_2^2| &\leq \frac{4(A-B)}{(2+\lambda)\pi^2},\end{aligned}$$

which is exactly the inequality (B).

To show that the bounds are sharp, we define the functions K_{p_n} $(n = 2, 3, \ldots)$ with $K_{p_n}(0) = 0 = [K_{p_n}]'(0) - 1$, by

$$(K'_{p_n}(z))^\lambda \left(\frac{zK'_{p_n}(z)}{K_{p_n}(z)}\right)^{1-\lambda} = p(z^{n-1}) = \frac{2 + \frac{2(A+1)}{\pi^2}\left(log\frac{1+\sqrt{z^{n-1}}}{1-\sqrt{z^{n-1}}}\right)^2}{2 + \frac{2(B+1)}{\pi^2}\left(log\frac{1+\sqrt{z^{n-1}}}{1-\sqrt{z^{n-1}}}\right)^2},$$

where $p(z)$ is as given in (1.9). Also, define the functions F_η and G_η, $0 \leq \eta \leq 1$, respectively, with $F_\eta(0) = 0 = F'_\eta(0) - 1$ and $G_\eta(0) = 0 = G'_\eta(0) - 1$ by

$$(F'_\eta(z))^\lambda \left(\frac{zF'_\eta(z)}{F_\eta(z)}\right)^{1-\lambda} = p\left(\frac{z(z+\eta)}{1+\eta z}\right)$$

and

$$(G'_\eta(z))^\lambda \left(\frac{zG'_\eta(z)}{G_\eta(z)}\right)^{1-\lambda} = p\left(-\frac{z(z+\eta)}{1+\eta z}\right).$$

Clearly,

$$K_{p_n}(z) := \begin{cases} \left[\lambda \int_0^z t^{\lambda-1} p(t^{n-1})dt\right]^{\frac{1}{\lambda}}, & \text{if } 0 < \lambda \leq 1, \\ \exp\left\{\int_0^z p(t^{n-1})\frac{dt}{t}\right\}, & \text{if } \lambda = 0. \end{cases}$$

Hence, $K_{p_n}, F_\eta,\ G_\eta \in \mathcal{N}[\lambda, A, B]$. Also, we write

$$K_{p_2}(z) := \begin{cases} \left[\lambda \int_0^z t^{\lambda-1} p(t)dt\right]^{\frac{1}{\lambda}}, & \text{if } 0 < \lambda \leq 1, \\ \exp\left\{\int_0^z p(t)\frac{dt}{t}\right\}, & \text{if } \lambda = 0. \end{cases}$$

If $\mu < \Upsilon_1$ or $\mu > \Upsilon_2$, then the equality holds if and only if f is $K_{p_2}(z)$ or one of its rotations. When $\Upsilon_1 < \mu < \Upsilon_2$, then the equality holds if and only if f is $K_{p_3}(z)$ satisfying

$$K_{p_3}(z) := \begin{cases} \left[\lambda \int_0^z t^{\lambda-1} p(t^2)dt\right]^{\frac{1}{\lambda}}, & \text{if } 0 < \lambda \leq 1, \\ \exp\left\{\int_0^z p(t^2)\frac{dt}{t}\right\}, & \text{if } \lambda = 0 \end{cases}$$

or one of its rotations. If $\mu = \Upsilon_1$ then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \Upsilon_2$ then the equality holds if and only if f is G_η or one of its rotations. □

As it is similar to the above result in Theorem 2.4, we state the following results without proof.

Theorem 2.5. *If the function $f \in \mathcal{M}[\lambda, A, B]$ is of the form* (1.1) *and $\mu \in \mathbb{R}$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{8(A-B)}{3(2-\lambda)\pi^2}\left\{-\frac{2(B+1)}{\pi^2} + \frac{[4(1-\lambda)-3\mu(2-\lambda)](A-B)}{2\pi^2} + \frac{1}{3}\right\}, & (\mu \leq \Gamma_1); \\ \frac{4(A-B)}{3(2-\lambda)\pi^2}, & (\Gamma_1 \leq \mu \leq \Gamma_2); \\ \frac{8(A-B)}{3(2-\lambda)\pi^2}\left\{\frac{2(B+1)}{\pi^2} - \frac{[4(1-\lambda)-3\mu(2-\lambda)](A-B)}{2\pi^2} - \frac{1}{3}\right\}, & (\mu \geq \Gamma_2), \end{cases}$$

where

$$\Gamma_1 = \frac{4(1-\lambda)}{3(2-\lambda)} - \frac{2[\frac{\pi^2}{6} + 2(B+1)]}{3(2-\lambda)(A-B)}$$

and

$$\Gamma_2 = \frac{4(1-\lambda)}{3(2-\lambda)} - \frac{2[-\frac{5\pi^2}{6} + 2(B+1)]}{3(2-\lambda)(A-B)}.$$

Further, we set

$$\Gamma_3 = \frac{4(1-\lambda)}{3(2-\lambda)} - \frac{2[-\frac{\pi^2}{3} + 2(B+1)]}{3(2-\lambda)(A-B)}.$$

Then, each of the following results holds:
(A) For $\mu \in [\Gamma_1, \Gamma_3]$,

$$|a_3 - \mu a_2^2| + \frac{2\pi^2(\widetilde{\Pi} + \frac{1}{2})}{3(2-\lambda)(A-B)}|a_2|^2 \leq \frac{4(A-B)}{3(2-\lambda)\pi^2};$$

(B) For $\mu \in [\Gamma_3, \Gamma_2]$,

$$|a_3 - \mu a_2^2| + \frac{2\pi^2(-\widetilde{\Pi} + \frac{1}{2})}{3(2-\lambda)(A-B)}|a_2|^2 \leq \frac{4(A-B)}{3(2-\lambda)\pi^2},$$

where

$$\widetilde{\Pi} = \frac{2(B+1)}{\pi^2} - \frac{(A-B)}{2\pi^2}[4(1-\lambda) - 3\mu(2-\lambda)] - \frac{1}{3}.$$

Theorem 2.6. *Let* $\mu \in \mathbb{R}$*. If* $f \in \mathcal{A}$ *is assumed to be in* $\mathcal{W}[\lambda, A, B]$*, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4(A-B)}{(3-2\lambda)\pi^2}\left\{-\frac{2(B+1)}{\pi^2} - \frac{[\lambda^2+5\lambda-8+4\mu(3-2\lambda)](A-B)}{(2-\lambda)^2\pi^2} + \frac{1}{3}\right\}, & (\mu \leq \aleph_1); \\ \frac{2(A-B)}{(3-2\lambda)\pi^2}, & (\aleph_1 \leq \mu \leq \aleph_2); \\ \frac{4(A-B)}{(3-2\lambda)\pi^2}\left\{\frac{2(B+1)}{\pi^2} + \frac{[\lambda^2+5\lambda-8+4\mu(3-2\lambda)](A-B)}{(2-\lambda)^2\pi^2} - \frac{1}{3}\right\}, & (\mu \geq \aleph_2), \end{cases}$$

where

$$\aleph_1 = -\frac{\lambda^2+5\lambda-8}{4(3-2\lambda)} - \frac{[\frac{\pi^2}{6} + 2(B+1)](2-\lambda)^2}{4(3-2\lambda)(A-B)}$$

and

$$\aleph_2 = -\frac{\lambda^2+5\lambda-8}{4(3-2\lambda)} - \frac{[-\frac{5\pi^2}{6} + 2(B+1)](2-\lambda)^2}{4(3-2\lambda)(A-B)}.$$

Besides, we let

$$\aleph_3 = -\frac{\lambda^2+5\lambda-8}{4(3-2\lambda)} - \frac{[-\frac{\pi^2}{3} + 2(B+1)](2-\lambda)^2}{4(3-2\lambda)(A-B)}.$$

Then, each of the following results holds:
(A) For $\mu \in [\aleph_1, \aleph_3]$,

$$|a_3 - \mu a_2^2| + \frac{(2-\lambda)^2\pi^2(\widetilde{\Psi} + \frac{1}{2})}{4(3-2\lambda)(A-B)}|a_2|^2 \leq \frac{2(A-B)}{(2+\lambda)\pi^2};$$

(B) For $\mu \in [\aleph_3, \aleph_2]$,

$$|a_3 - \mu a_2^2| + \frac{(2-\lambda)^2\pi^2(-\widetilde{\Psi} + \frac{1}{2})}{4(3-2\lambda)(A-B)}|a_2|^2 \leq \frac{2(A-B)}{(2+\lambda)\pi^2},$$

where

$$\widetilde{\Psi} = \frac{2(B+1)}{\pi^2} + \frac{(A-B)[\lambda^2+5\lambda-8+4\mu(3-2\lambda)]}{(2-\lambda)^2\pi^2} - \frac{1}{3}.$$

3. The logarithmic coefficients

In this section we determine the coefficient bounds and Fekete-Szegö problems associated with logarithmic function $\mathcal{H}(z)$ defined by

$$\mathcal{H}(z) = \log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} d_n z^n, \tag{3.1}$$

where the coefficient d_n of $\mathcal{H}(z)$ is called the logarithmic coefficient of $f \in \mathcal{A}$ defined in $\mathbb{D}$. Expanding (3.1) by series expansion of $\log(1+z)$ and equating the various coefficients, we assert that

$$d_1 = \frac{a_2}{2}, \tag{3.2}$$

$$d_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right). \tag{3.3}$$

Theorem 3.1. *Let $f \in \mathcal{N}[\lambda, A, B]$ with the logarithmic coefficients in (3.1), and $\nu \in \mathbb{C}$. Then*

$$|d_1| \leq \frac{2(A-B)}{(1+\lambda)\pi^2}, \tag{3.4}$$

$$|d_2| \leq \frac{2(A-B)}{(2+\lambda)\pi^2} \max\{1, 2|\Theta|\} \tag{3.5}$$

for

$$\Theta = \frac{2(B+1)}{\pi^2} + \frac{\lambda(2+\lambda)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3},$$

and

$$|d_2 - \nu d_1^2| \leq \frac{2(A-B)}{(2+\lambda)\pi^2} \max\{1, 2|\Lambda|\} \tag{3.6}$$

for

$$\Lambda = \frac{2(B+1)}{\pi^2} + \frac{(2+\lambda)(\lambda+\nu)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3}.$$

Proof. From (2.10) and (2.12), and (3.2) and (3.3) we get

$$\begin{aligned} d_1 &= \frac{(A-B)c_1}{(1+\lambda)\pi^2}, \\ d_2 &= \frac{(A-B)}{(2+\lambda)\pi^2}\left[c_2 - \left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{\lambda(2+\lambda)(A-B)}{(1+\lambda)^2\pi^2}\right)c_1^2\right]. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} d_2 - \nu d_1^2 &= \frac{(A-B)}{(2+\lambda)\pi^2}\left[c_2 - \left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{\lambda(2+\lambda)(A-B)}{(1+\lambda)^2\pi^2}\right)c_1^2\right] \\ &- \nu\left(\frac{(A-B)c_1}{(1+\lambda)\pi^2}\right)^2 = \frac{(A-B)}{(2+\lambda)\pi^2}\left[c_2 - \rho c_1^2\right], \end{aligned} \tag{3.7}$$

where

$$\rho = \frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{(2+\lambda)(\lambda+\nu)(A-B)}{(1+\lambda)^2\pi^2}.$$

In view of Lemmas 1.1 and 1.2, we get the desired results such that Theorem 3.1 holds true. □

Theorem 3.2. *Let $f \in \mathcal{M}[\lambda, A, B]$ with the logarithmic coefficients in (3.1), and $\nu \in \mathbb{C}$. Then*

$$|d_1| \leq \frac{A-B}{\pi^2}, \tag{3.8}$$

$$|d_2| \leq \frac{2(A-B)}{3(2-\lambda)\pi^2} \max\{1, 2|\Xi|\} \tag{3.9}$$

for

$$\Xi = \frac{2(B+1)}{\pi^2} + \frac{(5\lambda-2)(A-B)}{4\pi^2} - \frac{1}{3},$$

and

$$|d_2 - \nu d_1^2| \leq \frac{2(A-B)}{3(2-\lambda)\pi^2} \max\{1, 2|\Pi|\} \tag{3.10}$$

for

$$\Pi = \frac{2(B+1)}{\pi^2} + \frac{[5\lambda + 3\nu(2-\lambda) - 2](A-B)}{4\pi^2} - \frac{1}{3}.$$

Proof. According to (2.18), (2.19), (3.2) and (3.3) we get

$$\begin{aligned} d_1 &= \frac{(A-B)c_1}{2\pi^2}, \\ d_2 &= \frac{(A-B)}{3(2-\lambda)\pi^2}\left[c_2 - \left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{(5\lambda-2)(A-B)}{4\pi^2}\right)c_1^2\right]. \end{aligned}$$

Further, we have

$$\begin{aligned} d_2 - \nu d_1^2 &= \frac{(A-B)}{3(2-\lambda)\pi^2}\left[c_2 - \left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{(5\lambda-2)(A-B)}{4\pi^2}\right)c_1^2\right] \\ &- \frac{\nu(A-B)^2 c_1^2}{4\pi^4} = \frac{(A-B)}{3(2-\lambda)\pi^2}\left[c_2 - \varrho c_1^2\right], \end{aligned} \tag{3.11}$$

where

$$\varrho = \frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{[5\lambda + 3\nu(2-\lambda) - 2](A-B)}{4\pi^2}.$$

Applying Lemmas 1.2 and 1.1, we obtain the desire estimates and complete the proof of Theorem 3.2. □

Theorem 3.3. *Let $f \in \mathcal{W}[\lambda, A, B]$ with the logarithmic coefficients in* (3.1)*, and $\nu \in \mathbb{C}$. Then*

$$|d_1| \leq \frac{2(A-B)}{(2-\lambda)\pi^2}, \tag{3.12}$$

$$|d_2| \leq \frac{(A-B)}{(3-2\lambda)\pi^2} \max\{1, 2|\Phi|\} \tag{3.13}$$

for

$$\Phi = \frac{2(B+1)}{\pi^2} + \frac{(\lambda-1)(\lambda+2)(A-B)}{(2-\lambda)^2\pi^2} - \frac{1}{3},$$

and

$$|d_2 - \nu d_1^2| \leq \frac{(A-B)}{(3-2\lambda)\pi^2} \max\{1, 2|\Psi|\} \tag{3.14}$$

for

$$\Psi = \frac{2(B+1)}{\pi^2} + \frac{[(\lambda-1)(\lambda+2) + 2\nu(3-2\lambda)](A-B)}{(2-\lambda)^2\pi^2} - \frac{1}{3}.$$

Proof. From (2.26) and (2.27), and (3.2) and (3.3) we derive

$$\begin{aligned} d_1 &= \frac{(A-B)c_1}{(2-\lambda)\pi^2}, \\ d_2 &= \frac{(A-B)}{2(3-2\lambda)\pi^2}\left[c_2 - \left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{(\lambda-1)(\lambda+2)(A-B)}{(2-\lambda)^2\pi^2}\right)c_1^2\right]. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} d_2 - \nu d_1^2 &= \frac{(A-B)}{2(3-2\lambda)\pi^2}\left[c_2 - \left(\frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{(\lambda-1)(\lambda+2)(A-B)}{(2-\lambda)^2\pi^2}\right)c_1^2\right] \\ &- \frac{\nu(A-B)^2c_1^2}{(2-\lambda)^2\pi^4} = \frac{(A-B)}{2(3-2\lambda)\pi^2}\left[c_2 - \kappa c_1^2\right], \end{aligned} \tag{3.15}$$

where

$$\kappa = \frac{1}{6} + \frac{2(B+1)}{\pi^2} + \frac{[(\lambda-1)(\lambda+2) + 2\nu(3-2\lambda)](A-B)}{(2-\lambda)^2\pi^2}.$$

By applying Lemmas 1.1 and 1.2, we get the desired results, which prove Theorem 3.3. □

If we consider real ν, then by Lemma 1.3 we provide the several results for Fekete-Szegö problem with respect to the logarithmic coefficients.

Theorem 3.4. *Let $\nu \in \mathbb{R}$. If $f \in \mathcal{N}[\lambda, A, B]$ with the logarithmic coefficients is of the form* (3.1)*, then*

$$|d_2 - \nu d_1^2| \leq \begin{cases} \frac{4(A-B)}{(2+\lambda)\pi^2}\left\{-\frac{2(B+1)}{\pi^2} - \frac{(2+\lambda)(\lambda+\nu)(A-B)}{(1+\lambda)^2\pi^2} + \frac{1}{3}\right\}, & (\nu \leq \widetilde{\Upsilon}_1); \\ \frac{2(A-B)}{(2+\lambda)\pi^2}, & (\widetilde{\Upsilon}_1 \leq \nu \leq \widetilde{\Upsilon}_2); \\ \frac{4(A-B)}{(2+\lambda)\pi^2}\left\{\frac{2(B+1)}{\pi^2} + \frac{(2+\lambda)(\lambda+\nu)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3}\right\}, & (\nu \geq \widetilde{\Upsilon}_2), \end{cases}$$

where

$$\widetilde{\Upsilon}_1 = -\lambda - \frac{[\frac{\pi^2}{6} + 2(B+1)](1+\lambda)^2}{(2+\lambda)(A-B)}$$

and

$$\widetilde{\Upsilon}_2 = -\lambda - \frac{[-\frac{5\pi^2}{6} + 2(B+1)](1+\lambda)^2}{(2+\lambda)(A-B)}.$$

Moreover, we put

$$\widetilde{\Upsilon}_3 = -\lambda - \frac{[-\frac{\pi^2}{3} + 2(B+1)](1+\lambda)^2}{(2+\lambda)(A-B)}.$$

Then, each of the following results holds:
(i) For $\nu \in [\widetilde{\Upsilon}_1, \widetilde{\Upsilon}_3]$,

$$|d_2 - \nu d_1^2| + \frac{(1+\lambda)^2\pi^2(\Lambda + \frac{1}{2})}{(2+\lambda)(A-B)}|d_1|^2 \leq \frac{2(A-B)}{(2+\lambda)\pi^2};$$

(ii) For $\nu \in [\widetilde{\Upsilon}_3, \widetilde{\Upsilon}_2]$,

$$|d_2 - \nu d_1^2| + \frac{(1+\lambda)^2\pi^2(-\Lambda + \frac{1}{2})}{(2+\lambda)(A-B)}|d_1|^2 \leq \frac{2(A-B)}{(2+\lambda)\pi^2},$$

where

$$\Lambda = \frac{2(B+1)}{\pi^2} + \frac{(2+\lambda)(\lambda+\nu)(A-B)}{(1+\lambda)^2\pi^2} - \frac{1}{3}.$$

Theorem 3.5. *Let* $\nu \in \mathbb{R}$. *If* $f \in \mathcal{M}[\lambda, A, B]$ *with the logarithmic coefficients is of the form* (3.1)*, then*

$$|d_2 - \nu d_1^2| \leq \begin{cases} \frac{4(A-B)}{3(2-\lambda)\pi^2}\left\{-\frac{2(B+1)}{\pi^2} - \frac{[5\lambda+3\nu(2-\lambda)-2](A-B)}{4\pi^2} + \frac{1}{3}\right\}, & (\nu \leq \widetilde{\Gamma}_1); \\ \frac{2(A-B)}{3(2-\lambda)\pi^2}, & (\widetilde{\Gamma}_1 \leq \nu \leq \widetilde{\Gamma}_2); \\ \frac{4(A-B)}{3(2-\lambda)\pi^2}\left\{\frac{2(B+1)}{\pi^2} + \frac{[5\lambda)+3\nu(2-\lambda)-2](A-B)}{4\pi^2} - \frac{1}{3}\right\}, & (\nu \geq \widetilde{\Gamma}_2), \end{cases}$$

where

$$\widetilde{\Gamma}_1 = -\frac{5\lambda-2}{3(2-\lambda)} - \frac{4[\frac{\pi^2}{6} + 2(B+1)]}{3(2-\lambda)(A-B)}$$

and

$$\widetilde{\Gamma}_2 = -\frac{5\lambda-2}{3(2-\lambda)} - \frac{4[-\frac{5\pi^2}{6} + 2(B+1)]}{3(2-\lambda)(A-B)}.$$

Moreover, we put

$$\widetilde{\Gamma}_3 = -\frac{5\lambda-2}{3(2-\lambda)} - \frac{4[-\frac{\pi^2}{3} + 2(B+1)]}{3(2-\lambda)(A-B)}.$$

Then, each of the following results holds:
(i) For $\nu \in [\widetilde{\Gamma}_1, \widetilde{\Gamma}_3]$,

$$|d_2 - \nu d_1^2| + \frac{4\pi^2(\Pi + \frac{1}{2})}{3(2-\lambda)(A-B)}|d_1|^2 \leq \frac{2(A-B)}{3(2-\lambda)\pi^2};$$

(ii) For $\nu \in [\widetilde{\Gamma}_3, \widetilde{\Gamma}_2]$,

$$|d_2 - \nu d_1^2| + \frac{4\pi^2(-\Pi + \frac{1}{2})}{3(2-\lambda)(A-B)}|d_1|^2 \leq \frac{2(A-B)}{3(2-\lambda)\pi^2},$$

where

$$\Pi = \frac{2(B+1)}{\pi^2} + \frac{[5\lambda + 3\nu(2-\lambda) - 2](A-B)}{4\pi^2} - \frac{1}{3}.$$

Theorem 3.6. *Let $\nu \in \mathbb{R}$. If $f \in \mathcal{W}[\lambda, A, B]$ with the logarithmic coefficients is of the form* (3.1)*, then*

$$|d_2 - \nu d_1^2| \leq \begin{cases} \frac{2(A-B)}{(3-2\lambda)\pi^2}\left\{-\frac{2(B+1)}{\pi^2} - \frac{[(\lambda+1)(\lambda+2)+2\nu(3-2\lambda)](A-B)}{(2-\lambda)^2\pi^2} + \frac{1}{3}\right\}, \ (\nu \leq \widetilde{\aleph}_1); \\ \frac{(A-B)}{(3-2\lambda)\pi^2}, \ (\widetilde{\aleph}_1 \leq \nu \leq \widetilde{\aleph}_2); \\ \frac{2(A-B)}{(3-2\lambda)\pi^2}\left\{\frac{2(B+1)}{\pi^2} + \frac{[(\lambda+1)(\lambda+2)+2\nu(3-2\lambda)](A-B)}{(2-\lambda)^2\pi^2} - \frac{1}{3}\right\}, \ (\nu \geq \widetilde{\aleph}_2), \end{cases}$$

where

$$\widetilde{\aleph}_1 = -\frac{(\lambda+1)(\lambda+2)}{2(3-2\lambda)} - \frac{[\frac{\pi^2}{6} + 2(B+1)](2-\lambda)^2}{2(3-2\lambda)(A-B)}$$

and

$$\widetilde{\aleph}_2 = -\frac{(\lambda^2+1)(\lambda+2)}{2(3-2\lambda)} - \frac{[-\frac{5\pi^2}{6} + 2(B+1)](2-\lambda)^2}{2(3-2\lambda)(A-B)}.$$

Moreover, we let

$$\widetilde{\aleph}_3 = -\frac{(\lambda^2+1)(\lambda+2)}{2(3-2\lambda)} - \frac{[-\frac{\pi^2}{3} + 2(B+1)](2-\lambda)^2}{2(3-2\lambda)(A-B)}.$$

Then, each of the following results holds:
(i) For $\mu \in [\widetilde{\aleph}_1, \widetilde{\aleph}_3]$,

$$|d_2 - \nu d_1^2| + \frac{(2-\lambda)^2\pi^2(\Psi + \frac{1}{2})}{2(3-2\lambda)(A-B)}|d_1|^2 \leq \frac{(A-B)}{(3-2\lambda)\pi^2};$$

(ii) For $\nu \in [\widetilde{\aleph}_3, \widetilde{\aleph}_2]$,

$$|d_2 - \nu d_1^2| + \frac{(2-\lambda)^2\pi^2(-\Psi + \frac{1}{2})}{2(3-2\lambda)(A-B)}|d_1|^2 \leq \frac{(A-B)}{(3-2\lambda)\pi^2},$$

where

$$\Psi = \frac{2(B+1)}{\pi^2} + \frac{[(\lambda-1)(\lambda+2) + 2\nu(3-2\lambda)](A-B)}{(2-\lambda)^2\pi^2} - \frac{1}{3}.$$

Concluding Remark: By fixing $A = 1$ *and* $B = -1$ or $A = 1$ *and* $B = 1 - 2\alpha$, one can deduce some interesting results .

Acknowledgements

The authors are profoundly grateful to all the reviewers for their careful readings and insightful suggestions such that the original manuscript for this article is greatly improved.

This work is supported by Natural Science Foundation of Ningxia (Grant No. 2020AAC03066) and Natural Science Foundation of China (Grant Nos. 42064004; 11762017).

Conflict of interest

The authors declare no conflict of interest.

References

1. A. W. Goodman, *Univalent Functions*, vol. 1, Tempa: Mariner Publishing Company, 1983.
2. A. W. Goodman, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87–92.
3. S. Kanas, A. Wisniowska, Conic regions and k-uniform convexity, *J. Compt. Appl. Math.*, **105** (1999), 327–336.
4. S. Kanas, A. Wisniowska, Conic domains and starlike functions, *Rev. Roum. Math. Pure Appl.*, **45** (2000), 647–657.
5. S. Kanas, An unified approach to Fekete-Szegö problem, *Appl. Math. Compt.*, **218** (2012), 8453–8461.
6. S. Kanas, Coefficient estimates in subclasses of the Caratheodory class related to conical domains, *Acta Math. Univ. Comenian(N.S.)*, **74** (2005), 149–161.
7. B. Kowalczyk, A. Lecko, The Fekete-Szegö inequality for close-to-convex functions with respect to a certain starlike function dependent on a real parameter, *J. Inequal. Appl.*, **2014** (2014), 1–16.
8. W. Ma, D. Minda, Uniformly convex functions II, *Ann. Polon. Math.*, **8** (1993), 275–285.
9. W. Ma, D. Minda, *A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis*, Z. Li, F. Ren, L. Lang, S. Zhang (Editors), International Press, pp. 157-169, 1994.
10. S. N. Malik, *Some topics in Geometric functions theory*, Saarbrücken: LAP LAMBERT Academic Publishing, 2017.
11. S. N. Malik, S. Mahmood, M. Raza, S. Farman, S. Zainab, Coefficient Inequalities of functions associated with petal type domains, *Mathematics*, **6** (2018), 298.
12. S. N. Malik, M. Raza, M. Arif, S. Hussain, Coeffifficient estiates of some subclasses of analytic functions related with conic domains, *Anal. Univ. Ovidius Const. Ser. Mat.*, **21** (2013), 181–188.
13. K. I. Noor, S. N. Malik, On Coefficient inequalities of functions associated with conic domainsz, *Compt. Math. Appl.*, **62** (2011), 2209–2217.
14. M. Raza, S. N. Malik, Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli, *J. Inequal. Appl.*, **2013** (2013), 412.
15. A. Rønning, Uniformly convex functions and corresponding class of starlike functions, *Proc. Amer. Math. Soc.*, **118** (1993), 189–196.
16. F. Yousef, R. A. Frasin, T. A. Hawary, Fekete-Szegö inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials, *Filomat*, **32** (2018), 3229–3236.