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Research article

Pythagorean fuzzy sets in UP-algebras and approximations

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Abstract: The aim of this paper is to apply the concept of Pythagorean fuzzy sets to UP-algebras, and then we introduce five types of Pythagorean fuzzy sets in UP-algebras. In addition, we will also discuss the relationship between some assertions of Pythagorean fuzzy sets and Pythagorean fuzzy UP-subalgebras (resp., Pythagorean fuzzy near UP-filters, Pythagorean fuzzy UP-filters, Pythagorean fuzzy UP-ideals, Pythagorean fuzzy strong UP-ideals) in UP-algebras and study upper and lower approximations of Pythagorean fuzzy sets.

Keywords: UP-algebra; Pythagorean fuzzy set; Pythagorean fuzzy UP-subalgebra; Pythagorean fuzzy near UP-filter, Pythagorean fuzzy UP-filter; Pythagorean fuzzy UP-ideal; Pythagorean fuzzy strong UP-ideal; Upper approximation; lower approximation
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1. Introduction and preliminaries

For the study of many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [15], BCI-algebras [16], BE-algebras [19], UP-algebras [11], fully UP-semigroups [12], topological UP-algebras [27], UP-hyperalgebras [13], extension of KU/UP-algebras [24] and others.

The concept of rough sets was first introduced by Pawlak [22] in 1982. After the concept was introduced, several researchers were conducted on the generalizations of the concept of rough sets and application to many algebraic structures such as: in 2002, Jun [18] and Dudek et al. [7] applied rough set theory to BCK/BCI-algebras. In 2019-2020, Ansari et al. [2] and Klinseesook et al. [20] applied rough set theory to UP-algebras.

The concept of fuzzy sets was first introduced by Zadeh [35] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. After the introduction of the concept of fuzzy sets by Zadeh [35], Atanassov [3] defined new concept called intuitionistic fuzzy set which is a generalization of fuzzy set, Yager [33] introduce a new class of non-standard fuzzy subsets called Pythagorean fuzzy subsets and the related idea of Pythagorean membership grades, and Satirad and Iampan [25] introduced several types of subsets and of fuzzy sets of fully UP-semigroups, and investigated the algebraic properties of fuzzy sets under the operations of intersection and union.

The concept of Pythagorean fuzzy sets was applied to semigroups, ternary semigroups, and many logical algebras such as: In 2019, Hussain et al. [9] present the concept of rough Pythagorean fuzzy ideals in semigroups. Then, this idea is extended to the lower and upper approximations of Pythagorean fuzzy left (resp., right) ideals, bi-ideals, interior ideals, (1, 2)-ideals in semigroups and some important properties related to these concepts are given. Jansi and Mohana [17] introduced the concepts of bipolar Pythagorean fuzzy A-ideals of BCI-algebras and investigated their properties. Also, relationship between bipolar Pythagorean fuzzy subalgebras, bipolar Pythagorean fuzzy ideals, and bipolar Pythagorean fuzzy A-ideals are analyzed. In 2020, Chinram and Panityakul [5] studied rough Pythagorean fuzzy ideals in ternary semigroups. This idea is extended to the lower and upper approximations of Pythagorean fuzzy ideals.

In this paper, we apply the concept of Pythagorean fuzzy sets to UP-algebras and investigate their properties. Also, we discuss the relationship between the Pythagorean UP-subalgebras, Pythagorean fuzzy near UP-filters, Pythagorean fuzzy UP-filters, Pythagorean fuzzy UP-ideals, and Pythagorean fuzzy strong UP-ideals. This idea is extended to the lower and upper approximations of Pythagorean fuzzy sets in UP-algebras.

Before we begin our study, let's review the definition of UP-algebras.

Definition 1.1. [11] An algebra $U = (U, *, 0)$ of type $(2, 0)$ is said to be a *UP-algebra*, where U is a nonempty set, $*$ is a binary operation on U, and 0 is a fixed element of U if it fulfills the following axioms:

(UP-1) (for all $x, y, z \in U$)$((y * z) * ((x * y) * (x * z)) = 0)$,

(UP-2) (for all $x \in U$)$(0 * x = x)$,

(UP-3) (for all $x \in U$)$(x * 0 = 0)$, and

(UP-4) (for all $x, y \in U$)$(x * y = 0, y * x = 0 \Rightarrow x = y)$,

and is said to be a *KU-algebra* if it fulfills axioms (UP-2), (UP-3), (UP-4), and the following axiom:

(KU) (for all $x, y, z \in U$)$((x * y) * ((y * z) * (x * z)) = 0)$.

From [11], we know that the concept of UP-algebras is a generalization of KU-algebras (see [23]).

Example 1.2. [29] Let U be a nonempty set and let $X \in \mathcal{P}(U)$, where $\mathcal{P}(U)$ means the power set of U. Let $\mathcal{P}_X(U) = \{A \in \mathcal{P}(U) \mid X \subseteq A\}$. Define a binary operation $\vartriangle$ on $\mathcal{P}_X(U)$ by putting $A \vartriangle B = B \cap (A^C \cup X)$ for all $A, B \in \mathcal{P}_X(U)$, where A^C means the complement of a subset A. Then $(\mathcal{P}_X(U), \vartriangle, X)$ is a UP-algebra. Let $\mathcal{P}^X(U) = \{A \in \mathcal{P}(U) \mid A \subseteq X\}$. Define a binary operation $\blacktriangle$ on $\mathcal{P}^X(U)$ by putting $A \blacktriangle B = B \cup (A^C \cap X)$ for all $A, B \in \mathcal{P}^X(U)$. Then $(\mathcal{P}^X(U), \blacktriangle, X)$ is a UP-algebra.

Example 1.3. [6] Let $\mathbb{N}_0$ be the set of all natural numbers with zero. Define two binary operations $\circ$ and $\star$ on $\mathbb{N}_0$ by

$$\text{(for all } m, n \in \mathbb{N}_0)\left(m \circ n = \begin{cases} n & \text{if } m < n, \\ 0 & \text{otherwise} \end{cases}\right)$$

and

$$\text{(for all } m, n \in \mathbb{N}_0)\left(m \star n = \begin{cases} n & \text{if } m > n \text{ or } m = 0, \\ 0 & \text{otherwise} \end{cases}\right).$$

Then $(\mathbb{N}_0, \circ, 0)$ and $(\mathbb{N}_0, \star, 0)$ are UP-algebras.

For more examples of UP-algebras, see [1, 2, 4, 12, 14, 28–31].

In a UP-algebra $U = (U, *, 0)$, the following axioms are valid (see [11, 12]).

$$\text{(for all } x \in U)(x * x = 0), \tag{1}$$
$$\text{(for all } x, y, z \in U)(x * y = 0, y * z = 0 \Rightarrow x * z = 0),$$
$$\text{(for all } x, y, z \in U)(x * y = 0 \Rightarrow (z * x) * (z * y) = 0), \tag{2}$$
$$\text{(for all } x, y, z \in U)(x * y = 0 \Rightarrow (y * z) * (x * z) = 0), \tag{3}$$
$$\text{(for all } x, y \in U)(x * (y * x) = 0), \tag{4}$$
$$\text{(for all } x, y \in U)((y * x) * x = 0 \Leftrightarrow x = y * x),$$
$$\text{(for all } x, y \in U)(x * (y * y) = 0),$$
$$\text{(for all } a, x, y, z \in U)((x * (y * z)) * (x * ((a * y) * (a * z))) = 0),$$
$$\text{(for all } a, x, y, z \in U)((((a * x) * (a * y)) * z) * ((x * y) * z) = 0),$$
$$\text{(for all } x, y, z \in U)(((x * y) * z) * (y * z) = 0),$$
$$\text{(for all } x, y, z \in U)(x * y = 0 \Rightarrow x * (z * y) = 0), \tag{5}$$
$$\text{(for all } x, y, z \in U)(((x * y) * z) * (x * (y * z)) = 0), \text{ and}$$
$$\text{(for all } a, x, y, z \in U)(((x * y) * z) * (y * (a * z)) = 0).$$

From [11], the binary relation $\leq$ on a UP-algebra $U = (U, *, 0)$ is defined as follows:

$$\text{(for all } x, y \in U)(x \leq y \Leftrightarrow x * y = 0).$$

In a KU-algebra $U = (U, *, 0)$, the following axioms are valid (see [21]).

$$\text{(for all } x, y, z \in U)(x * (y * z) = y * (x * z)), \text{ and} \tag{6}$$
$$\text{(for all } x, y \in U)(y * ((y * x) * x) = 0).$$

Theorem 1.4. *[11] In a UP-algebra $U = (U, *, 0)$, the following statements are equivalent:*

(1) *U is a KU-algebra,*

(2) *(for all $x, y, z \in U$)$(x * (y * z) = y * (x * z))$, and*

(3) $(\textit{for all } x, y, z \in U)(x * (y * z) = 0 \Rightarrow y * (x * z) = 0)$.

For a nonempty subset S of a UP-algebra $U = (U, *, 0)$ which fulfills the following assertion:

$$(\text{for all } x, y \in U)(y \in S \Rightarrow x * y \in S). \tag{7}$$

Then the constant 0 of U is in S. Indeed, let $x \in S$. By (1) and (7), we have $0 = x * x \in S$.

Definition 1.5. [8, 10, 11, 32] A nonempty subset S of a UP-algebra $U = (U, *, 0)$ is said to be

(1) a *UP-subalgebra* of U if it fulfills the following assertion:

$$(\text{for all } x, y \in S)(x * y \in S),$$

(2) a *near UP-filter* of U if it fulfills the assertion (7),

(3) a *UP-filter* of U if it fulfills the following assertions:

$$\text{the constant 0 of } U \text{ is in } S, \tag{8}$$
$$(\text{for all } x, y \in U)(x * y \in S, x \in S \Rightarrow y \in S),$$

(4) a *UP-ideal* of U if it fulfills the assertion (8) and the following assertion:

$$(\text{for all } x, y, z \in U)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S),$$

(5) a *strong UP-ideal* of U if it fulfills the assertion (8) and the following assertion:

$$(\text{for all } x, y, z \in U)((z * y) * (z * x) \in S, y \in S \Rightarrow x \in S).$$

Guntasow et al. [8] and Iampan [10] proved that the concept of UP-subalgebras is a generalization of near UP-filters, near UP-filters is a generalization of UP-filters, UP-filters is a generalization of UP-ideals, and UP-ideals is a generalization of strong UP-ideals. Furthermore, they proved that the only strong UP-ideal of a UP-algebra U is U.

Definition 1.6. [35] A *fuzzy set* F in a nonempty set U (or a *fuzzy subset* of U) is described by its membership function $\mathrm{f_F}$. To every point $x \in U$, this function associates a real number $\mathrm{f_F}(x)$ in the closed interval $[0, 1]$. The real number $\mathrm{f_F}(x)$ is interpreted for the point as a degree of membership of an object $x \in U$ to the fuzzy set F, that is, $\mathrm{F} := \{(x, \mathrm{f_F}(x)) \mid x \in U\}$. We say that a fuzzy set F in U is *constant* if its membership function $\mathrm{f_F}$ is constant.

Definition 1.7. [35] Let F be a fuzzy set in a nonempty set U. The *complement* of F, denoted by $\widetilde{\mathrm{F}}$, is described by its membership function $\mathrm{f}_{\widetilde{\mathrm{F}}}$ which defined as follows:

$$(\text{for all } x \in U)(\mathrm{f}_{\widetilde{\mathrm{F}}}(x) = 1 - \mathrm{f_F}(x)).$$

The following two propositions are easy to verify.

Proposition 1.8. *Let* F *be a fuzzy set in a nonempty set* U*. Then following assertions are valid:*

(1) (*for all* $x, y \in U$)($\mathrm{f_F}(x) \leq \mathrm{f_F}(y) \Leftrightarrow \mathrm{f}_{\widetilde{\mathrm{F}}}(x) \geq \mathrm{f}_{\widetilde{\mathrm{F}}}(y)$),

(2) (*for all* $x, y \in U$)($\mathrm{f_F}(x) = \mathrm{f_F}(y) \Leftrightarrow \mathrm{f}_{\widetilde{\mathrm{F}}}(x) = \mathrm{f}_{\widetilde{\mathrm{F}}}(y)$),

(3) $\widetilde{\widetilde{\mathrm{F}}} = \mathrm{F}$, *and*

(4) (*for all* $x, y \in U$)($1 - \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\} = \max\{\mathrm{f}_{\widetilde{\mathrm{F}}}(x), \mathrm{f}_{\widetilde{\mathrm{F}}}(y)\}$).

Proposition 1.9. *Let* $\{\mathrm{F}_i\}_{i\in I}$ *be a nonempty family of fuzzy sets in a nonempty set* U*, where* I *is an arbitrary index set. Then following assertions are valid:*

(1) (*for all* $x, y \in U$)($\inf_{i\in I}\{\min\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\} = \min\{\inf_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)\}, \inf_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(y)\}\}$),

(2) (*for all* $x, y \in U$)($\sup_{i\in I}\{\max\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\} = \max\{\sup_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)\}, \sup_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(y)\}\}$),

(3) (*for all* $x, y \in U$)($\inf_{i\in I}\{\max\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\} \geq \max\{\inf_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)\}, \inf_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(y)\}\}$),

(4) (*for all* $x, y \in U$)($\sup_{i\in I}\{\min\{\mathrm{f}_{\mathrm{F}_i}(x), \mathrm{f}_{\mathrm{F}_i}(y)\}\} \leq \min\{\sup_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)\}, \sup_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(y)\}\}$),

(5) (*for all* $x \in U$)($(\sup_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)\})^2 = \sup_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)^2\}$),

(6) (*for all* $x \in U$)($(\inf_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)\})^2 = \inf_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)^2\}$),

(7) (*for all* $x \in U$)($1 - \sup_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)\} = \inf_{i\in I}\{1 - \mathrm{f}_{\mathrm{F}_i}(x)\}$), *and*

(8) (*for all* $x \in U$)($1 - \inf_{i\in I}\{\mathrm{f}_{\mathrm{F}_i}(x)\} = \sup_{i\in I}\{1 - \mathrm{f}_{\mathrm{F}_i}(x)\}$).

For a fuzzy set F in a UP-algebra $U = (U, *, 0)$ which fulfills the following assertion:

$$(\text{for all } x, y \in U)(\mathrm{f_F}(x * y) \geq \mathrm{f_F}(y)). \tag{9}$$

Then

$$(\text{for all } x \in U)(\mathrm{f_F}(0) \geq \mathrm{f_F}(x)).$$

Indeed, let $x \in U$. By (1) and (9), we have $\mathrm{f_F}(0) = \mathrm{f_F}(x * x) \geq \mathrm{f_F}(x)$.

Definition 1.10. [8, 26, 32] A fuzzy set F in a UP-algebra $U = (U, *, 0)$ is said to be

(1) a *fuzzy UP-subalgebra* of U if it fulfills the following assertion:

$$(\text{for all } x, y \in U)(\mathrm{f_F}(x * y) \geq \min\{\mathrm{f_F}(x), \mathrm{f_F}(y)\}), \tag{10}$$

(2) a *fuzzy near UP-filter* of U if it fulfills the assertion (9),

(3) a *fuzzy UP-filter* of U if it fulfills the following assertions:

$$(\text{for all } x \in U)(\mathrm{f_F}(0) \geq \mathrm{f_F}(x)), \tag{11}$$

$$(\text{for all } x, y \in U)(\mathrm{f_F}(y) \geq \min\{\mathrm{f_F}(x * y), \mathrm{f_F}(x)\}), \tag{12}$$

(4) a *fuzzy UP-ideal* of U if it fulfills the assertion (11) and the following assertion:

$$(\text{for all } x, y, z \in U)(\mathrm{f_F}(x * z) \geq \min\{\mathrm{f_F}(x * (y * z)), \mathrm{f_F}(y)\}), \tag{13}$$

(5) a *fuzzy strong UP-ideal* of U if it fulfills the assertion (11) and the following assertion:

$$(\text{for all } x, y, z \in U)(\mathrm{f_F}(x) \geq \min\{\mathrm{f_F}((z * y) * (z * x)), \mathrm{f_F}(y)\}). \tag{14}$$

Guntasow et al. [8], and Satirad and Iampan [26] proved that the concept of fuzzy UP-subalgebras is a generalization of fuzzy near UP-filters, fuzzy near UP-filters is a generalization of fuzzy UP-filters, fuzzy UP-filters is a generalization of fuzzy UP-ideals, and fuzzy UP-ideals is a generalization of fuzzy strong UP-ideals. Furthermore, they proved that fuzzy strong UP-ideals and constant fuzzy sets coincide in a UP-algebras U.

2. Pythagorean fuzzy sets in UP-algebras

In 2013, Yager [33] and Yager and Abbasov [34] introduced the concept of Pythagorean fuzzy sets for the first time.

Definition 2.1. [33, 34] A *Pythagorean fuzzy set* P in a nonempty set U is described by their membership function μ_{P} and non-membership function ν_{P}. To every point $x \in U$, these functions associate real numbers $\mu_{\mathrm{P}}(x)$ and $\nu_{\mathrm{P}}(x)$ in the closed interval $[0, 1]$, with the following assertion:

$$(\text{for all } x \in U)(0 \leq \mu_{\mathrm{P}}(x)^2 + \nu_{\mathrm{P}}(x)^2 \leq 1).$$

The real numbers $\mu_{\mathrm{P}}(x)$ and $\nu_{\mathrm{P}}(x)$ are interpreted for the point as a degree of membership and non-membership of an object $x \in U$, respectively, to the Pythagorean fuzzy set P, that is, $\mathrm{P} := \{(x, \mu_{\mathrm{P}}(x), \nu_{\mathrm{P}}(x)) \mid x \in U\}$. For the sake of simplicity, a Pythagorean fuzzy set P is denoted by $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$. We say that a Pythagorean fuzzy set P in U is *constant* if their membership function μ_{P} and non-membership function ν_{P} are constant.

We apply the concept of Pythagorean fuzzy sets to UP-algebras and introduce the five types of Pythagorean fuzzy sets in UP-algebras.

Definition 2.2. A Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ in a UP-algebra $U = (U, *, 0)$ is said to be

(1) a *Pythagorean fuzzy UP-subalgebra* of U if it fulfills the following assertions:

$$(\text{for all } x, y \in U)(\mu_{\mathrm{P}}(x * y) \geq \min\{\mu_{\mathrm{P}}(x), \mu_{\mathrm{P}}(y)\}), \tag{15}$$
$$(\text{for all } x, y \in U)(\nu_{\mathrm{P}}(x * y) \leq \max\{\nu_{\mathrm{P}}(x), \nu_{\mathrm{P}}(y)\}), \tag{16}$$

(2) a *Pythagorean fuzzy near UP-filter* of U if it fulfills the following assertions:

$$(\text{for all } x, y \in U)(\mu_{\mathrm{P}}(x * y) \geq \mu_{\mathrm{P}}(y)), \tag{17}$$
$$(\text{for all } x, y \in U)(\nu_{\mathrm{P}}(x * y) \leq \nu_{\mathrm{P}}(y)), \tag{18}$$

(3) a *Pythagorean fuzzy UP-filter* of U if it fulfills the following assertions:

$$(\text{for all } x \in U)(\mu_P(0) \geq \mu_P(x)), \tag{19}$$

$$(\text{for all } x \in U)(\nu_P(0) \leq \nu_P(x)), \tag{20}$$

$$(\text{for all } x, y \in U)(\mu_P(y) \geq \min\{\mu_P(x * y), \mu_P(x)\}), \tag{21}$$

$$(\text{for all } x, y \in U)(\nu_P(y) \leq \max\{\nu_P(x * y), \nu_P(x)\}), \tag{22}$$

(4) a *Pythagorean fuzzy UP-ideal* of U if it fulfills the assertions (19) and (20) and the following assertions:

$$(\text{for all } x, y, z \in U)(\mu_P(x * z) \geq \min\{\mu_P(x * (y * z)), \mu_P(y)\}), \tag{23}$$

$$(\text{for all } x, y, z \in U)(\nu_P(x * z) \leq \max\{\nu_P(x * (y * z)), \nu_P(y)\}), \tag{24}$$

(5) a *Pythagorean strong fuzzy UP-ideal* of U if it fulfills the assertions (19) and (20) and the following assertions:

$$(\text{for all } x, y, z \in U)(\mu_P(x) \geq \min\{\mu_P((z * y) * (z * x)), \mu_P(y)\}), \tag{25}$$

$$(\text{for all } x, y, z \in U)(\nu_P(x) \leq \max\{\nu_P((z * y) * (z * x)), \nu_P(y)\}). \tag{26}$$

From now on, we shall let U be a UP-algebra $U = (U, *, 0)$.

Theorem 2.3. *A Pythagorean fuzzy set in U is a Pythagorean fuzzy strong UP ideal if and only if it is constant.*

Proof. Assume that $P = (\mu_P, \nu_P)$ is a Pythagorean fuzzy strong UP ideal of U. Then it fulfills (19) and (20). Thus for all $x \in U$,

$$\begin{aligned}
\mu_P(x) &\geq \min\{\mu_P((x * 0) * (x * x)), \mu_P(0)\} && \text{by (25)} \\
&= \min\{\mu_P(0 * (x * x)), \mu_P(0)\} && \text{by (UP-3)} \\
&= \min\{\mu_P(x * x), \mu_P(0)\} && \text{by (UP-2)} \\
&= \min\{\mu_P(0), \mu_P(0)\} && \text{by (1)} \\
&= \mu_P(0)
\end{aligned}$$

and

$$\begin{aligned}
\nu_P(x) &\leq \max\{\nu_P((x * 0) * (x * x)), \nu_P(0)\} && \text{by (26)} \\
&= \max\{\nu_P(0 * (x * x)), \nu_P(0)\} && \text{by (UP-3)} \\
&= \max\{\nu_P(x * x), \nu_P(0)\} && \text{by (UP-2)} \\
&= \max\{\nu_P(0), \nu_P(0)\} && \text{by (1)} \\
&= \nu_P(0).
\end{aligned}$$

Since $\mu_P(0) \geq \mu_P(x)$ and $\nu_P(0) \leq \nu_P(x)$, we have $\mu_P(x) = \mu_P(0)$ and $\nu_P(x) = \nu_P(0)$ for all $x \in U$. Hence, μ_P and ν_P are constant, that is, P is constant.

The converse is evident because P is constant. □

Theorem 2.4. *Every Pythagorean fuzzy near UP-filter of U is a Pythagorean fuzzy UP-subalgebra.*

Proof. Let $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ be a Pythagorean fuzzy near UP-filter of U. Then for all $x, y \in U$,

$$\mu_{\mathrm{P}}(x * y) \geq \mu_{\mathrm{P}}(y) \geq \min\{\mu_{\mathrm{P}}(x), \mu_{\mathrm{P}}(y)\} \qquad \text{by (17)}$$

and

$$\nu_{\mathrm{P}}(x * y) \leq \nu_{\mathrm{P}}(y) \leq \max\{\nu_{\mathrm{P}}(x), \nu_{\mathrm{P}}(y)\}. \qquad \text{by (18)}$$

Therefore, P is a Pythagorean fuzzy UP-subalgebra of U. □

The converse of Theorem 2.4 does not hold in general. This is shown by the following example.

Example 2.5. Let $U = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

We define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	0.9	0.7	0.8	0.5
ν_{P}	0	0.4	0.1	0.6

Then P is a Pythagorean fuzzy UP-subalgebra of U. Since $\mu_{\mathrm{P}}(3 * 2) = \mu_{\mathrm{P}}(1) = 0.7 \ngeq 0.8 = \mu_{\mathrm{P}}(2)$, we have P is not a Pythagorean fuzzy near UP-filter of U.

Theorem 2.6. *Every Pythagorean fuzzy UP-filter of U is a Pythagorean fuzzy near UP-filter.*

Proof. Let $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ be a Pythagorean fuzzy UP-filter of U. Then for all $x, y \in U$,

$$\mu_{\mathrm{P}}(x * y) \geq \min\{\mu_{\mathrm{P}}(y * (x * y)), \mu_{\mathrm{P}}(y)\} = \min\{\mu_{\mathrm{P}}(0), \mu_{\mathrm{P}}(y)\} = \mu_{\mathrm{P}}(y) \qquad \text{by (21), (4)}$$

and

$$\nu_{\mathrm{P}}(x * y) \leq \max\{\nu_{\mathrm{P}_\nu}(y * (x * y)), \nu_{\mathrm{P}}(y)\} = \max\{\nu_{\mathrm{P}}(0), \nu_{\mathrm{P}}(y)\} = \nu_{\mathrm{P}}(y). \qquad \text{by (22), (4)}$$

Therefore, P is a Pythagorean fuzzy near UP-filter of U. □

The converse of Theorem 2.6 does not hold in general. This is shown by the following example.

Example 2.7. Let $U = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

We define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	1	0.7	0.8	0.75
ν_{P}	0	0.6	0.3	0.4

Then P is a Pythagorean fuzzy near UP-filter of U. Since $\mu_{\mathrm{P}}(1) = 0.7 \not\geq 0.75 = \min\{1, 0.75\} = \min\{\mu_{\mathrm{P}}(0), \mu_{\mathrm{P}}(3)\} = \min\{\mu_{\mathrm{P}}(3 * 1), \mu_{\mathrm{P}}(3)\}$, we have P is not a Pythagorean fuzzy UP-filter of U.

Theorem 2.8. *Every Pythagorean fuzzy UP-ideal of U is a Pythagorean fuzzy UP-filter.*

Proof. Let $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ be a Pythagorean fuzzy UP-ideal of U. It is sufficient to prove the assertions (21) and (22). Then for all $x, y \in U$,

$$\mu_{\mathrm{P}}(y) = \mu_{\mathrm{P}}(0 * y) \geq \min\{\mu_{\mathrm{P}}(0 * (x * y)), \mu_{\mathrm{P}}(x)\} = \min\{\mu_{\mathrm{P}}(x * y), \mu_{\mathrm{P}}(x)\} \qquad \text{by (UP-2), (23)}$$

and

$$\nu_{\mathrm{P}}(y) = \nu_{\mathrm{P}}(0 * y) \leq \max\{\nu_{\mathrm{P}}(0 * (x * y)), \nu_{\mathrm{P}}(x)\} = \max\{\nu_{\mathrm{P}}(x * y), \nu_{\mathrm{P}}(x)\}. \qquad \text{by (UP-2), (24)}$$

Therefore, P is a Pythagorean fuzzy UP-filter of U. □

The converse of Theorem 2.8 does not hold in general. This is shown by the following example.

Example 2.9. Let $U = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

We define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	0.9	0.5	0.2	0.2
ν_{P}	0.1	0.4	0.5	0.5

Then P is a Pythagorean fuzzy UP-filter of U. Since $\mu_{\mathrm{P}}(2 * 3) = \mu_{\mathrm{P}}(2) = 0.2 \not\geq 0.5 = \min\{0.9, 0.5\} = \min\{\mu_{\mathrm{P}}(0), \mu_{\mathrm{P}}(1)\} = \min\{\mu_{\mathrm{P}}(2 * (1 * 3)), \mu_{\mathrm{P}}(1)\}$, we have P is not a Pythagorean fuzzy UP-ideal of U.

Theorem 2.10. *Every Pythagorean fuzzy strong UP-ideal of U is a Pythagorean fuzzy UP-ideal.*

Proof. Let $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ be a Pythagorean fuzzy strong UP-ideal of U. By Theorem 2.3, we have P is constant. Therefore, it is evident that P is a Pythagorean fuzzy UP-ideal of U. □

The converse of Theorem 2.10 does not hold in general. This is shown by the following example.

Example 2.11. Let $U = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

We define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	1	0.5	0.2	0.7
ν_{P}	0	0.6	0.8	0.4

Then P is a Pythagorean fuzzy UP-ideal of U. But P is not constant and by Theorem 2.3, we have P is not a Pythagorean fuzzy strong UP-ideal of U.

Theorem 2.12. *Let* F *be a fuzzy set in U. Then the following statements hold:*

(1) $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ *is a Pythagorean fuzzy set in U,*

(2) F *is a fuzzy UP-subalgebra of U if and only if* $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ *is a Pythagorean fuzzy UP-subalgebra of U,*

(3) F *is a fuzzy near UP-filter of U if and only if* $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ *is a Pythagorean fuzzy near UP-filter of U,*

(4) F *is a fuzzy UP-filter of U if and only if* $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ *is a Pythagorean fuzzy UP-filter of U,*

(5) F *is a fuzzy UP-ideal of U if and only if* $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ *is a Pythagorean fuzzy UP-ideal of U, and*

(6) F *is a fuzzy strong UP-ideal of U if and only if* $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ *is a Pythagorean fuzzy strong UP-ideal of U.*

Proof. (1) Let $x \in U$. Then $0 \leq \mathrm{f}_{\mathrm{F}}(x)^2 + \mathrm{f}_{\widetilde{\mathrm{F}}}(x)^2 = \mathrm{f}_{\mathrm{F}}(x)^2 + (1 - \mathrm{f}_{\mathrm{F}}(x))^2 \leq \mathrm{f}_{\mathrm{F}}(x) + (1 - \mathrm{f}_{\mathrm{F}}(x)) = 1$. Hence, $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy set in U.

(2) Assume that F is a fuzzy UP-subalgebra of U. Then for all $x, y \in U$,

$$\mathrm{f}_{\mathrm{F}}(x * y) \geq \min\{\mathrm{f}_{\mathrm{F}}(x), \mathrm{f}_{\mathrm{F}}(y)\} \qquad \text{by (10)}$$

and

$$\mathrm{f}_{\widetilde{\mathrm{F}}}(x * y) = 1 - \mathrm{f}_{\mathrm{F}}(x * y) \leq 1 - \min\{\mathrm{f}_{\mathrm{F}}(x), \mathrm{f}_{\mathrm{F}}(y)\} = \max\{\mathrm{f}_{\widetilde{\mathrm{F}}}(x), \mathrm{f}_{\widetilde{\mathrm{F}}}(y)\}. \qquad \text{by Proposition 1.8 (4), (10)}$$

This implies that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy UP-subalgebra of U.

Conversely, assume that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy UP-subalgebra of U. Then F fulfills the assertion (15). Hence, F is a fuzzy UP-subalgebra of U.

(3) Assume that F is a fuzzy near UP-filter of U. Then for all $x, y \in U$,

$$\mathrm{f}_{\mathrm{F}}(x * y) \geq \mathrm{f}_{\mathrm{F}}(y) \qquad \text{by (9)}$$

and

$$\mathrm{f}_{\widetilde{\mathrm{F}}}(x*y) \leq \mathrm{f}_{\widetilde{\mathrm{F}}}(y). \qquad \text{by Proposition 1.8 (1)}$$

This implies that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy near UP-filter of U.

Conversely, assume that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy near UP-filter of U. Then F fulfills the assertion (17). Hence, F is a fuzzy near UP-filter of U.

(4) Assume that F is a fuzzy UP-filter of U. Then for all $x, y \in U$,

$$\begin{aligned} \mathrm{f}_{\mathrm{F}}(0) &\geq \mathrm{f}_{\mathrm{F}}(x), && \text{by (11)} \\ \mathrm{f}_{\widetilde{\mathrm{F}}}(0) &\leq \mathrm{f}_{\widetilde{\mathrm{F}}}(x), && \text{by Proposition 1.8 (1)} \\ \mathrm{f}_{\mathrm{F}}(y) &\geq \min\{\mathrm{f}_{\mathrm{F}}(x*y), \mathrm{f}_{\mathrm{F}}(x)\}, && \text{by (12)} \end{aligned}$$

and

$$\mathrm{f}_{\widetilde{\mathrm{F}}}(y) = 1 - \mathrm{f}_{\mathrm{F}}(y) \leq 1 - \min\{\mathrm{f}_{\mathrm{F}}(x*y), \mathrm{f}_{\mathrm{F}}(x)\} = \max\{\mathrm{f}_{\widetilde{\mathrm{F}}}(x*y), \mathrm{f}_{\widetilde{\mathrm{F}}}(x)\}. \qquad \text{by (12), Proposition 1.8 (4)}$$

This implies that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy UP-filter of U.

Conversely, assume that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy UP-filter of U. Then F fulfills the assertions (19) and (21). Hence, F is a fuzzy UP-filter of U.

(5) Assume that F is a fuzzy UP-ideal of U. Then for all $x, y \in U$,

$$\begin{aligned} \mathrm{f}_{\mathrm{F}}(0) &\geq \mathrm{f}_{\mathrm{F}}(x), && \text{by (11)} \\ \mathrm{f}_{\widetilde{\mathrm{F}}}(0) &\leq \mathrm{f}_{\widetilde{\mathrm{F}}}(x), && \text{by Proposition 1.8 (1)} \\ \mathrm{f}_{\mathrm{F}}(x*z) &\geq \min\{\mathrm{f}_{\mathrm{F}}(x*(y*z)), \mathrm{f}_{\mathrm{F}}(y)\}, && \text{by (13)} \end{aligned}$$

and

$$\mathrm{f}_{\widetilde{\mathrm{F}}}(x*z) = 1 - \mathrm{f}_{\mathrm{F}}(x*z) \leq 1 - \min\{\mathrm{f}_{\mathrm{F}}(x*(y*z)), \mathrm{f}_{\mathrm{F}}(y)\} = \max\{\mathrm{f}_{\widetilde{\mathrm{F}}}(x*(y*z)), \mathrm{f}_{\widetilde{\mathrm{F}}}(y)\}.$$
$$\text{by (13), Proposition 1.8 (4)}$$

This implies that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy UP-ideal of U.

Conversely, assume that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy UP-ideal of U. Then F fulfills the assertions (19) and (23). Hence, F is a fuzzy UP-ideal of U.

(6) Assume that F is a fuzzy strong UP-ideal of U. Then f_{F} is constant and so $\mathrm{f}_{\widetilde{\mathrm{F}}}$ is constant. By Theorem 2.3, we have $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy strong UP-ideal of U.

Conversely, assume that $(\mathrm{f}_{\mathrm{F}}, \mathrm{f}_{\widetilde{\mathrm{F}}})$ is a Pythagorean fuzzy strong UP-ideal of U. By Theorem 2.3, we have f_{F} is constant. Hence, F is a fuzzy strong UP-ideal of U. □

3. Properties of Pythagorean fuzzy sets in UP-algebras

In this section, we shall find some properties and examples for study the generalizations of Pythagorean fuzzy sets in UP-algebras.

Proposition 3.1. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy UP-subalgebra of U, then it fulfills the assertions* (19) *and* (20).

Proof. Let $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ be a Pythagorean fuzzy UP-subalgebra of U. Then for all $x \in U$,

$$\mu_{\mathrm{P}}(0) = \mu_{\mathrm{P}}(x * x) \geq \min\{\mu_{\mathrm{P}}(x), \mu_{\mathrm{P}}(x)\} = \mu_{\mathrm{P}}(x) \qquad \text{by (1) and (15)}$$

and

$$\nu_{\mathrm{P}}(0) = \nu_{\mathrm{P}}(x * x) \leq \max\{\nu_{\mathrm{P}}(x), \nu_{\mathrm{P}}(x)\} = \nu_{\mathrm{P}}(x). \qquad \text{by (1) and (16)}$$

Hence, P fulfills the assertions (19) and (20). □

Proposition 3.2. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy UP-filter of U, then*

$$(\textit{for all } x, y \in U)(x \leq y \Rightarrow \mu_{\mathrm{P}}(x) \leq \mu_{\mathrm{P}}(y)), \tag{27}$$

$$(\textit{for all } x, y \in U)(x \leq y \Rightarrow \nu_{\mathrm{P}}(x) \geq \nu_{\mathrm{P}}(y)). \tag{28}$$

Proof. Let $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ be a Pythagorean fuzzy UP-filter of U and let $x, y \in U$ be such that $x \leq y$. Then $x * y = 0$, so

$$\mu_{\mathrm{P}}(y) \geq \min\{\mu_{\mathrm{P}}(x * y), \mu_{\mathrm{P}}(x)\} = \min\{\mu_{\mathrm{P}}(0), \mu_{\mathrm{P}}(x)\} = \mu_{\mathrm{P}}(x) \qquad \text{by (21)}$$

and

$$\nu_{\mathrm{P}}(y) \leq \max\{\nu_{\mathrm{P}}(x * y), \nu_{\mathrm{P}}(x)\} = \max\{\nu_{\mathrm{P}}(0), \nu_{\mathrm{P}}(x)\} = \nu_{\mathrm{P}}(x). \qquad \text{by (22)}$$

Hence, P fulfills the assertions (27) and (28). □

Corollary 3.3. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy UP-filter of U, then*

$$(\textit{for all } x, y \in U)(\mu_{\mathrm{P}}(y) \leq \mu_{\mathrm{P}}(x * y)),$$

$$(\textit{for all } x, y \in U)(\nu_{\mathrm{P}}(y) \geq \nu_{\mathrm{P}}(x * y)).$$

Proof. By (4), we have $y * (x * y) = 0$, that is, $y \leq x * y$. By (27) and (28), we have $\mu_{\mathrm{P}}(y) \leq \mu_{\mathrm{P}}(x * y)$ and $\nu_{\mathrm{P}}(y) \geq \nu_{\mathrm{P}}(x * y)$. Hence, P fulfills the assertions (3.3) and (3.3). □

Proposition 3.4. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the following assertions:*

$$(\textit{for all } x, y, z \in U)(z \leq x \Rightarrow \mu_{\mathrm{P}}(x * y) \geq \min\{\mu_{\mathrm{P}}(z), \mu_{\mathrm{P}}(y)\}), \tag{29}$$

$$(\textit{for all } x, y, z \in U)(z \leq x \Rightarrow \nu_{\mathrm{P}}(x * y) \leq \max\{\nu_{\mathrm{P}}(z), \nu_{\mathrm{P}}(y)\}), \tag{30}$$

then it is a Pythagorean fuzzy UP-subalgebra of U.

Proof. Let $x, y \in U$. By (1), we have $x \leq x$. It follows from (29) and (30) that $\mu_{\mathrm{P}}(x * y) \geq \min\{\mu_{\mathrm{P}}(x), \mu_{\mathrm{P}}(y)\}$ and $\nu_{\mathrm{P}}(x * y) \leq \max\{\nu_{\mathrm{P}}(x), \nu_{\mathrm{P}}(y)\}$. Hence, P is a Pythagorean fuzzy UP-subalgebra of U. □

Theorem 3.5. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the assertions* (29) *and* (30)*, then it fulfills the assertions* (19) *and* (20).

Proof. It is straightforward by Proposition 3.4. □

In general, the converse of Theorem 3.5 may be not true by the following example.

Example 3.6. From Example 2.9, we define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	1	0.5	0.1	0.7
ν_{P}	0	0.5	0.6	0.4

Then P fulfills the assertions (19) and (20) but it does not satisfy the assertions (29) and (30). Indeed, $1 \leq 1$ but $\mu_{\mathrm{P}}(1 * 3) = \mu_{\mathrm{P}}(2) = 0.1 \ngeq 0.5 = \min\{0.5, 0.7\} = \min\{\mu_{\mathrm{P}}(1), \mu_{\mathrm{P}}(3)\}$ and $\nu_{\mathrm{P}}(1 * 3) = \nu_{\mathrm{P}}(2) = 0.6 \nleq 0.5 = \max\{0.5, 0.4\} = \max\{\nu_{\mathrm{P}}(1), \nu_{\mathrm{P}}(3)\}$.

Proposition 3.7. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the following assertions:*

$$(\textit{for all } x, y, z \in U)(\mu_{\mathrm{P}}(x * y) \geq \min\{\mu_{\mathrm{P}}(z), \mu_{\mathrm{P}}(y)\}), \tag{31}$$

$$(\textit{for all } x, y, z \in U)(\nu_{\mathrm{P}}(x * y) \leq \max\{\nu_{\mathrm{P}}(z), \nu_{\mathrm{P}}(y)\}), \tag{32}$$

then it fulfills the assertions (29) *and* (30).

In general, the converse of Proposition 3.7 may be not true by the following example.

Example 3.8. Let $U = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

We define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	0.8	0.1	0.3	0.2
ν_{P}	0.4	0.9	0.6	0.8

Then P fulfills the assertions (29) and (30) but it does not satisfy the assertions (31) and (32). Indeed, $\mu_{\mathrm{P}}(1 * 2) = \mu_{\mathrm{P}}(3) = 0.2 \ngeq 0.3 = \min\{0.8, 0.3\} = \min\{\mu_{\mathrm{P}}(0), \mu_{\mathrm{P}}(2)\}$ and $\nu_{\mathrm{P}}(1 * 2) = \nu_{\mathrm{P}}(3) = 0.8 \nleq 0.6 = \max\{0.4, 0.6\} = \max\{\nu_{\mathrm{P}}(0), \nu_{\mathrm{P}}(2)\}$.

Proposition 3.9. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the assertions* (27) *and* (28), *then it is a Pythagorean fuzzy near UP-filter of U.*

Proof. Let $x, y \in U$. By (4), we have $y \leq x * y$. It follows from (27) and (28) that $\mu_{\mathrm{P}}(x * y) \geq \mu_{\mathrm{P}}(y)$ and $\nu_{\mathrm{P}}(x * y) \leq \nu_{\mathrm{P}}(y)$. Hence, P is a Pythagorean fuzzy near UP-filter of U. □

Theorem 3.10. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the assertions* (27) *and* (28), *then it fulfills the assertions* (31) *and* (32).

Proof. Let $x, y, z \in U$. By (4), we have $y \leq x * y$. It follows from (27) and (28) that $\mu_P(x * y) \geq \mu_P(y) \geq \min\{\mu_P(z), \mu_P(y)\}$ and $\nu_P(x * y) \leq \nu_P(y) \leq \max\{\nu_P(z), \nu_P(y)\}$. Hence, P fulfills the assertions (31) and (32). □

In general, the converse of Theorem 3.10 may be not true by the following example.

Example 3.11. From Example 2.7, we define a Pythagorean fuzzy set $P = (\mu_P, \nu_P)$ with μ_P and ν_P as follows:

U	0	1	2	3
μ_P	0.8	0.3	0.4	0.7
ν_P	0.2	0.7	0.5	0.4

Then P fulfills the assertions (31) and (32) but it does not satisfy the assertions (27) and (28). Indeed, $3 \leq 1$ but $\mu_P(3) = 0.7 \nleq 0.3 = \mu_P(1)$ and $\nu_P(3) = 0.4 \ngeq 0.7 = \nu_P(1)$.

Theorem 3.12. *If* $P = (\mu_P, \nu_P)$ *is a Pythagorean fuzzy UP-subalgebra of U fulfilling the following assertions:*

$$(\textit{for all } x, y \in U)(x * y \neq 0 \Rightarrow \mu_P(x) \geq \mu_P(y)), \tag{33}$$
$$(\textit{for all } x, y \in U)(x * y \neq 0 \Rightarrow \nu_P(x) \leq \nu_P(y)), \tag{34}$$

then it is a Pythagorean fuzzy near UP-filter of U.

Proof. Let $x, y \in U$.

Case 1: $x*y = 0$. By Proposition 3.1, we have $\mu_P(x*y) = \mu_P(0) \geq \mu_P(y)$ and $\nu_P(x*y) = \nu_P(0) \leq \nu_P(y)$.

Case 2: $x * y \neq 0$. By (33) and (34), we have $\mu_P(x * y) \geq \min\{\mu_P(x), \mu_P(y)\} = \mu_P(y)$ and $\nu_P(x * y) \leq \max\{\nu_P(x), \nu_P(y)\} = \nu_P(y)$. Hence, P is a Pythagorean fuzzy near UP-filter of U. □

Proposition 3.13. *A Pythagorean fuzzy set* $P = (\mu_P, \nu_P)$ *in U fulfills the following assertions:*

$$(\textit{for all } x, y, z \in U)(z \leq x * y \Rightarrow \mu_P(y) \geq \min\{\mu_P(z), \mu_P(x)\}), \tag{35}$$
$$(\textit{for all } x, y, z \in U)(z \leq x * y \Rightarrow \nu_P(y) \leq \max\{\nu_P(z), \nu_P(x)\}) \tag{36}$$

if and only if it is a Pythagorean fuzzy UP-filter of U.

Proof. Let $x \in U$. By (UP-3), we have $x \leq x * 0$. It follows from (35) and (36) that $\mu_P(0) \geq \min\{\mu_P(x), \mu_P(x)\} = \mu_P(x)$ and $\nu_P(0) \leq \max\{\nu_P(x), \nu_P(x)\} = \nu_P(x)$. Next, let $x, y \in U$. By (1), we have $x * y \leq x * y$. It follows from (35) and (36) that $\mu_P(y) \geq \min\{\mu_P(x * y), \mu_P(x)\}$ and $\nu_P(y) \leq \max\{\nu_P(x * y), \nu_P(x)\}$. Hence, P is a Pythagorean fuzzy UP-filter of U.

Conversely, let $x, y, z \in U$ be such that $z \leq x * y$. Then $z * (x * y) = 0$, so

$$\mu_P(x * y) \geq \min\{\mu_P(z * (x * y)), \mu_P(z)\} = \min\{\mu_P(0), \mu_P(z)\} = \mu_P(z) \qquad \text{by (21)}$$

and

$$\nu_P(x * y) \leq \max\{\nu_P(z * (x * y)), \nu_P(z)\} = \max\{\nu_P(0), \nu_P(z)\} = \nu_P(z). \qquad \text{by (22)}$$

Thus

$$\mu_P(y) \geq \min\{\mu_P(x * y), \mu_P(x)\} \geq \min\{\mu_P(z), \mu_P(x)\}$$

and

$$\nu_P(y) \leq \max\{\nu_P(x * y), \nu_P(x)\} \leq \max\{\nu_P(z), \nu_P(x)\}.$$

Hence, P fulfills the assertions (35) and (36). □

Theorem 3.14. *If* $P = (\mu_P, \nu_P)$ *is a Pythagorean fuzzy set in U fulfilling the assertions* (35) *and* (36)*, then it fulfills the assertions* (27) *and* (28).

Proof. Let $x, y \in U$ be such that $x \leq y$. By (5), we have $x \leq x * y$. It follows from (35) and (36) that $\mu_P(y) \geq \min\{\mu_P(x), \mu_P(x)\} = \mu_P(x)$ and $\nu_P(y) \leq \max\{\nu_P(x), \nu_P(x)\} = \nu_P(x)$. Hence, P fulfills the assertions (27) and (28). □

In general, the converse of Theorem 3.14 may be not true by the following example.

Example 3.15. Let $U = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

We define a Pythagorean fuzzy set $P = (\mu_P, \nu_P)$ with μ_P and ν_P as follows:

U	0	1	2	3
μ_P	0.7	0.3	0.5	0.1
ν_P	0.3	0.7	0.5	0.8

Then P fulfills the assertions (27) and (28) but it does not satisfy the assertions (35) and (36). Indeed, $2 \leq 1 * 3$ but $\mu_P(3) = 0.1 \ngeq 0.3 = \min\{0.5, 0.3\} = \min\{\mu_P(2), \mu_P(1)\}$ and $\nu_P(3) = 0.8 \nleq 0.7 = \max\{0.5, 0.7\} = \max\{\nu_P(2), \nu_P(1)\}$.

Theorem 3.16. *If* $P = (\mu_P, \nu_P)$ *is a Pythagorean fuzzy near UP-filter of U fulfilling the following assertions:*

$$(\text{for all } x, y \in U)(\mu_P(x * y) = \mu_P(y)), \tag{37}$$

$$(\text{for all } x, y \in U)(\nu_P(x * y) = \nu_P(y)), \tag{38}$$

then it is a Pythagorean fuzzy UP-filter of U.

Proof. Let $x, y \in U$. By Theorem 2.4 and Proposition 3.1, we have P is a Pythagorean fuzzy UP-subalgebra of U which fulfills the assertions (19) and (20). By (37) and (38), we have $\mu_P(y) \geq \min\{\mu_P(y), \mu_P(x)\} = \min\{\mu_P(x * y), \mu_P(x)\}$ and $\nu_P(y) \leq \max\{\nu_P(y), \nu_P(x)\} = \max\{\nu_P(x * y), \nu_P(x)\}$. Hence, P is a Pythagorean fuzzy UP-filter of U. □

Proposition 3.17. *A Pythagorean fuzzy set* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *in* U *fulfills the following assertions:*

$$(\text{for all } a, x, y, z \in U)(a \leq x * (y * z) \Rightarrow \mu_{\mathrm{P}}(x * z) \geq \min\{\mu_{\mathrm{P}}(a), \mu_{\mathrm{P}}(y)\}), \tag{39}$$

$$(\text{for all } a, x, y, z \in U)(a \leq x * (y * z) \Rightarrow \nu_{\mathrm{P}}(x * z) \leq \max\{\nu_{\mathrm{P}}(a), \nu_{\mathrm{P}}(y)\}) \tag{40}$$

if and only if it is a Pythagorean fuzzy UP-ideal of U*.*

Proof. Let $x \in U$. By (UP-3), we have $x \leq x * (x * 0)$. Then

$$\mu_{\mathrm{P}}(0) = \mu_{\mathrm{P}}(x * 0) \geq \min\{\mu_{\mathrm{P}}(x), \mu_{\mathrm{P}}(x)\} = \mu_{\mathrm{P}}(x) \qquad \text{by (UP-3) and (39)}$$

and

$$\nu_{\mathrm{P}}(0) = \nu_{\mathrm{P}}(x * 0) \leq \max\{\nu_{\mathrm{P}}(x), \nu_{\mathrm{P}}(x)\} = \nu_{\mathrm{P}}(x). \qquad \text{by (UP-3) and (40)}$$

Let $x, y, z \in U$. By (1), we have $x * (y * z) \leq x * (y * z)$. Then

$$\mu_{\mathrm{P}}(x * z) \geq \min\{\mu_{\mathrm{P}}(x * (y * z)), \mu_{\mathrm{P}}(y)\} \qquad \text{by (39)}$$

and

$$\nu_{\mathrm{P}}(x * z) \leq \max\{\nu_{\mathrm{P}}(x * (y * z)), \nu_{\mathrm{P}}(y)\}. \qquad \text{by (40)}$$

Hence, P is a Pythagorean fuzzy UP-ideal of U.

Conversely, let $a, x, y, z \in U$ be such that $a \leq x*(y*z)$. By (27) and (28), we have $\mu_{\mathrm{P}}(a) \leq \mu_{\mathrm{P}}(x*(y*z))$ and $\nu_{\mathrm{P}}(a) \geq \nu_{\mathrm{P}}(x * (y * z))$. Thus

$$\mu_{\mathrm{P}}(x * z) \geq \min\{\mu_{\mathrm{P}}(x * (y * z)), \mu_{\mathrm{P}}(y)\} \geq \min\{\mu_{\mathrm{P}}(a), \mu_{\mathrm{P}}(y)\} \qquad \text{by (23)}$$

and

$$\nu_{\mathrm{P}}(x * z) \leq \max\{\nu_{\mathrm{P}}(x * (y * z)), \nu_{\mathrm{P}}(y)\} \leq \max\{\nu_{\mathrm{P}}(a), \nu_{\mathrm{P}}(y)\}. \qquad \text{by (24)}$$

Hence, P fulfills the assertions (39) and (40). □

Proposition 3.18. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy UP-ideal of U, then*

$$(\text{for all } a, x, y, z \in U)(a \leq x * (y * z) \Rightarrow \mu_{\mathrm{P}}(a * z) \geq \min\{\mu_{\mathrm{P}}(x), \mu_{\mathrm{P}}(y)\}), \tag{41}$$

$$(\text{for all } a, x, y, z \in U)(a \leq x * (y * z) \Rightarrow \nu_{\mathrm{P}}(a * z) \leq \max\{\nu_{\mathrm{P}}(x), \nu_{\mathrm{P}}(y)\}). \tag{42}$$

Proof. Let $a, x, y, z \in U$ such that $a \leq x * (y * z)$. Then $a * (x * (y * z)) = 0$, so

$$\mu_{\mathrm{P}}(a * (y * z)) \geq \min\{\mu_{\mathrm{P}}(a * (x * (y * z))), \mu_{\mathrm{P}}(x)\} = \min\{\mu_{\mathrm{P}}(0), \mu_{\mathrm{P}}(x)\} = \mu_{\mathrm{P}}(x) \qquad \text{by (23)}$$

and

$$\nu_{\mathrm{P}}(a * (y * z)) \leq \max\{\nu_{\mathrm{P}}(a * (x * (y * z))), \nu_{\mathrm{P}}(x)\} = \max\{\nu_{\mathrm{P}}(0), \nu_{\mathrm{P}}(x)\} = \nu_{\mathrm{P}}(x). \qquad \text{by (24)}$$

Thus

$$\mu_{\mathrm{P}}(a * z) \geq \min\{\mu_{\mathrm{P}}(a * (y * z)), \mu_{\mathrm{P}}(y)\} \geq \min\{\mu_{\mathrm{P}}(x), \mu_{\mathrm{P}}(y)\} \qquad \text{by (23)}$$

and

$$\nu_{\mathrm{P}}(a * z) \leq \max\{\nu_{\mathrm{P}}(a * (y * z)), \nu_{\mathrm{P}}(y)\} \leq \max\{\nu_{\mathrm{P}}(x), \nu_{\mathrm{P}}(y)\}. \qquad \text{by (24)}$$

Hence, P fulfills the assertions (41) and (42). □

Corollary 3.19. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the assertions* (39) *and* (40)*, then it fulfills the assertions* (41) *and* (42)*.*

Proof. It is straightforward by Propositions 3.17 and 3.18. □

Theorem 3.20. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the assertions* (6)*,* (41)*, and* (42)*, then it fulfills the assertions* (39) *and* (40)*.*

Proof. Let $a, x, y, z \in U$ be such that $a \leq x*(y*z)$. By (6), we have $0 = a*(x*(y*z)) = x*(a*(y*z))$, that is, $x \leq a*(y*z)$. It follows from (41) and (42) that $\mu_{\mathrm{P}}(x*z) \geq \min\{\mu_{\mathrm{P}}(a), \mu_{\mathrm{P}}(y)\}$ and $\nu_{\mathrm{P}}(x*z) \leq \max\{\nu_{\mathrm{P}}(a), \nu_{\mathrm{P}}(y)\}$. Hence, P fulfills the assertions (39) and (40). □

Theorem 3.21. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the assertions* (41) *and* (42)*, then it fulfills the assertions* (35) *and* (36)*.*

Proof. Let $x, y, z \in U$ be such that $z \leq x*y$. By (1) and (2), we have $0 = z*z \leq z*(x*y)$. By (UP-2), (41), and (42), we have $\mu_{\mathrm{P}}(y) = \mu_{\mathrm{P}}(0*y) \geq \min\{\mu_{\mathrm{P}}(z), \mu_{\mathrm{P}}(x)\}$ and $\nu_{\mathrm{P}}(y) = \nu_{\mathrm{P}}(0*y) \leq \max\{\nu_{\mathrm{P}}(z), \nu_{\mathrm{P}}(x)\}$. Hence, P fulfills the assertions (35) and (36). □

Corollary 3.22. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy set in U fulfilling the assertions* (39) *and* (40)*, then it fulfills the assertions* (35) *and* (36)*.*

Proof. It is straightforward by Corollary 3.19 and Theorem 3.21. □

In general, the converse of Theorem 3.21 may be not true by the following example.

Example 3.23. From Example 3.8, we define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	0.7	0.3	0.2	0.2
ν_{P}	0.3	0.7	0.75	0.75

Then P fulfills the assertions (35) and (36) but it does not satisfy the assertions (41) and (42). Indeed, $3 \leq 1*(0*2)$ but $\mu_{\mathrm{P}}(3*2) = \mu_{\mathrm{P}}(2) = 0.2 \ngeq 0.3 = \min\{0.3, 0.7\} = \min\{\mu_{\mathrm{P}}(1), \mu_{\mathrm{P}}(0)\}$ and $\nu_{\mathrm{P}}(3*2) = \nu_{\mathrm{P}}(2) = 0.75 \nleq 0.7 = \max\{0.7, 0.3\} = \max\{\nu_{\mathrm{P}}(1), \nu_{\mathrm{P}}(0)\}$.

The following example shows that Pythagorean fuzzy set in a UP-algebra which fulfills the assertions (39) and (40) is not constant.

Example 3.24. From Example 2.11, we define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	1	0.8	0.5	0.5
ν_{P}	0	0.3	0.6	0.6

Then P fulfills the assertions (39) and (40) but it is not constant.

Theorem 3.25. *If* $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ *is a Pythagorean fuzzy UP-filter of U fulfilling the assertion* (6)*, then it is a Pythagorean fuzzy UP-ideal of U.*

Proof. Let P be a Pythagorean fuzzy UP-filter of U. Then for all $x, y, z \in U$,

$$\mu_P(x * z) \geq \min\{\mu_P(y * (x * z)), \mu_P(y)\} = \min\{\mu_P(x * (y * z)), \mu_P(y)\} \qquad \text{by (21) and (6)}$$

and

$$\nu_P(x * z) \leq \max\{\nu_P(y * (x * z)), \nu_P(y)\} = \max\{\nu_P(x * (y * z)), \nu_P(y)\}. \qquad \text{by (22) and (6)}$$

Hence, P is a Pythagorean fuzzy UP-ideal of U. □

Proposition 3.26. *A Pythagorean fuzzy set* $P = (\mu_P, \nu_P)$ *in U fulfills the following assertions:*

$$(\textit{for all } a, x, y, z \in U)(a \leq (z * y) * (z * x) \Rightarrow \mu_P(x) \geq \min\{\mu_P(a), \mu_P(y)\}), \tag{43}$$

$$(\textit{for all } a, x, y, z \in U)(a \leq (z * y) * (z * x) \Rightarrow \nu_P(x) \leq \max\{\nu_P(a), \nu_P(y)\}) \tag{44}$$

if and only if it is a Pythagorean fuzzy strong UP-ideal of U.

Proof. Let $x \in U$. By (UP-3), we have $x \leq 0 = x * 0 = (0 * x) * (0 * 0)$. By (43) and (44), we have $\mu_P(0) \geq \min\{\mu_P(x), \mu_P(x)\} = \mu_P(x)$ and $\nu_P(0) \leq \max\{\nu_P(x), \nu_P(x)\} = \nu_P(x)$. Next, let $x, y, z \in U$. By (1), we have $(z*y)*(z*x) \leq (z*y)*(z*x)$. By (43) and (44), we have $\mu_P(x) \geq \min\{\mu_P((z*y)*(z*x)), \mu_P(y)\}$ and $\nu_P(x) \leq \max\{\nu_P((z * y) * (z * x)), \nu_P(y)\}$. Hence, P is a Pythagorean fuzzy strong UP-ideal of U.

The converse is evident because P is constant by Theorem 2.3. □

Theorem 3.27. *If* $P = (\mu_P, \nu_P)$ *is a Pythagorean fuzzy set in U fulfilling the following assertions:*

$$(\textit{for all } x, y, z \in U)(z \leq x * y \Rightarrow \mu_P(z) \geq \min\{\mu_P(x), \mu_P(y)\}), \tag{45}$$

$$(\textit{for all } x, y, z \in U)(z \leq x * y \Rightarrow \nu_P(z) \leq \max\{\nu_P(x), \nu_P(y)\}), \tag{46}$$

then it fulfills the assertions (29) *and* (30).

Proof. Let $x, y, z \in U$ be such that $z \leq x$. By (3), we have $x * y \leq z * y$. By (45) and (46), we have $\mu_P(x * y) \geq \min\{\mu_P(z), \mu_P(y)\}$ and $\nu_P(x * y) \leq \max\{\nu_P(z), \nu_P(y)\}$. Hence, P fulfills the assertions (29) and (30). □

Proposition 3.28. *A Pythagorean fuzzy set* $P = (\mu_P, \nu_P)$ *in U fulfills the assertions* (45) *and* (46) *if and only if it is a Pythagorean fuzzy strong UP-ideal of U.*

Proof. Let $x \in U$. By (UP-3), we have $x \leq 0 = 0 * 0$. By (45) and (46), we have $\mu_P(x) \geq \min\{\mu_P(0), \mu_P(0)\} = \mu_P(0)$ and $\nu_P(x) \leq \max\{\nu_P(0), \nu_P(0)\} = \nu_P(0)$. By Theorem 3.27, we have P fulfills (29) and (30). Thus P is a Pythagorean fuzzy UP-subalgebra of U by Proposition 3.4. It follows from Proposition 3.1 that $\mu_P(0) \geq \mu_P(x)$ and $\nu_P(0) \leq \nu_P(x)$, so $\mu_P(x) = \mu_P(0)$ and $\nu_P(x) = \nu_P(0)$ for all $x \in U$, that is, P is constant. By Theorem 2.3, we have P is a Pythagorean fuzzy strong UP-ideal of U.

The converse is evident because P is constant by Theorem 2.3. □

Theorem 3.29. *If* $P = (\mu_P, \nu_P)$ *is a Pythagorean fuzzy set in U fulfilling the following assertions:*

$$(\textit{for all } x, y, z \in U)(z \leq x * y \Rightarrow \mu_P(z) \geq \mu_P(y)), \tag{47}$$

$$(\textit{for all } x, y, z \in U)(z \leq x * y \Rightarrow \nu_P(z) \leq \nu_P(y)), \tag{48}$$

then it fulfills the assertions (29) *and* (30).

Proof. Let $x, y, z \in U$ be such that $z \leq x$. By (3), we have $x * y \leq z * y$. It follows from (47) and (48) that $\mu_P(x * y) \geq \mu_P(y) \geq \min\{\mu_P(z), \mu_P(y)\}$ and $\nu_P(x * y) \leq \nu_P(y) \leq \max\{\nu_P(z), \nu_P(y)\}$. Hence, P fulfills the assertions (29) and (30). □

Proposition 3.30. *A Pythagorean fuzzy set* $P = (\mu_P, \nu_P)$ *in U fulfills the assertions* (47) *and* (48) *if and only if it is a Pythagorean fuzzy strong UP-ideal of U.*

Proof. Let $x \in U$. By (UP-3), we have $x \leq 0 = 0 * 0$. By (47) and (48), we have $\mu_P(x) \geq \mu_P(0)$ and $\nu_P(x) \leq \nu_P(0)$. By Theorem 3.27, we have P fulfills (29) and (30). Thus P is a Pythagorean fuzzy UP-subalgebra of U by Proposition 3.4. It follows from Proposition 3.1 that $\mu_P(0) \geq \mu_P(x)$ and $\nu_P(0) \leq \nu_P(x)$, so $\mu_P(x) = \mu_P(0)$ and $\nu_P(x) = \nu_P(0)$ for all $x \in U$, that is, P is constant. By Theorem 2.3, we have P is a Pythagorean fuzzy strong UP-ideal of U.

The converse is evident because P is constant by Theorem 2.3. □

We get the diagram of the properties of Pythagorean fuzzy sets in UP-algebras, which is shown with Figure 1.

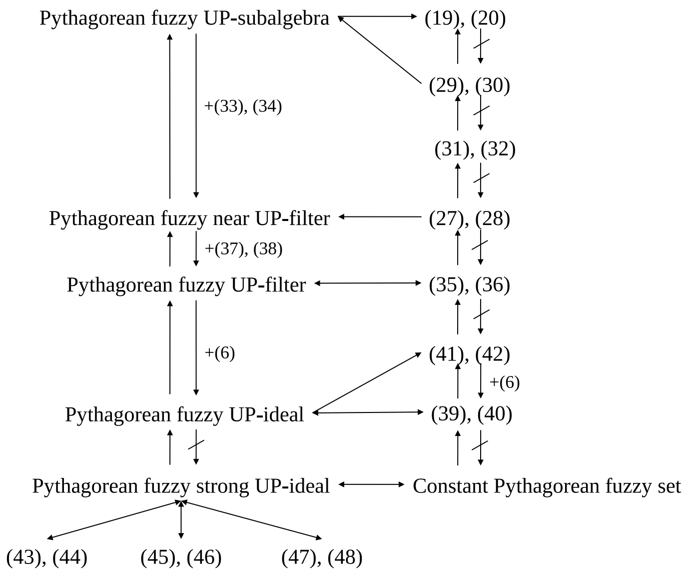

Figure 1. Properties of Pythagorean fuzzy sets in UP-algebras.

4. Approximations

Let U be a set and ρ an equivalence relation on U. If $x \in U$, then the ρ-class of U is the set $(x)_\rho$ defined as follows:

$$(x)_\rho = \{y \in U \mid (x, y) \in \rho\}.$$

An equivalence relation ρ on a UP-algebra $U = (U, *, 0)$ is said to be a *congruence relation* if

$$(\text{for all } x, y, z \in U)((x, y) \in \rho \Rightarrow (x * z, y * z) \in \rho, (z * x, z * y) \in \rho).$$

Definition 4.1. For nonempty subsets A and B of a UP-algebra $U = (U, *, 0)$, we denote

$$AB = A * B = \{a * b \mid a \in A \text{ and } b \in B\}.$$

If ρ is a congruence on a UP-algebra $U = (U, *, 0)$, then

$$(\text{for all } x, y \in U)((x)_\rho(y)_\rho \subseteq (x * y)_\rho). \qquad \text{see [20]}$$

A congruence relation ρ on a UP-algebra $U = (U, *, 0)$ is said to be *complete* if

$$(\text{for all } x, y \in U)((x)_\rho(y)_\rho = (x * y)_\rho).$$

Example 4.2. Let $U = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	1
3	0	0	0	0

Let

$$\rho = \{(0, 0), (1, 1), (2, 2), (3, 3), (0, 1), (1, 0), (2, 3), (3, 2)\}.$$

Then ρ is a congruence relation on U. Thus

$$(0)_\rho = (1)_\rho = \{0, 1\}, (2)_\rho = (3)_\rho = \{2, 3\}.$$

We consider

$$\{0, 1\} = \{0, 1\}\{0, 1\} = (0)_\rho(0)_\rho = (0 * 0)_\rho = (0)_\rho = \{0, 1\},$$
$$\{0, 1\} = \{0, 1\}\{0, 1\} = (0)_\rho(1)_\rho = (0 * 1)_\rho = (1)_\rho = \{0, 1\},$$
$$\{2.3\} = \{0, 1\}\{2, 3\} = (0)_\rho(2)_\rho = (0 * 2)_\rho = (2)_\rho = \{2, 3\},$$
$$\{2, 3\} = \{0, 1\}\{2, 3\} = (0)_\rho(3)_\rho = (0 * 3)_\rho = (3)_\rho = \{2, 3\},$$
$$\{0, 1\} = \{0, 1\}\{0, 1\} = (1)_\rho(0)_\rho = (1 * 0)_\rho = (0)_\rho = \{0, 1\},$$
$$\{0, 1\} = \{0, 1\}\{0, 1\} = (1)_\rho(1)_\rho = (1 * 1)_\rho = (0)_\rho = \{0, 1\},$$
$$\{2.3\} = \{0, 1\}\{2, 3\} = (1)_\rho(2)_\rho = (1 * 2)_\rho = (2)_\rho = \{2, 3\},$$
$$\{2, 3\} = \{0, 1\}\{2, 3\} = (1)_\rho(3)_\rho = (1 * 3)_\rho = (3)_\rho = \{2, 3\},$$
$$\{0, 1\} = \{2, 3\}\{0, 1\} = (2)_\rho(0)_\rho = (2 * 0)_\rho = (0)_\rho = \{0, 1\},$$
$$\{0, 1\} = \{2, 3\}\{0, 1\} = (2)_\rho(1)_\rho = (2 * 1)_\rho = (0)_\rho = \{0, 1\},$$
$$\{0, 1\} = \{2, 3\}\{2, 3\} = (2)_\rho(2)_\rho = (2 * 2)_\rho = (0)_\rho = \{0, 1\},$$
$$\{0, 1\} = \{2, 3\}\{2, 3\} = (2)_\rho(3)_\rho = (2 * 3)_\rho = (1)_\rho = \{0, 1\},$$
$$\{0, 1\} = \{2, 3\}\{0, 1\} = (3)_\rho(0)_\rho = (3 * 0)_\rho = (0)_\rho = \{0, 1\},$$

$$\{0,1\} = \{2,3\}\{0,1\} = (3)_\rho(1)_\rho = (3*1)_\rho = (0)_\rho = \{0,1\},$$
$$\{0,1\} = \{2,3\}\{2,3\} = (3)_\rho(2)_\rho = (3*2)_\rho = (0)_\rho = \{0,1\},$$
$$\{0,1\} = \{2,3\}\{2,3\} = (3)_\rho(3)_\rho = (3*3)_\rho = (0)_\rho = \{0,1\}.$$

Hence, ρ is a complete congruence relation on U.

Definition 4.3. Let ρ be an equivalence relation on a nonempty set U and $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ a Pythagorean fuzzy set in U. The *upper approximation* is defined by

$$\rho^+(\mathrm{P}) = \{(x, \overline{\mu}_\mathrm{P}(x), \overline{\nu}_\mathrm{P}(x)) \mid x \in U\},$$

where $\overline{\mu}_\mathrm{P}(x) = \sup_{a\in(x)_\rho}\{\mu_\mathrm{P}(a)\}$ and $\overline{\nu}_\mathrm{P}(x) = \inf_{a\in(x)_\rho}\{\nu_\mathrm{P}(a)\}$. The *lower approximation* is defined by

$$\rho^-(\mathrm{P}) = \{(x, \underline{\mu}_\mathrm{P}(x), \underline{\nu}_\mathrm{P}(x)) \mid x \in U\},$$

where $\underline{\mu}_\mathrm{P}(x) = \inf_{a\in(x)_\rho}\{\mu_\mathrm{P}(a)\}$ and $\underline{\nu}_\mathrm{P}(x) = \sup_{a\in(x)_\rho}\{\nu_\mathrm{P}(a)\}$.

Theorem 4.4. *Let ρ be an equivalence relation on a nonempty set U and* $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ *a Pythagorean fuzzy set in U. Then the following statements hold:*

(1) $\rho^+(\mathrm{P})$ *is a Pythagorean fuzzy set in U, and*

(2) $\rho^-(\mathrm{P})$ *is a Pythagorean fuzzy set in U.*

Proof. Let $x \in U$.

(1) We consider

$$\begin{aligned}
0 &\le \overline{\mu}_\mathrm{P}(x)^2 + \overline{\nu}_\mathrm{P}(x)^2 \\
&= (\sup_{a\in(x)_\rho}\{\mu_\mathrm{P}(a)\})^2 + (\inf_{a\in(x)_\rho}\{\nu_\mathrm{P}(a)\})^2 \\
&= \sup_{a\in(x)_\rho}\{\mu_\mathrm{P}(a)^2\} + \inf_{a\in(x)_\rho}\{\nu_\mathrm{P}(a)^2\} && \text{by Proposition 1.9 (6)} \\
&\le \sup_{a\in(x)_\rho}\{\mu_\mathrm{P}(a)^2\} + \inf_{a\in(x)_\rho}\{1 - \mu_\mathrm{P}(a)^2\} \\
&= \sup_{a\in(x)_\rho}\{\mu_\mathrm{P}(a)^2\} + 1 - \sup_{a\in(x)_\rho}\{\mu_\mathrm{P}(a)^2\} && \text{by Proposition 1.9 (7)} \\
&= 1.
\end{aligned}$$

This implies that $0 \le \overline{\mu}_\mathrm{P}(x)^2 + \overline{\nu}_\mathrm{P}(x)^2 \le 1$. Therefore, $\rho^+(\mathrm{P})$ is a Pythagorean fuzzy set in U.

(2) The proof is similar to the proof of (1). □

Lemma 4.5. *If ρ is an equivalence relation on a nonempty set U and* $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ *a Pythagorean fuzzy set in U, then*

$$(\textit{for all } x, y \in U)(x\rho y \Rightarrow \overline{\mu}_\mathrm{P}(x) = \overline{\mu}_\mathrm{P}(y)), \tag{49}$$
$$(\textit{for all } x, y \in U)(x\rho y \Rightarrow \overline{\nu}_\mathrm{P}(x) = \overline{\nu}_\mathrm{P}(y)), \tag{50}$$
$$(\textit{for all } x, y \in U)(x\rho y \Rightarrow \underline{\mu}_\mathrm{P}(x) = \underline{\mu}_\mathrm{P}(y)),$$
$$(\textit{for all } x, y \in U)(x\rho y \Rightarrow \underline{\nu}_\mathrm{P}(x) = \underline{\nu}_\mathrm{P}(y)).$$

Proof. Let $x, y \in U$ be such that $x\rho y$. Then

$$\begin{aligned}
\overline{\mu}_{\mathrm{P}}(x) = \sup_{a\in(x)_\rho}\{\mu_{\mathrm{P}}(a)\} &= \sup_{b\in(y)_\rho}\{\mu_{\mathrm{P}}(b)\} = \overline{\mu}_{\mathrm{P}}(y),\\
\overline{\nu}_{\mathrm{P}}(x) = \inf_{a\in(x)_\rho}\{\nu_{\mathrm{P}}(a)\} &= \inf_{b\in(y)_\rho}\{\nu_{\mathrm{P}}(b)\} = \overline{\nu}_{\mathrm{P}}(y),\\
\underline{\mu}_{\mathrm{P}}(x) = \inf_{a\in(x)_\rho}\{\mu_{\mathrm{P}}(a)\} &= \inf_{b\in(y)_\rho}\{\mu_{\mathrm{P}}(b)\} = \underline{\mu}_{\mathrm{P}}(y),\\
\underline{\nu}_{\mathrm{P}}(x) = \sup_{a\in(x)_\rho}\{\nu_{\mathrm{P}}(a)\} &= \sup_{b\in(y)_\rho}\{\nu_{\mathrm{P}}(b)\} = \underline{\nu}_{\mathrm{P}}(y).
\end{aligned}$$

We complete the proof. □

Theorem 4.6. *Let ρ be an congruence relation on a UP-algebra $U = (U, *, 0)$ and $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ a Pythagorean fuzzy set in U. Then the following statements hold:*

(1) *if P is a Pythagorean fuzzy UP-subalgebra of U and ρ is complete, then $\rho^{-}(\mathrm{P})$ is a Pythagorean fuzzy UP-subalgebra of U,*

(2) *if P is a Pythagorean fuzzy near UP-filter of U and ρ is complete, then $\rho^{-}(\mathrm{P})$ is a Pythagorean fuzzy near UP-filter of U,*

(3) *if P is a Pythagorean fuzzy UP-filter of U and $(0)_\rho = \{0\}$, then $\rho^{-}(\mathrm{P})$ is a Pythagorean fuzzy UP-filter of U,*

(4) *if P is a Pythagorean fuzzy UP-ideal of U, $(0)_\rho = \{0\}$, and ρ is complete, then $\rho^{-}(\mathrm{P})$ is a Pythagorean fuzzy UP-ideal of U, and*

(5) *if P is a Pythagorean fuzzy strong UP-ideal of U, then $\rho^{-}(\mathrm{P})$ is a Pythagorean fuzzy strong UP-ideal of U.*

Proof. (1) Assume that P is a Pythagorean fuzzy UP-subalgebra of U and ρ is complete. Then for all $x, y \in U$,

$$\begin{aligned}
\underline{\mu}_{\mathrm{P}}(x*y) &= \inf_{c\in(x*y)_\rho}\{\mu_{\mathrm{P}}(c)\}\\
&= \inf_{c\in(x)_\rho(y)_\rho}\{\mu_{\mathrm{P}}(c)\}\\
&= \inf_{a*b\in(x)_\rho(y)_\rho}\{\mu_{\mathrm{P}}(a*b)\}\\
&\geq \inf_{a\in(x)_\rho, b\in(y)_\rho}\{\min\{\mu_{\mathrm{P}}(a), \mu_{\mathrm{P}}(b)\}\} && \text{by (15)}\\
&= \min\{\inf_{a\in(x)_\rho}\{\mu_{\mathrm{P}}(a)\}, \inf_{b\in(y)_\rho}\{\mu_{\mathrm{P}}(b)\}\} && \text{by Proposition 1.9 (1)}\\
&= \min\{\underline{\mu}_{\mathrm{P}}(x), \underline{\mu}_{\mathrm{P}}(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\underline{\nu}_{\mathrm{P}}(x*y) &= \sup_{c\in(x*y)_\rho}\{\nu_{\mathrm{P}}(c)\}\\
&= \sup_{c\in(x)_\rho(y)_\rho}\{\nu_{\mathrm{P}}(c)\}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{a*b\in(x)_\rho(y)_\rho}\{\nu_P(a*b)\} \\
&\leq \sup_{a\in(x)_\rho, b\in(y)_\rho}\{\max\{\nu_P(a),\nu_P(b)\}\} && \text{by (16)} \\
&= \max\{\sup_{a\in(x)_\rho}\{\nu_P(a)\}, \sup_{b\in(y)_\rho}\{\nu_P(b)\}\} && \text{by Proposition 1.9 (2)} \\
&= \max\{\underline{\nu}_P(x), \underline{\nu}_P(y)\}.
\end{aligned}
$$

Hence, $\rho^-(\mathrm{P})$ is a Pythagorean fuzzy UP-subalgebra of U.

(2) Assume that P is a Pythagorean fuzzy near UP-filter of U and ρ is complete. Then for all $x, y \in U$,

$$\underline{\mu}_P(x*y) = \inf_{c\in(x*y)_\rho}\{\mu_P(c)\} = \inf_{c\in(x)_\rho(y)_\rho}\{\mu_P(c)\} = \inf_{a*b\in(x)_\rho(y)_\rho}\{\mu_P(a*b)\} \geq \inf_{b\in(y)_\rho}\{\mu_P(b)\} = \underline{\mu}_P(y) \qquad \text{by (17)}$$

and

$$\underline{\nu}_P(x*y) = \sup_{c\in(x*y)_\rho}\{\nu_P(c)\} = \sup_{c\in(x)_\rho(y)_\rho}\{\nu_P(c)\} = \sup_{a*b\in(x)_\rho(y)_\rho}\{\nu_P(a*b)\} \leq \sup_{b\in(y)_\rho}\{\nu_P(b)\} = \underline{\nu}_P(y). \qquad \text{by (18)}$$

Hence, $\rho^-(\mathrm{P})$ is a Pythagorean fuzzy near UP-filter of U.

(3) Assume that P is a Pythagorean fuzzy UP-filter of U and $(0)_\rho = \{0\}$. Then for all $x, y \in U$,

$$\underline{\mu}_P(0) = \inf_{a\in(0)_\rho}\{\mu_P(a)\} = \mu_P(0) \geq \mu_P(b) \geq \inf_{b\in(x)_\rho}\{\mu_P(b)\} = \underline{\mu}_P(x),$$

$$\underline{\nu}_P(0) = \sup_{a\in(0)_\rho}\{\nu_P(a)\} = \nu_P(0) \leq \nu_P(b) \leq \sup_{b\in(x)_\rho}\{\nu_P(b)\} = \underline{\nu}_P(x),$$

$$
\begin{aligned}
\underline{\mu}_P(y) &= \inf_{b\in(y)_\rho}\{\mu_P(b)\} \\
&\geq \inf_{a*b\in(x)_\rho(y)_\rho, a\in(x)_\rho}\{\min\{\mu_P(a*b),\mu_P(a)\}\} && \text{by (21)} \\
&\geq \inf_{a*b\in(x*y)_\rho, a\in(x)_\rho}\{\min\{\mu_P(a*b),\mu_P(a)\}\} \\
&= \min\{\inf_{a*b\in(x*y)_\rho}\{\mu_P(a*b)\}, \inf_{a\in(x)_\rho}\{\mu_P(a)\}\} && \text{by Proposition 1.9 (1)} \\
&= \min\{\underline{\mu}_P(x*y), \underline{\mu}_P(x)\},
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{\nu}_P(y) &= \sup_{b\in(y)_\rho}\{\nu_P(b)\} \\
&\leq \sup_{a*b\in(x)_\rho(y)_\rho, a\in(x)_\rho}\{\max\{\nu_P(a*b),\nu_P(a)\}\} && \text{by (22)} \\
&\leq \sup_{a*b\in(x*y)_\rho, a\in(x)_\rho}\{\max\{\nu_P(a*b),\nu_P(a)\}\} \\
&= \max\{\sup_{a*b\in(x*y)_\rho}\{\nu_P(a*b)\}, \sup_{a\in(x)_\rho}\{\nu_P(a)\}\} && \text{by Proposition 1.9 (2)} \\
&= \max\{\underline{\nu}_P(x*y), \underline{\nu}_P(x)\}.
\end{aligned}
$$

Hence, $\rho^{-}(\mathrm{P})$ is a Pythagorean fuzzy UP-filter of U.

(4) Assume that P is a Pythagorean fuzzy UP-ideal of U, ρ is complete, and $(0)_\rho = \{0\}$. Then for all $x, y, z \in U$,

$$\underline{\mu}_{\mathrm{P}}(0) = \inf_{a\in(0)_\rho}\{\mu_{\mathrm{P}}(a)\} = \mu_{\mathrm{P}}(0) \geq \mu_{\mathrm{P}}(b) \geq \inf_{b\in(x)_\rho}\{\mu_{\mathrm{P}}(b)\} = \underline{\mu}_{\mathrm{P}}(x),$$

$$\underline{\nu}_{\mathrm{P}}(0) = \sup_{a\in(0)_\rho}\{\nu_{\mathrm{P}}(a)\} = \nu_{\mathrm{P}}(0) \leq \nu_{\mathrm{P}}(b) \leq \sup_{b\in(x)_\rho}\{\nu_{\mathrm{P}}(b)\} = \underline{\nu}_{\mathrm{P}}(x),$$

$$\begin{aligned}
\underline{\mu}_{\mathrm{P}}(x * z) &= \inf_{d\in(x*z)_\rho}\{\mu_{\mathrm{P}}(d)\} \\
&= \inf_{d\in(x)_\rho(z)_\rho}\{\mu_{\mathrm{P}}(d)\} \\
&= \inf_{a*c\in(x)_\rho(z)_\rho}\{\mu_{\mathrm{P}}(a * c)\} \\
&\geq \inf_{a*(b*c)\in(x)_\rho((y)_\rho(z)_\rho),b\in(y)_\rho}\{\min\{\mu_{\mathrm{P}}(a * (b * c)), \mu_{\mathrm{P}}(b)\}\} && \text{by (23)} \\
&= \inf_{a*(b*c)\in(x*(y*z))_\rho,b\in(y)_\rho}\{\min\{\mu_{\mathrm{P}}(a * (b * c)), \mu_{\mathrm{P}}(b)\}\} \\
&= \min\{\inf_{a*(b*c)\in(x*(y*z))_\rho}\{\mu_{\mathrm{P}}(a * (b * c))\}, \inf_{b\in(y)_\rho}\{\mu_{\mathrm{P}}(b)\}\} && \text{by Proposition 1.9 (1)} \\
&= \min\{\underline{\mu}_{\mathrm{P}}(x * (y * z)), \underline{\mu}_{\mathrm{P}}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\underline{\nu}_{\mathrm{P}}(x * z) &= \sup_{d\in(x*z)_\rho}\{\nu_{\mathrm{P}}(d)\} \\
&= \sup_{d\in(x)_\rho(z)_\rho}\{\nu_{\mathrm{P}}(d)\} \\
&= \sup_{a*c\in(x)_\rho(z)_\rho}\{\nu_{\mathrm{P}}(a * c)\} \\
&\leq \sup_{a*(b*c)\in(x)_\rho((y)_\rho(z)_\rho),b\in(y)_\rho}\{\max\{\nu_{\mathrm{P}}(a * (b * c)), \nu_{\mathrm{P}}(b)\}\} && \text{by (24)} \\
&= \sup_{a*(b*c)\in(x*(y*z))_\rho,b\in(y)_\rho}\{\max\{\nu_{\mathrm{P}}(a * (b * c)), \nu_{\mathrm{P}}(b)\}\} \\
&= \max\{\sup_{a*(b*c)\in(x*(y*z))_\rho}\{\nu_{\mathrm{P}}(a * (b * c))\}, \sup_{b\in(y)_\rho}\{\nu_{\mathrm{P}}(b)\}\} && \text{by Proposition 1.9 (2)} \\
&= \max\{\underline{\nu}_{\mathrm{P}}(x * (y * z)), \underline{\nu}_{\mathrm{P}}(y)\}.
\end{aligned}$$

Hence, $\rho^{-}(\mathrm{P})$ is a Pythagorean fuzzy UP-ideal of U.

(5) Assume that P is a Pythagorean fuzzy strong UP-ideal of U. By Theorem 2.3, we have P is constant. Then for all $x, y, z \in U$,

$$\underline{\mu}_{\mathrm{P}}(0) = \inf_{a\in(0)_\rho}\{\mu_{\mathrm{P}}(a)\} = \inf_{b\in(x)_\rho}\{\mu_{\mathrm{P}}(b)\} = \underline{\mu}_{\mathrm{P}}(x),$$

$$\underline{\nu}_{\mathrm{P}}(0) = \sup_{a\in(0)_\rho}\{\nu_{\mathrm{P}}(a)\} = \sup_{b\in(x)_\rho}\{\nu_{\mathrm{P}}(b)\} = \underline{\nu}_{\mathrm{P}}(x),$$

$$\underline{\mu}_{\mathrm{P}}(x) = \inf_{a\in(x)_\rho}\{\mu_{\mathrm{P}}(a)\}$$

$$
\begin{aligned}
&\geq \inf_{(c*b)*(c*a)\in((z)_\rho(y)_\rho)((z)_\rho(x)_\rho),b\in(y)_\rho}\{\min\{\mu_P((c*b)*(c*a)),\mu_P(b)\}\} && \text{by (25)}\\
&\geq \inf_{(c*b)*(c*a)\in((z*y)*(z*x))_\rho,b\in(y)_\rho}\{\min\{\mu_P((c*b)*(c*a)),\mu_P(b)\}\}\\
&= \min\{\inf_{(c*b)*(c*a)\in((z*y)*(z*x))_\rho}\{\mu_P((c*b)*(c*a))\},\inf_{b\in(y)_\rho}\{\mu_P(b)\}\} && \text{by Proposition 1.9 (1)}\\
&= \min\{\underline{\mu}_P((z*y)*(z*x)),\underline{\mu}_P(y)\},
\end{aligned}
$$

and

$$
\begin{aligned}
\underline{\nu}_P(x) &= \sup_{a\in(x)_\rho}\{\nu_P(a)\}\\
&\leq \sup_{(c*b)*(c*a)\in((z)_\rho(y)_\rho)((z)_\rho(x)_\rho),b\in(y)_\rho}\{\max\{\nu_P((c*b)*(c*a)),\nu_P(b)\}\} && \text{by (26)}\\
&\leq \sup_{(c*b)*(c*a)\in((z*y)*(z*x))_\rho,b\in(y)_\rho}\{\max\{\nu_P((c*b)*(c*a)),\nu_P(b)\}\}\\
&= \max\{\sup_{(c*b)*(c*a)\in((z*y)*(z*x))_\rho}\{\nu_P((c*b)*(c*a))\},\sup_{b\in(y)_\rho}\{\nu_P(b)\}\} && \text{by Proposition 1.9 (2)}\\
&= \max\{\underline{\nu}_P((z*y)*(z*x)),\underline{\nu}_P(y)\}.
\end{aligned}
$$

Hence, $\rho^-(\mathrm{P})$ is a Pythagorean fuzzy strong UP-ideal of U. □

The following example shows that Theorem 4.6 (3) may be not true if $(0)_\rho \neq \{0\}$.

Example 4.7. Let $U = \{0,1,2,3\}$ be a UP-algebra with a fixed element 0 and a binary operation $*$ defined by the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	0
2	0	1	0	3
3	0	1	2	0

We define a Pythagorean fuzzy set $\mathrm{P} = (\mu_P, \nu_P)$ with μ_P and ν_P as follows:

U	0	1	2	3
μ_P	0.7	0.4	0.6	0.6
ν_P	0.2	0.6	0.3	0.3

Then $\mathrm{P} = (\mu_P, \nu_P)$ is a Pythagorean fuzzy UP-filter of U. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,0),(0,3),(3,0)\}.$$

Then ρ is a congruence relation on U. Thus

$$(0)_\rho = (1)_\rho = (3)_\rho = \{0,1,3\}, (2)_\rho = \{2\}.$$

Since $\underline{\mu}_P(0) = \min\{\mu_P(0),\mu_P(1),\mu_P(3)\} = \min\{0.7,0.4,0.6\} = 0.4 \ngeq 0.6 = \mu_P(2) = \underline{\mu}_P(2)$ and $\underline{\nu}_P(0) = \max\{\nu_P(0),\nu_P(1),\nu_P(3)\} = \max\{0.2,0.6,0.3\} = 0.6 \nleq 0.3 = \nu_P(2) = \underline{\nu}_P(2)$, we have $\rho^-(\mathrm{P})$ is not a Pythagorean fuzzy UP-filter of U.

The following example shows that Theorem 4.6 (4) may be not true if $(0)_\rho \neq \{0\}$ and ρ is not complete.

Example 4.8. From Example 2.11, we define a Pythagorean fuzzy set $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ with μ_P and ν_P as follows:

U	0	1	2	3
μ_P	1	0.2	0.1	0.5
ν_P	0	0.6	0.9	0.4

Then $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ is a Pythagorean fuzzy UP-ideal of U. Let

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (0,2), (2,0)\}.$$

Then ρ is a congruence relation on U. Thus

$$(0)_\rho = (2)_\rho = \{0, 2\}, (1)_\rho = \{1\}, (3)_\rho = \{3\}.$$

Since $\underline{\mu}_\mathrm{P}(0) = \min\{\mu_\mathrm{P}(0), \mu_\mathrm{P}(2)\} = \min\{1, 0.1\} = 0.1 \ngeq 0.2 = \mu_\mathrm{P}(1) = \underline{\mu}_\mathrm{P}(1)$ and $\underline{\nu}_\mathrm{P}(0) = \max\{\nu_\mathrm{P}(0), \nu_\mathrm{P}(2)\} = \max\{0, 0.9\} = 0.9 \nleq 0.6 = \nu_\mathrm{P}(1) = \underline{\nu}_\mathrm{P}(1)$, we have $\rho^-(\mathrm{P})$ is not a Pythagorean fuzzy UP-ideal of U.

Problem 4.9. Is the lower approximation $\rho^-(\mathrm{P})$ a Pythagorean fuzzy UP-ideal of U if P is a Pythagorean fuzzy UP-ideal, $(0)_\rho \neq \{0\}$, and ρ is complete?

Lemma 4.10. *If ρ is an congruence relation on a UP-algebra $U = (U, *, 0)$ and $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ a Pythagorean fuzzy UP-subalgebra of U, then the upper approximation $\rho^+(\mathrm{P})$ fulfills the following assertions:*

$$(\text{for all } x \in U)(\overline{\mu}_\mathrm{P}(0) \geq \overline{\mu}_\mathrm{P}(x)), \tag{51}$$

$$(\text{for all } x \in U)(\overline{\nu}_\mathrm{P}(0) \leq \overline{\nu}_\mathrm{P}(x)). \tag{52}$$

Proof. Let $x \in U$. Then

$$\overline{\mu}_\mathrm{P}(0) = \sup_{a \in (0)_\rho}\{\mu_\mathrm{P}(a)\} \geq \mu_\mathrm{P}(0) \geq \sup_{b \in (x)_\rho}\{\mu_\mathrm{P}(b)\} = \overline{\mu}_\mathrm{P}(x) \qquad \text{by (19)}$$

and

$$\overline{\nu}_\mathrm{P}(0) = \inf_{a \in (0)_\rho}\{\nu_\mathrm{P}(a)\} \leq \nu_\mathrm{P}(0) \leq \inf_{b \in (x)_\rho}\{\nu_\mathrm{P}(b)\} = \overline{\nu}_\mathrm{P}(x). \qquad \text{by (20)}$$

Hence, $\rho^+(\mathrm{P})$ fulfills the assertions (51) and (52). □

Theorem 4.11. *Let ρ be an congruence relation on a UP-algebra $U = (U, *, 0)$ and $\mathrm{P} = (\mu_\mathrm{P}, \nu_\mathrm{P})$ a Pythagorean fuzzy set in U. Then the following statements hold:*

(1) *If P is a Pythagorean fuzzy UP-subalgebra of U, then $\rho^+(\mathrm{P})$ is a Pythagorean fuzzy UP-subalgebra of U,*

(2) *If P is a Pythagorean fuzzy near UP-filter of U, then $\rho^+(\mathrm{P})$ is a Pythagorean fuzzy near UP-filter of U, and*

(3) *If* P *is a Pythagorean fuzzy strong UP-ideal of* U*, then* $\rho^{+}(\mathrm{P})$ *is a Pythagorean fuzzy strong UP-ideal of* U*.*

Proof. (1) Assume that P is a Pythagorean fuzzy UP-subalgebra of U. Then for all $x, y \in U$,
Case 1: $x = y$. Then

$$\overline{\mu}_{\mathrm{P}}(x * y) = \overline{\mu}_{\mathrm{P}}(0) \geq \overline{\mu}_{\mathrm{P}}(x) \geq \min\{\overline{\mu}_{\mathrm{P}}(x), \overline{\mu}_{\mathrm{P}}(y)\} \qquad \text{by (1), (51)}$$

and

$$\overline{\nu}_{\mathrm{P}}(x * y) = \overline{\nu}_{\mathrm{P}}(0) \leq \overline{\nu}_{\mathrm{P}}(x) \leq \max\{\overline{\nu}_{\mathrm{P}}(x), \overline{\nu}_{\mathrm{P}}(y)\}. \qquad \text{by (1), (52)}$$

Case 2: $x \neq y$.
Case 2.1: $x * y = x$ or y. It is sufficient to assume that $x * y = x$. Then

$$\overline{\mu}_{\mathrm{P}}(x * y) = \overline{\mu}_{\mathrm{P}}(x) \geq \min\{\overline{\mu}_{\mathrm{P}}(x), \overline{\mu}_{\mathrm{P}}(y)\}$$

and

$$\overline{\nu}_{\mathrm{P}}(x * y) = \overline{\nu}_{\mathrm{P}}(x) \leq \max\{\overline{\nu}_{\mathrm{P}}(x), \overline{\nu}_{\mathrm{P}}(y)\}.$$

Case 2.2: $x * y \neq x$ and $x * y \neq y$. Assume that there exists $z \in U$ be such that $x * y = z$. If $z\rho 0$, then

$$\overline{\mu}_{\mathrm{P}}(x * y) = \overline{\mu}_{\mathrm{P}}(z) = \overline{\mu}_{\mathrm{P}}(0) \geq \min\{\overline{\mu}_{\mathrm{P}}(x), \overline{\mu}_{\mathrm{P}}(y)\} \qquad \text{by (49)}$$

and

$$\overline{\nu}_{\mathrm{P}}(x * y) = \overline{\nu}_{\mathrm{P}}(z) = \overline{\nu}_{\mathrm{P}}(0) \leq \max\{\overline{\nu}_{\mathrm{P}}(x), \overline{\nu}_{\mathrm{P}}(y)\}. \qquad \text{by (50)}$$

If $x\rho 0$ or $y\rho 0$, it is sufficient to assume that $x\rho 0$. Since ρ is a congruence relation on U, we have $xy\rho 0y$, that is, $z\rho y$. Therefore,

$$\overline{\mu}_{\mathrm{P}}(x * y) = \overline{\mu}_{\mathrm{P}}(z) = \overline{\mu}_{\mathrm{P}}(y) = \min\{\overline{\mu}_{\mathrm{P}}(0), \overline{\mu}_{\mathrm{P}}(y)\} = \min\{\overline{\mu}_{\mathrm{P}}(x), \overline{\mu}_{\mathrm{P}}(y)\} \qquad \text{by (49), (51)}$$

and

$$\overline{\nu}_{\mathrm{P}}(x * y) = \overline{\nu}_{\mathrm{P}}(z) = \overline{\nu}_{\mathrm{P}}(y) = \min\{\overline{\nu}_{\mathrm{P}}(0), \overline{\nu}_{\mathrm{P}}(y)\} = \max\{\overline{\nu}_{\mathrm{P}}(x), \overline{\nu}_{\mathrm{P}}(y)\}. \qquad \text{by (50), (52)}$$

Hence, $\rho^{+}(\mathrm{P})$ is a Pythagorean fuzzy UP-subalgebra of U.
(2) Assume that P is a Pythagorean fuzzy near UP-filter of U. Then for all $x, y \in U$,

$$\overline{\mu}_{\mathrm{P}}(x * y) = \sup_{c \in (x*y)_{\rho}} \{\mu_{\mathrm{P}}(c)\} \geq \sup_{c \in (x)_{\rho}(y)_{\rho}} \{\mu_{\mathrm{P}}(c)\} = \sup_{a*b \in (x)_{\rho}(y)_{\rho}} \{\mu_{\mathrm{P}}(a * b)\} \geq \sup_{b \in (y)_{\rho}} \{\mu_{\mathrm{P}}(b)\} = \overline{\mu}_{\mathrm{P}}(y) \qquad \text{by (17)}$$

and

$$\overline{\nu}_{\mathrm{P}}(x * y) = \inf_{c \in (x*y)_{\rho}} \{\nu_{\mathrm{P}}(c)\} \leq \inf_{c \in (x)_{\rho}(y)_{\rho}} \{\nu_{\mathrm{P}}(c)\} = \inf_{a*b \in (x)_{\rho}(y)_{\rho}} \{\nu_{\mathrm{P}}(a * b)\} \leq \inf_{b \in (y)_{\rho}} \{\nu_{\mathrm{P}}(b)\} = \overline{\nu}_{\mathrm{P}}(y). \qquad \text{by (18)}$$

Hence, $\rho^{+}(\mathrm{P})$ is a Pythagorean fuzzy near UP-filter of U.

(3) Assume that P is a Pythagorean fuzzy strong UP-ideal of U. By Theorem 2.3, we have P is constant. Then for all $x, y, z \in U$,

$$\overline{\mu}_{\mathrm{P}}(0) = \sup_{a\in(0)_\rho}\{\mu_{\mathrm{P}}(a)\} = \sup_{b\in(x)_\rho}\{\mu_{\mathrm{P}}(b)\} = \overline{\mu}_{\mathrm{P}}(x),$$

$$\overline{\nu}_{\mathrm{P}}(0) = \inf_{a\in(0)_\rho}\{\nu_{\mathrm{P}}(a)\} = \inf_{b\in(x)_\rho}\{\nu_{\mathrm{P}}(b)\} = \overline{\nu}_{\mathrm{P}}(x),$$

$$\begin{aligned}
\overline{\mu}_{\mathrm{P}}(x) &= \sup_{a\in(x)_\rho}\{\mu_{\mathrm{P}}(a)\} \\
&\geq \sup_{(c*b)*(c*a)\in((z)_\rho(y)_\rho)((z)_\rho(x)_\rho), b\in(y)_\rho}\{\min\{\mu_{\mathrm{P}}((c*b)*(c*a)), \mu_{\mathrm{P}}(b)\}\} && \text{by (25)} \\
&= \sup_{(c*b)*(c*a)\in((z*y)*(z*x))_\rho, b\in(y)_\rho}\{\min\{\mu_{\mathrm{P}}((c*b)*(c*a)), \mu_{\mathrm{P}}(b)\}\} \\
&= \min\{\sup_{(c*b)*(c*a)\in((z*y)*(z*x))_\rho}\{\mu_{\mathrm{P}}((c*b)*(c*a))\}, \sup_{b\in(y)_\rho}\{\mu_{\mathrm{P}}(b)\}\} && \text{by P is constant} \\
&= \min\{\overline{\mu}_{\mathrm{P}}((z*y)*(z*x)), \overline{\mu}_{\mathrm{P}}(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\overline{\nu}_{\mathrm{P}}(x) &= \inf_{a\in(x)_\rho}\{\nu_{\mathrm{P}}(a)\} \\
&\leq \inf_{(c*b)*(c*a)\in((z)_\rho(y)_\rho)((z)_\rho(x)_\rho), b\in(y)_\rho}\{\max\{\nu_{\mathrm{P}}((c*b)*(c*a)), \nu_{\mathrm{P}}(b)\}\} && \text{by (26)} \\
&= \inf_{(c*b)*(c*a)\in((z*y)*(z*x))_\rho, b\in(y)_\rho}\{\max\{\nu_{\mathrm{P}}((c*b)*(c*a)), \nu_{\mathrm{P}}(b)\}\} \\
&= \max\{\inf_{(c*b)*(c*a)\in((z*y)*(z*x))_\rho}\{\nu_{\mathrm{P}}((c*b)*(c*a))\}, \inf_{b\in(y)_\rho}\{\nu_{\mathrm{P}}(b)\}\} && \text{by P is constant} \\
&= \max\{\overline{\nu}_{\mathrm{P}}((z*y)*(z*x)), \overline{\nu}_{\mathrm{P}}(y)\}.
\end{aligned}$$

Hence, $\rho^+(\mathrm{P})$ is a Pythagorean fuzzy strong UP-ideal of U. □

The following example shows that if P is a Pythagorean fuzzy UP-filter of U, then the upper approximation $\rho^+(\mathrm{P})$ is not a Pythagorean fuzzy UP-filter in general.

Example 4.12. From Example 3.8, we define a Pythagorean fuzzy set $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ with μ_{P} and ν_{P} as follows:

U	0	1	2	3
μ_{P}	0.6	0.5	0.3	0.3
ν_{P}	0.3	0.4	0.7	0.7

Then $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ is a Pythagorean fuzzy UP-filter of U. Let

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (3,0), (0,3)\}.$$

Then ρ is a congruence relation on U. Thus

$$(0)_\rho = (3)_\rho = \{0,3\}, (1)_\rho = \{1\}, (2)_\rho = \{2\}.$$

Since $\overline{\mu}_{\mathrm{P}}(2) = \mu_{\mathrm{P}}(2) = 0.3 \ngeq 0.5 = \min\{\max\{\mu_{\mathrm{P}}(0), \mu_{\mathrm{P}}(3)\}, \mu_{\mathrm{P}}(1)\}\} = \min\{\overline{\mu}_{\mathrm{P}}(3), \overline{\mu}_{\mathrm{P}}(1)\} = \min\{\overline{\mu}_{\mathrm{P}}(1 * 2), \overline{\mu}_{\mathrm{P}}(1)\}$. we have $\rho^+(\mathrm{P})$ is not a Pythagorean fuzzy UP-filter of U.

Problem 4.13. Is the upper approximation $\rho^+(\mathrm{P})$ a Pythagorean fuzzy UP-filter of U if P is a Pythagorean fuzzy UP-filter of U?

5. Conclusions and future works

In this paper, we have introduced the concept of Pythagorean fuzzy sets in UP-algebras, and then we have introduced five types of Pythagorean fuzzy sets in UP-algebras, namely Pythagorean fuzzy UP-subalgebras, Pythagorean fuzzy near UP-filters, Pythagorean fuzzy UP-filters, Pythagorean fuzzy UP-ideals, and Pythagorean fuzzy strong UP-ideals. Further, we have discussed the relationship between some assertions of Pythagorean fuzzy sets and Pythagorean fuzzy UP-subalgebras (resp., Pythagorean fuzzy near UP-filters, Pythagorean fuzzy UP-filters, Pythagorean fuzzy UP-ideals, Pythagorean fuzzy strong UP-ideals) in UP-algebras and have studied upper and lower approximations of Pythagorean fuzzy sets. Hence, we get the diagram of generalization of Pythagorean fuzzy sets in UP-algebras, which is shown with Figure 2.

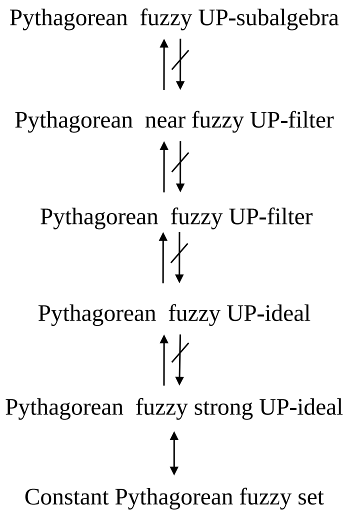

Figure 2. Pythagorean fuzzy sets in UP-algebras.

Some important topics for our future study of UP-algebras are as follows:

(1) to get more results in Pythagorean fuzzy sets,

(2) to define new types of Pythagorean fuzzy sets,

(3) to get more results and examples in upper approximation and lower approximation,

(4) to study the roughness of Pythagorean fuzzy sets, and

(5) to study the soft set theory of Pythagorean fuzzy sets.

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Conflict of interest

The authors declare no conflict of interest.

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