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Research article

Optimal error estimates of a class of system of two quasi-variational inequalities

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Abstract: In the present paper, the finite element approximation of a class of system of two quasi-variational inequalities with terms sources and obstacles depending on solution is analyzed. An optimal L^∞-error estimate is derived, combining a modified algorithm of Bensoussan-Lions type and standard uniform error estimates known for elliptic variational inequalities (VIs).

Keywords: elliptic variational inequalities; system of quasi-variational inequalities with nonlinear source terms; iterative scheme; finite element; L^∞-error estimate
Mathematics Subject Classification: 65K15, 65N30, 65N15

1. Introduction

This paper is concerned with the finite element approximation of system of J = 2 quasi-variational inequalities QVIs with term sources and obstacles depending on solution: Find a vector $U = (u^1, u^2) \in (H_0^1(\Omega))^2$ satisfying

$$a^i(u^i, v - u^i) \geq (f^i(u^i), v - u^i);\ v \in H_0^1(\Omega) \tag{1.1}$$

$$v, u^i \leq Mu^i;\ u^i \geq 0.$$

Where Ω is a bounded smooth domain of $\mathbb{R}^N$ with $N \geq 1$, each $a^i(.,.)$ is a continuous elliptic bilinear form, $(.,.)$ is the inner product in $L^2(\Omega)$ and each f^i is a regular, nonlinear functional depending on solutions. The obstacle M provide the coupling between the unknowns $u^1; u^2$

$$Mu^i = k + inf_{\mu \neq i} u^{\mu};$$

k is a positive number.We point out that in the case where f^i are independent of the solution, the system (1.1) coincides with that introduced by Bensoussan and Lions in [1] which arises in the management of energy production problems.

It is easy to note that the structure of system (1.1) is analogous to that of the classical obstacle problem [2] where the term source and obstacle are depending upon the solution sought. The terminology QVI being chosen is a result of this remark.

Numerical analysis of system of quasi-variational inequalities where term sources not depending on solutions were achieved in several works, we refer to [3–8] for system of quasi-variational inequalities with coercive or noncoercive operators.

For results on systems related to evolutionary Hamilton-Jacobi-Bellman equation we refer to [9–11].

The main objective of this paper is to show that problem (1.1) can be properly approximated by a finite element method and an optimal L^∞–error estimates is derived, which coincides with the optimal convergence order of elliptic variational inequalities of an obstacle type problem [12].

The approximation is carried out by first introducing a modified Bensoussan-Lions type iterative scheme depending on parameters which is shown to converge geometrically to the continuous solution. By a symmetrical approach, using the standard finite element method and a discrete maximum principle (DMP), the geometric convergence of the discrete modified Bensoussan-Lions type iterative scheme depending upon parameters is given as well. An L^∞-error estimates is then established combining the geometric convergence of both the continuous and discrete iterative schemes and the known uniform error estimates in elliptic VIs.

It is worth mentioning that even the guiding idea of this paper rests on the algorithmic approach followed in many papers cited above, the treatment of the geometric convergence of both continuous and discrete schemes is totally different because of the nonlinear nature of terms sources. Also, it is used for the first time for a system of QVIs.

An outline of this paper is as follows: In section 2, we lay down some definitions and classical results related to variational inequalities and prove a Lipschitz continuous and discrete dependency with respect to the source term, the boundary condition and the obstacle. Section 3 discusses the continuous Bensoussan-Lions type iterative scheme and proves its geometrical convergence. In Section 4, we establish the finite element counter parts of the continuous system and the continuous Bensoussan-Lions type iterative scheme respectively and the geometrical convergence of the discrete scheme. Section 5 is devoted the L^∞-error analysis of the method.

2. Preliminaries

We are given functions $a^i_{jk}(x), a^i_k(x), a^i_0(x), 1 \leq\ i \leq 2$ sufficiently smooth functions such that $1 \leq j, k \leq N$

$$\sum_{1 \leq j,k \leq N} a^i_{jk}(x) \xi_j\ \xi_k\ \geq \alpha\, |\xi|^2, \xi\ \in \mathbb{R}^N, \alpha > 0$$

$$a^i_0(x) \geq \beta^i > 0, (x \in\ \Omega) \tag{2.1}$$

where β^i is a positive constant. We define the bilinear forms: For all $u, v \in H^1_0(\Omega)$

$$a^i(u,v) = \int_\Omega \left(\sum_{1\le j,k\le N} a^i_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} + \sum_{k=1}^N a^i_k(x)\frac{\partial u}{\partial x_k}v + a^i_0(x)uv\right)dx \quad (2.2)$$

We are given right-hand sides

$$f^i \text{ such that } f^i \in L^\infty(\Omega), f^i \ge f^0 > 0,$$

a nonlinear functional and Lipschitz continuous on $\mathbb{R}$; that is

$$\left|f^i(x) - f^i(y)\right| \le k^i|x-y|, \forall x,y \in \mathbb{R},$$

such that

$$\alpha^i = \frac{k^i}{\beta^i} < 1, \quad (2.3)$$

where β^i is a constant defined in (2.1). For $W = (w^1, w^2) \in \left(L^\infty_+(\Omega)\right)^2$ we introduce the norm

$$\|W\|_\infty = \max_{1\le i\le 2}\left\|w^i\right\|_{L^\infty(\Omega)}.$$

2.1. Elliptic variational inequalities

Let be Ω a bounded polyhedral domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. We consider the bilinear form of the same form of those defined in (2.2), the linear form

$$(f,v) = \int_\Omega f(x)\,v(x)dx, \quad (2.4)$$

The right hand side

$$f \in L^\infty(\Omega), \quad (2.5)$$

the obstacle

$$\psi \in W^{2,\infty}(\Omega) \text{ and } \psi \ge 0, \quad (2.6)$$

the boundary condition $g \in L^\infty(\partial\Omega)$ and the nonempty convex set

$$K^g = \{v \in H^1(\Omega) \text{ such that } v = g \text{ on } \partial\Omega \text{ and } v \le \psi \text{ on } \Omega\}. \quad (2.7)$$

We consider the variational inequality V.I.: Find $u \in K^g$ such that

$$a(u, v-u) \ge (f, v-u), \forall v \in K^g. \quad (2.8)$$

2.2. A monotonicity property

Proposition 1 *Let (f,g,ψ); $(\tilde f,\tilde g,\tilde\psi)$ be a pair of data and $\zeta = \sigma(f,g,\psi)$; $\tilde\zeta = \sigma(\tilde f,\tilde g,\tilde\psi)$ the corresponding solution to (2.8). If $f \le \tilde f$ in Ω, $g \le \tilde g$ on $\partial\Omega$ and $\psi \le \tilde\psi$ then, $\zeta \le \tilde\zeta$ in Ω.*

Proof. The proof is an adaptation of the proof of the monotonicity property of the solution of VI with nonlinear source term (see [13]). According to [14], $\zeta = \max\left\{\underline{\zeta}\right\}$ where $\left\{\underline{\zeta}\right\}$ is the set of all the subsolutions of ζ. Hence, $\forall\underline{\zeta} \in \left\{\underline{\zeta}\right\}$, $\underline{\zeta}$ satisfies

$$a\left(\underline{\zeta}, v\right) \le (f, v), \forall v \ge 0 \; with \; \underline{\zeta} \; \le \psi \; and \; \underline{\zeta} \le \; g.$$

By using the conditions $f \le \tilde{f}$ in Ω, $g \le \tilde{g}$ on $\partial\Omega$ and $\psi \le \tilde{\psi}$, we get

$$a\left(\underline{\zeta}, v\right) \le (f, v) \le \left(\tilde{f}, v\right),$$

with

$$\underline{\zeta} \le \psi \le \tilde{\psi} \; and \; \underline{\zeta} \le g \le \tilde{g} \; on \; \partial\Omega.$$

Thus, ζ is a subsolution of $\tilde{\zeta} = \sigma\left(\tilde{f}, \tilde{g}, \tilde{\psi}\right)$, that is $\zeta \le \tilde{\zeta}$ in Ω.

2.3. A Lipschitz continuous dependency with respect to the boundary condition, the source term and the obstacle

This subsection is devoted to the establishment of a Lipschitz continuous dependence property of the solution with respect to the source term, the boundary condition and the obstacle by which we first, set out and demonstrate.

Proposition 2 *Let* (f, g, ψ); $\left(\tilde{f}, \tilde{g}, \tilde{\psi}\right)$ *be a pair of data and* $\zeta = \sigma(f, g, \psi)$ *;* $\tilde{\zeta} = \sigma\left(\tilde{f}, \tilde{g}, \tilde{\psi}\right)$ *the corresponding solution to (2.8). Then, we have*

$$\left\|\zeta - \tilde{\zeta}\right\|_{L^\infty(\Omega)} \le \max \; \left\{\left(\tfrac{1}{\beta}\right)\left\|f - \tilde{f}\right\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}, \left\|\psi - \tilde{\psi}\right\|_{L^\infty(\Omega)}\right\}. \quad (2.9)$$

Proof. The proof is an adaptation of the proof of a Lipschitz property of the solution of VI with nonlinear source term (see [13]). First, set

$$\phi = \max \; \left\{\left(\tfrac{1}{\beta}\right)\left\|f - \tilde{f}\right\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}, \left\|\psi - \tilde{\psi}\right\|_{L^\infty(\Omega)}\right\}. \quad (2.10)$$

Then,

$$\begin{aligned}\tilde{f} &\le f + \left\|f - \tilde{f}\right\|_{L^\infty(\Omega)} \\ &\le f + (1)\left\|f - \tilde{f}\right\|_{L^\infty(\Omega)} \\ &\le f + \left(\frac{a_0(x)}{\beta}\right)\left\|f - \tilde{f}\right\|_{L^\infty(\Omega)} \\ &\le f + a_0(x)\max\left\{\left(\frac{1}{\beta}\right)\left\|f - \tilde{f}\right\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}, \left\|\psi - \tilde{\psi}\right\|_{L^\infty(\Omega)}\right\} .\end{aligned}$$

So,

$$\le f + a_0(x)\phi \; in \; \Omega. \quad (2.11)$$

Thus, for all $0 < v$,

$$\left(\tilde{f}, v\right) \le (f + a_0(x)\phi, v),$$

with

$$\tilde{\zeta} \le \tilde{g} \le g + \phi \text{ } on \text{ } \partial\Omega,$$

$$\tilde{\zeta} \le \tilde{\psi} \le \psi + \phi \text{ } in \text{ } \Omega.$$

So, according to the property $\tilde{\zeta}$ is a subsolution of $\sigma(f + a_0(x)\phi, g + \phi, \psi + \phi) = \sigma(f, g, \psi) + \phi$, that is

$$\tilde{\zeta} \le \zeta + \phi \text{ } in \text{ } \overline{\Omega}$$

or

$$\tilde{\zeta} - \zeta \le \phi \text{ } in \text{ } \overline{\Omega}. \tag{2.12}$$

Similarly, interchanging the roles of the couples (f, g, ψ); $(\tilde{f}, \tilde{g}, \tilde{\psi})$, we obtain

$$\zeta - \tilde{\zeta} \le \phi \text{ } in \text{ } \overline{\Omega}, \tag{2.13}$$

which completes the proof.

Let τ_h be a triangulation of Ω with meshsize h, V_h be the space of finite elements consisting of continuous piecewise linear functions v vanishing on $\partial\Omega$ and ϕ_s; $s = 1,2,\ldots,m(h)$ be the basis functions of V_h.

The discrete counterpart of (2.8) consists of finding $u_h \in K_h^g$ such that

$$a(u_h, v - u_h) \ge (f, v - u_h), \forall v \in K_h^g. \tag{2.14}$$

Where

$$K_h^g = \{v \in V_h \text{ } such \text{ } that \text{ } v = \pi_h g \text{ } on \text{ } \partial\Omega \text{ } and \text{ } v \le r_h\psi \text{ } on \text{ } \Omega\}, \tag{2.15}$$

π_h is an interpolation operator on $\partial\Omega$ and r_h is the usual finite element restriction operator on Ω.

Theorem 3 *(See* [12] *Under conditions (2.5) and (2.6), there exists a constant C independent of h such that*

$$\|\zeta - \zeta_h\|_{L^\infty(\Omega)} \le Ch^2|logh|^2. \tag{2.16}$$

2.4. A Lipschitz discrete dependency with respect to the boundary condition, the source term and the obstacle

Assuming that the DMP is satisfied, i.e. the matrix resulting from the finite element discretization is an M-matrix (see [15,16]), we prove the Lipschitz discrete dependence with respect to the boundary condition, the source term and the obstacle by a similar study to that undertaken previously for the Lipschitz continuous dependence property.

Proposition 4 *Let* $(f, g, r_h\psi)$; $(\tilde{f}, \tilde{g}, r_h\tilde{\psi})$ *be a pair of data and* $\zeta_h = \sigma_h(f, g, r_h\psi)$; $\tilde{\zeta}_h = \sigma_h(\tilde{f}, \tilde{g}, r_h\tilde{\psi})$ *the corresponding solution to (2.14). If* $f \le \tilde{f}$ *in* Ω, $g \le \tilde{g}$ *on* $\partial\Omega$ *and* $r_h\psi \le r_h\tilde{\psi}$ *then,* $\zeta_h \le \tilde{\zeta}_h$ *in* Ω.

Proof. The proof is similar to that of the continuous case.

The proposition below establishes a Lipschitz discrete dependence of the solution with respect to the data.

Proposition 5 *Let the* (*d.m.p*) *holds. Then, we have*

$$\|\zeta_h - \tilde{\zeta}_h\|_{L^\infty(\Omega)} \leq \max \left\{\left(\frac{1}{\beta}\right)\|f - \tilde{f}\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}, \|r_h\psi - r_h\tilde{\psi}\|_{L^\infty(\Omega)}\right\} \quad (2.17)$$

Proof. The proof is similar to that of the continuous case.

3. The continuous problem

We define the following fixed-point mapping

$$\mathrm{T}: \left(\mathrm{L}_+^\infty(\Omega)\right)^2 \to \left(\mathrm{L}_+^\infty(\Omega)\right)^2$$

$$\mathrm{Z} = (\mathrm{z}^1, \mathrm{z}^2) \to \mathrm{TZ} = \zeta = (\zeta^1, \zeta^2).$$

Where $\zeta^{\mathrm{i}} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution to the following variational inequality

$$a^i(\zeta^{\mathrm{i}}, \mathrm{v} - \zeta^{\mathrm{i}}) \geq (\mathrm{f}^{\mathrm{i}}(\mathrm{z}^{\mathrm{i}}), \mathrm{v} - \zeta^{\mathrm{i}}); v \in H_0^1(\Omega) \quad (3.1)$$

$$v, \zeta^{\mathrm{i}} \leq M\zeta^{\mathrm{i}} = \mathrm{k} + \mathrm{z}^{\mathrm{j}};\ \zeta^{\mathrm{i}} \geq 0 \text{ with } \mathrm{i} \neq \mathrm{j}.$$

Thanks to [1,2], ζ^{i} is the unique solution to coercive variational inequality (3.1).

Remark 1 *We remark that the solution* $U = (u^1, u^2)$ *of the system* (1.1) *is the fixed point of the mapping T; that is* $TU = U$.

3.1. A continuous iterative scheme

Starting from $U^0 = (u^{1,0}, u^{2,0})$ where $u^{i,0}$; $i = 1;2$ is solution of the variational equation

$$a^i(u^{i,0}, v) = (f^i(u^{i,0}), v), \forall v \in H_0^1(\Omega),$$

and for all $0 < w_i < 1$; $i = 1,2$ we define the sequences $(u^{1,n+1})$ and $(u^{2,n+1})$ such that $u^{1,n+1}$ and $u^{2,n+1}$ the components of the vector U^{n+1}, solve the following elliptic variational inequalities respectively

$$(u^{1,n+1}, v - u^{1,n+1}) \geq (w_1 f^1(u^{1,n+1}) + (1 - w_1) f^1(u^{1,n}), v - u^{1,n+1}) \quad (3.2)$$

$$v, u^{1,n+1} \leq M u^{1,n+1} = k + u^{2,n}, \quad (3.3)$$

$$a^2(u^{2,n+1}, v - u^{2,n+1}) \geq (w_2 f^2(u^{2,n+1}) + (1 - w_2) f^2(u^{2,n}), v - u^{2,n+1}) \quad (3.4)$$

$$v, u^{2,n+1} \leq M u^{2,n+1} = k + u^{1,n+1}. \quad (3.5)$$

3.2. Convergence of the continuous iterative scheme

Theorem 2 *The sequences* $(u^{1,n+1})$ *and* $(u^{2,n+1})$ *converge geometrically to the solution* $U = (u^1, u^2)$ *of the system* (1.1)*; there exist a positive real* $\rho \in (0,1)$ *which depends on* α_i *and* w_i *such that for all* $n \geq 0$

$$\|U^{n+1} - U\|_\infty \leq \rho^{n+1}\|U^0 - U\|_\infty \quad (3.6)$$

where

$$\rho = \max_{1\le i\le 2} \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1} < 1. \tag{3.7}$$

Proof. The proof will carry out by induction.

- We first deal with the case

$$\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} = \max_{1\le i\le 2} \left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}. \tag{3.8}$$

- Indeed for $n = 0$; using (1.1), (3.2), (3.3) and (2.9), we have

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \max\left\{\begin{matrix}\left(\frac{1}{\beta^1}\right)\left\|f^1(u^1) - \left(w_1 f^1(u^{1,1}) + (1-w_1)f^1(u^{1,0})\right)\right\|_{L^\infty(\Omega)}; \\ \|u^2 - u^{2,0}\|_{L^\infty(\Omega)}\end{matrix}\right\}$$

$$\le max\left\{\begin{matrix}\left(\frac{1}{\beta^1}\right)\left\|w_1\left(f^1(u^1) - f^1(u^{1,1})\right) + (1-w_1)\left(f^1(u^1) - f^1(u^{1,0})\right)\right\|_{L^\infty(\Omega)}; \\ \|u^2 - u^{2,0}\|_{L^\infty(\Omega)}\end{matrix}\right\}$$

$$\le max \left\{\begin{matrix}\left(\frac{k^1}{\beta^1}\right)\left(w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + (1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}\right); \\ \|u^2 - u^{2,0}\|_{L^\infty(\Omega)}\end{matrix}\right\}.$$

So,

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \max \left\{\begin{matrix}\alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}; \\ \|u^2 - u^{2,0}\|_{L^\infty(\Omega)}\end{matrix}\right\} \tag{3.9}$$

We distinguish two cases

$$\max\left\{\alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}; \|u^2 - u^{2,0}\|_{L^\infty(\Omega)}\right\}$$

$$= \alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \tag{3.10}$$

or

$$\max\left\{\alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)};\ \|u^2 - u^{2,0}\|_{L^\infty(\Omega)}\right\}$$

$$= \|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \tag{3.11}$$

(3.9) in conjunction with case (3.10) implies

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \tag{3.12}$$

with

$$\|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \le \alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}, \tag{3.13}$$

which implies

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}. \tag{3.14}$$

By replacing (3.14) in (3.13), we get

$$\|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}$$
$$\le \rho \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)},$$

which coincides with (3.8).
(3.9) in conjunction with (3.11) implies

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \tag{3.15}$$

with

$$\alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,0}\|_{L^\infty(\Omega)}. \tag{3.16}$$

$\|u^2 - u^{2,0}\|_{L^\infty(\Omega)}$ is bounded below by both $\|u^1 - u^{1,1}\|_{L^\infty(\Omega)}$
and

$$\alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}.$$

So,

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}$$

or

$$\alpha_1 w_1\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,1}\|_{L^\infty(\Omega)}.$$

Then,

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \tfrac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \tag{3.17}$$

or

$$\tfrac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,1}\|_{L^\infty(\Omega)}. \tag{3.18}$$

(3.15), (3.17) and (3.18) generate the following three possibilities

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \le \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}$$

or

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \le \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}$$

or

$$\frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \le \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}.$$

All possibilities are true in the same time because they coincide with (3.8). So, there is either a contradiction and thus case (3.11) is impossible or case (3.11) is possible if and only if

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} = \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}.$$

Hence, both cases (3.10) and (3.11) imply (3.14).

- Let us now discuss the second case

$$\|u^2 - u^{2,0}\|_{L^\infty(\Omega)} = \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}. \tag{3.19}$$

(3.9) in conjunction with (3.10) implies (3.14) with

$$\|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}$$
$$\le \rho \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)} < \|u^2 - u^{2,0}\|_{L^\infty(\Omega)},$$

which contradicts (3.19) which means that (3.10) is impossible. (3.9) in conjunction with (3.11) we get (3.17) and (3.18). So,

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \le \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}$$

or

$$\frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}.$$

We remark that both alternatives are true in same time because both coincide with (3.19) which implies that in case (3.11), we must have

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} = \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,0}\|_{L^\infty(\Omega)}.$$

Hence, in both cases (3.8) and (3.19), we obtain (3.14). Hence,

$$\|u^1 - u^{1,1}\|_{L^\infty(\Omega)} \le \rho \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}. \tag{3.20}$$

- As

$$U^1 = (u^{1,1}, u^{2,1})\ and\ U = (u^1, u^2),$$

we need to deal also with $\|u^2 - u^{2,1}\|_{L^\infty(\Omega)}$, by following the same reasoning as that adopted for u^1 and $u^{1,1}$, we get

$$\|u^2 - u^{2,1}\|_{L^\infty(\Omega)} \le \max\begin{Bmatrix}\alpha_2 w_2\|u^2 - u^{2,1}\|_{L^\infty(\Omega)} + \alpha_2(1-w_2)\|u^2 - u^{2,0}\|_{L^\infty(\Omega)}; \\ \|u^1 - u^{1,1}\|_{L^\infty(\Omega)}\end{Bmatrix} \tag{3.21}$$

Again we distinguish two possibilities

$$\max\left\{\alpha_2 w_2\|u^2 - u^{2,1}\|_{L^\infty(\Omega)} + \alpha_2(1-w_2)\|u^2 - u^{2,0}\|_{L^\infty(\Omega)}; \|u^1 - u^{1,1}\|_{L^\infty(\Omega)}\right\}$$
$$= \alpha_2 w_2\|u^2 - u^{2,1}\|_{L^\infty(\Omega)} + \alpha_2(1-w_2)\|u^2 - u^{2,0}\|_{L^\infty(\Omega)}; \tag{3.22}$$

or

$$\max\{\alpha_2 w_2\|u^2-u^{2,1}\|_{L^\infty(\Omega)}+\alpha_2(1-w_2)\|u^2-u^{2,0}\|_{L^\infty(\Omega)};\ \|u^1-u^{1,1}\|_{L^\infty(\Omega)}\}$$

$$=\|u^1-u^{1,1}\|_{L^\infty(\Omega)}. \tag{3.23}$$

(3.21) and (3.22) imply

$$\|u^2-u^{2,1}\|_{L^\infty(\Omega)}\leq\frac{\alpha_2(1-w_2)}{(1-\alpha_2 w_2)}\|u^2-u^{2,0}\|_{L^\infty(\Omega)} \tag{3.24}$$

with

$$\|u^1-u^{1,1}\|_{L^\infty(\Omega)}\leq\alpha_2 w_2\|u^2-u^{2,1}\|_{L^\infty(\Omega)}+\alpha_2(1-w_2)\|u^2-u^{2,0}\|_{L^\infty(\Omega)}. \tag{3.25}$$

By substituting (3.24) in (3.25), we get

$$\|u^1-u^{1,1}\|_{L^\infty(\Omega)}\leq\frac{\alpha_2(1-w_2)}{1-\alpha_2 w_2}\|u^2-u^{2,0}\|_{L^\infty(\Omega)}\leq\rho\max_{1\leq i\leq 2}\|u^i-u^{i,0}\|_{L^\infty(\Omega)},$$

which coincides with (3.20). (3.21) and (3.23) imply

$$\|u^2-u^{2,1}\|_{L^\infty(\Omega)}\leq\|u^1-u^{1,1}\|_{L^\infty(\Omega)}, \tag{3.26}$$

with

$$\alpha_2 w_2\|u^2-u^{2,1}\|_{L^\infty(\Omega)}+\alpha_2(1-w_2)\|u^2-u^{2,0}\|_{L^\infty(\Omega)}\leq\|u^1-u^{1,1}\|_{L^\infty(\Omega)}.$$

It is clear that $\|u^1-u^{1,1}\|_{L^\infty(\Omega)}$ is bounded below by both

$$\|u^2-u^{2,1}\|_{L^\infty(\Omega)}$$

and

$$\alpha_2 w_2\|u^2-u^{2,1}\|_{L^\infty(\Omega)}+\alpha_2(1-w_2)\|u^2-u^{2,0}\|_{L^\infty(\Omega)},$$

which leads us to distinguish the following possibilities

$$\|u^2-u^{2,1}\|_{L^\infty(\Omega)}\leq\alpha_2 w_2\|u^2-u^{2,1}\|_{L^\infty(\Omega)}+\alpha_2(1-w_2)\|u^2-u^{2,0}\|_{L^\infty(\Omega)}$$

or

$$\alpha_2 w_2\|u^2-u^{2,1}\|_{L^\infty(\Omega)}+\alpha_2(1-w_2)\|u^2-u^{2,0}\|_{L^\infty(\Omega)}\leq\|u^2-u^{2,1}\|_{L^\infty(\Omega)}.$$

Then,

$$\|u^2-u^{2,1}\|_{L^\infty(\Omega)}\leq\frac{\alpha_2(1-w_2)}{1-\alpha_2 w_2}\|u^2-u^{2,0}\|_{L^\infty(\Omega)} \tag{3.27}$$

or

$$\frac{\alpha_2(1-w_2)}{1-\alpha_2 w_2}\|u^2-u^{2,0}\|_{L^\infty(\Omega)}\leq\|u^2-u^{2,1}\|_{L^\infty(\Omega)}. \tag{3.28}$$

Thus, (3.26)–(3.28) imply that the three following alternatives are required

$$\|u^2-u^{2,1}\|_{L^\infty(\Omega)}\leq\|u^1-u^{1,1}\|_{L^\infty(\Omega)}\leq\frac{\alpha_2(1-w_2)}{1-\alpha_2 w_2}\|u^2-u^{2,0}\|_{L^\infty(\Omega)}$$

or

$$\|u^2 - u^{2,1}\|_{L^\infty(\Omega)} \le \frac{\alpha_2(1-w_2)}{1-\alpha_2 w_2}\|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,1}\|_{L^\infty(\Omega)}$$

or

$$\frac{\alpha_2(1-w_2)}{1-\alpha_2 w_2}\|u^2 - u^{2,0}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,1}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,1}\|_{L^\infty(\Omega)}.$$

It is clear that all alternatives coincide with (3.20). So, we must have

$$\|u^2 - u^{2,1}\|_{L^\infty(\Omega)} = \frac{\alpha_2(1-w_2)}{1-\alpha_2 w_2}\|u^2 - u^{2,0}\|_{L^\infty(\Omega)}.$$

Thus, in both cases (3.22) and (3.23) we obtain (3.24). Hence,

$$\|u^2 - u^{2,1}\|_{L^\infty(\Omega)} \le \rho \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}. \tag{3.29}$$

(3.20) and (3.29) imply

$$\|U^1 - U\|_\infty \le \rho\|U^0 - U\|_\infty.$$

- Let us assume that, for $n \ge 0$

$$\left\|u^i - u^{i,n}\right\|_{L^\infty(\Omega)} \le \rho^n \max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}, i = 1,2. \tag{3.30}$$

- We prove

$$\left\|u^i - u^{i,n+1}\right\|_{L^\infty(\Omega)} \le \rho^{n+1} \max_{1\le i\le 2}\left\|u^i - u^{i,n}\right\|_{L^\infty(\Omega)}, i = 1,2. \tag{3.31}$$

By adopting the same arguments for (1.1), (3.2), (3.3) and (2.9) as that applied for the previous iterates, we get

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \max\left\{\begin{matrix}\left(\frac{1}{\beta^1}\right)\left\|f^1(u^1) - \left(w_1 f^1(u^{1,n+1}) + (1-w_1) f^1(u^{1,n})\right)\right\|_{L^\infty(\Omega)}; \\ \|u^2 - u^{2,n}\|_{L^\infty(\Omega)}\end{matrix}\right\}$$

So,

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \max\left\{\begin{matrix}\alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}; \\ \|u^2 - u^{2,n}\|_{L^\infty(\Omega)}\end{matrix}\right\} \tag{3.32}$$

Also we distinguish two cases:

$$\begin{aligned}&\max\left\{\begin{matrix}\alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}; \\ \|u^2 - u^{2,n}\|_{L^\infty(\Omega)}\end{matrix}\right\} \\ &= \alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}\end{aligned} \tag{3.33}$$

or

$$\max\left\{\begin{matrix}\alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}; \\ \|u^2 - u^{2,n}\|_{L^\infty(\Omega)}\end{matrix}\right\} = \|u^2 - u^{2,n}\|_{L^\infty(\Omega)} \tag{3.34}$$

(3.32) in conjunction with (3.33) implies

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}, \tag{3.35}$$

with

$$\|u^2 - u^{2,n}\|_{L^\infty(\Omega)} \le \alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}. \tag{3.36}$$

By replacing (3.35) in (3.36) we get, according to (3.30); $i = 1$

$$\|u^2 - u^{2,n}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,n}\|_{L^\infty(\Omega)} \le \rho^{n+1}\max_{1\le i\le 2}\left\|u^i - u^{i,0}\right\|_{L^\infty(\Omega)}$$

which matches with (3.30); $i = 2$. (3.32) in conjunction with (3.34) implies

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,n}\|_{L^\infty(\Omega)} \tag{3.37}$$

with

$$\alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,n}\|_{L^\infty(\Omega)}.$$

$\|u^2 - u^{2,n}\|_{L^\infty(\Omega)}$ is bounded below by both $\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)}$
and

$$\alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}$$

So,

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}$$

or

$$\alpha_1 w_1\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^1 - u^{1,n}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)}.$$

Thus,

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}$$

or

$$\frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,n}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)}.$$

By taking into account (3.37), we get

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,n}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}$$

or

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,n}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,n}\|_{L^\infty(\Omega)}$$

or

$$\frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,n}\|_{L^\infty(\Omega)} \le \|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \|u^2 - u^{2,n}\|_{L^\infty(\Omega)}.$$

Three possibilities are true because all coincide with (3.30). So, we necessarily get

$$\|u^1 - u^{1,n+1}\|_{L^\infty(\Omega)} \le \frac{\alpha_1(1-w_1)}{1-\alpha_1 w_1}\|u^1 - u^{1,n}\|_{L^\infty(\Omega)}.$$

Thus, both cases (3.33) and (3.34) imply (3.35). Hence, by using (3.30) we get (3.31) for $i = 1$. The proof for (3.31); $i = 2$ is obtain in similar way by using (3.31); $i = 1$ and (3.35) so, it will be omitted. The desired result (3.6) follows naturally from (3.31).

4. Statement of discrete problem

This section, we will handle the discrete problem by a perfect symmetry in the treatment of that the continuous one. Indeed, we define the discrete system of QVIs: Find a vector $U_h = (u_h^1, u_h^2) \in (V_h)^2$ such that

$$\begin{cases} a^i(u_h^i, v - u_h^i) \ge (f^i(u_h^i), v - u_h^i); v \in V_h \\ v,\ u_h^i \le r_h(Mu_h^i) = r_h(k + u_h^j); i \ne j. u_h^i \ge 0 \ and\ u_h^i = \pi_h g\ on\ \partial\Omega. \end{cases} \tag{4.1}$$

The related discrete fixed-point mapping

$$T_h: (V_h)^2 \to (V_h)^2$$
$$Z_h = (z_h^1, z_h^2) \to T_h Z_h = \zeta_h = (\zeta_h^1, \zeta_h^2),$$

where $\zeta_h^i \in V_h$ is the unique solution to the following discrete variational inequality

$$a^i(\zeta_h^i, v - \zeta_h^i) \ge (f^i(z_h^i), v - \zeta_h^i); v \in V_h \tag{4.2}$$
$$v, \zeta_h^i \le r_h(M\zeta_h^i) = r_h(k + z_h^j);\ \zeta_h^i \ge 0\ with\ i \ne j\ and\ \zeta_h^i = \pi_h g\ on\ \partial\Omega.$$

Remark 1 *We remark that the solution* $U_h = (u_h^1, u_h^2)$ *of the system* (4.1) *is the fixed point of the mapping* T_h*; that is* $T_h U_h = U_h$.

4.1. A discrete iterative scheme

Starting from $U_h^0 = (u_h^{1,0}, u_h^{2,0})$ where $u_h^{i,0} = r_h u^{i,0}; i = 1,2$ is the discrete analog of $u^{i,0}$ then,

$$\|u^{i,0} - u_h^{i,0}\|_{L^\infty(\Omega)} \le Ch^2|logh|^2. \tag{4.3}$$

For all $0 < w_i < 1; i = 1,2$ we define the discrete sequences $(u_h^{1,n+1})$ and $(u_h^{2,n+1})$ such that $u_h^{1,n+1}$ and $u_h^{2,n+1}$ components of the vector U_h^{n+1} solve discrete elliptic variational inequalities

$$a^1(u_h^{1,n+1}, v - u_h^{1,n+1}) \ge (w_1 f^1(u_h^{1,n+1}) + (1-w_1)f^1(u_h^{1,n}), v - u_h^{1,n+1}) \tag{4.4}$$
$$v, u_h^{1,n+1} \le r_h(Mu_h^{1,n+1}) = r_h(k + u_h^{2,n}), \tag{4.5}$$
$$a^2(u_h^{2,n+1}, v - u_h^{2,n+1}) \ge (w_2 f^2(u_h^{2,n+1}) + (1-w_2)f^2(u_h^{2,n}), v - u_h^{2,n+1}) \tag{4.6}$$
$$v, u_h^{2,n+1} \le r_h(Mu_h^{2,n+1}) = r_h(k + u_h^{1,n+1}). \tag{4.7}$$

4.2. Convergence of the discrete iterative scheme

Theorem 2 *The discrete sequences* $\left(u_h^{1,n+1}\right)$ *and* $\left(u_h^{2,n+1}\right)$ *converge geometrically to the discrete solution* $U_h = (u_h^1, u_h^2)$ *of the system* (4.1)*; there exist a positive real* $\rho \in (0,1)$ *defined in* (3.7) *such that for all* $n \geq 0$

$$\|U_h^{n+1} - U_h\|_\infty \leq \rho^{n+1}\left\|U_h^0 - U_h\right\|_\infty. \tag{4.8}$$

Proof. The proof is similar to that of the continuous case.

5. L^∞-error analysis

This section is devoted to the proof of the main result of this paper. For that purpose we need to introduce an auxiliary system.

5.1. Auxiliary system

Let $w_h^{i,0} = u_h^{i,0}; i = 1,2$ be an initialization. For all $0 < w_i < 1; i = 1,2$ we define the discrete sequences $\left(w_h^{1,n+1}\right)$ and $\left(w_h^{2,n+1}\right)$ such that $w_h^{1,n+1}$ and $w_h^{2,n+1}$ solve coercive variational inequalities

$$a^1\left(w_h^{1,n+1}, v - w_h^{1,n+1}\right) \geq \left(w_1 f^1\left(u^{1,n+1}\right) + (1 - w_1) f^1\left(u^{1,n}\right), v - w_h^{1,n+1}\right) \tag{5.1}$$

$$v, w_h^{1,n+1} \leq r_h\left(Mu^{1,n+1}\right) = r_h\left(k + u^{2,n}\right), \tag{5.2}$$

$$a^2\left(w_h^{2,n+1}, v - w_h^{2,n+1}\right) \geq \left(w_2 f^2\left(u^{2,n+1}\right) + (1 - w_2) f^2\left(u^{2,n}\right), v - w_h^{2,n+1}\right) \tag{5.3}$$

$$v, w_h^{2,n+1} \leq r_h\left(Mu^{2,n+1}\right) = r_h\left(k + u^{1,n+1}\right). \tag{5.4}$$

It is clear that $w_h^{i,n+1}; i = 1,2$ components of the vector W_h^{n+1} are finite element approximation of $u^{i,n+1}$ defined in (3.2)–(3.4). Thus, making use of (2.16)*;* we get

$$\left\|w_h^{i,n+1} - u^{i,n+1}\right\|_{L^\infty(\Omega)} \leq Ch^2|log|^2; i = 1,2 \; and \; n \geq 0. \tag{5.5}$$

The algorithmic approach used in the present paper rests on the following crucial lemma, where the error estimate between the nth iterate U^n and its discrete counter parts U_h^{n+1} is established.

Lemma 1 *Let* (U^{n+1}) *and* (U_h^{n+1}) *be the vectors whose components are sequences defined in* (3.2)–(3.5) *and* (4.4)–(4.7) *respectively. Then,*

$$\|U^{n+1} - U_h^{n+1}\|_\infty \leq \left(\gamma\left(\tfrac{1-\rho^{n+1}}{1-\rho}\right) + \rho^{n+1}\right) \max_{n\geq 0}\|U^n - W_h^n\|_\infty . \tag{5.6}$$

Where

$$\gamma = \max_{1\leq i\leq 2}\left\{\tfrac{1}{(1-\alpha_i w_i)}\right\}. \tag{5.7}$$

Proof. The proof of the lemma rests on the discrete Lipschitz continuous dependency with respect to source term and obstacle and will carry out by induction.

- For $n = 0$, we have

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} \leq \left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} + \left\|w_h^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}.$$

(5.1), (5.2), (4.4), (4.5) and (2.17) imply

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} \leq \left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)}$$

$$+\max\left\{\begin{matrix}\left(\dfrac{1}{\beta^1}\right)\left\|f^1(u^{1,1})-\left(w_1 f^1\left(u_h^{1,1}\right)+(1-w_1)f^1\left(u_h^{1,0}\right)\right)\right\|_{L^\infty(\Omega)}; \\ \left\|r_h(k+u^{2,0})-r_h\left(k+u_h^{2,0}\right)\right\|_{L^\infty(\Omega)}\end{matrix}\right\}$$

So,

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} \leq \left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)}$$

$$+\max\left\{\begin{matrix}\left(\frac{k^1}{\beta^1}\right)w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}+\left(\frac{k^1}{\beta^1}\right)(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}; \\ \left\|r_h(k+u^{2,0})-r_h\left(k+u_h^{2,0}\right)\right\|_{L^\infty(\Omega)}\end{matrix}\right\}.$$

Therefore,

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} \leq \left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} \tag{5.8}$$

$$+\max\left\{\begin{matrix}\alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}+\alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}; \\ \left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}\end{matrix}\right\}.$$

We distinguish two cases

$$\max\left\{\begin{matrix}\alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}+\alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}; \\ \left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}\end{matrix}\right\}$$

$$=\alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}+\alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)} \tag{5.9}$$

or

$$\max\left\{\begin{matrix}\alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}+\alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}; \\ \left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}\end{matrix}\right\}=\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)} \tag{5.10}$$

(5.8) in conjunction with (5.9) imply

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} \leq \left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)}+\alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}$$

$$+\alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}$$

with

$$\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)} \leq \alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}+\alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}. \tag{5.11}$$

So,

$$(1-\alpha_1 w_1)\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} \le \left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)},$$

with (5.11). Then,

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} \le \frac{1}{(1-\alpha_1 w_1)}\left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} + \frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}. \quad (5.12)$$

By replacing (5.12) in (5.11) we obtain

$$\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)} \le \frac{\alpha_1 w_1}{(1-\alpha_1 w_1)}\left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} + \frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}.$$

According to (5.5) and (4.3) we get,

$$\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)} \le \frac{\alpha_1}{(1-\alpha_1 w_1)} Ch^2|logh|^2,$$

which coincides with (4.3).
(5.8) and (5.10) imply

$$\left\|u^{1,1}-u_{\mathrm{h}}^{1,1}\right\|_{\mathrm{L}^\infty(\Omega)} \le \left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} + \left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)} \quad (5.13)$$

with

$$\alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)} \le \left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}.$$

Then, multiplying (5.13) by $\alpha_1 w_1$ and adding $\alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}$, we obtain

$$\begin{aligned}&\alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}\\ &\le \alpha_1 w_1\left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} + \alpha_1 w_1\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}\\ &+ \alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}.\end{aligned}$$

We note that

$$\alpha_1 w_1\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}$$

is bounded by both

$$\alpha_1 w_1\left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} + \alpha_1 w_1\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}$$

and

$$\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}.$$

So,

$$\begin{aligned}&\alpha_1 w_1\left\|u^{1,1}-w_h^{1,1}\right\|_{L^\infty(\Omega)} + \alpha_1 w_1\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,0}-u_h^{1,0}\right\|_{L^\infty(\Omega)}\\ &\le \left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}\end{aligned}$$

or

$$\|u^{2,0}-u_h^{2,0}\|_{L^\infty(\Omega)} \le \alpha_1 w_1 \|u^{1,1}-w_h^{1,1}\|_{L^\infty(\Omega)} + \alpha_1 w_1 \|u^{2,0}-u_h^{2,0}\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\|u^{1,0}-u_h^{1,0}\|_{L^\infty(\Omega)}.$$

Therefore, according to (5.5) and (4.3), we get

$$\frac{\alpha_1 w_1}{(1-\alpha_1 w_1)}\|u^{1,1}-w_h^{1,1}\|_{L^\infty(\Omega)} + \frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\|u^{1,0}-u_h^{1,0}\|_{L^\infty(\Omega)} \le \|u^{2,0}-u_h^{2,0}\|_{L^\infty(\Omega)} \le Ch^2|logh|^2$$

or

$$\|u^{2,0}-u_h^{2,0}\|_{L^\infty(\Omega)} \le \frac{\alpha_1 w_1}{(1-\alpha_1 w_1)}\|u^{1,1}-w_h^{1,1}\|_{L^\infty(\Omega)} + \frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\|u^{1,0}-u_h^{1,0}\|_{L^\infty(\Omega)} \le \frac{\alpha_1}{(1-\alpha_1 w_1)}Ch^2|logh|^2.$$

So, the last two alternatives are true at the same time because both coincide with (4.3). We necessarily deduce that

$$\|u^{2,0}-u_h^{2,0}\|_{L^\infty(\Omega)} = \frac{\alpha_1 w_1}{(1-\alpha_1 w_1)}\|u^{1,1}-w_h^{1,1}\|_{L^\infty(\Omega)} + \frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\|u^{1,0}-u_h^{1,0}\|_{L^\infty(\Omega)}. \quad (5.14)$$

By replacing (5.14) in (5.13); we get (5.12). Hence, in both cases (5.9) and (5.10); we can write

$$\|u^{1,1}-u_h^{1,1}\|_{L^\infty(\Omega)} \le \max_{1\le i\le 2}\left\{\frac{1}{(1-\alpha_i w_i)}\right\}\max_{1\le i\le 2}\|u^{i,1}-w_h^{i,1}\|_{L^\infty(\Omega)} + \max_{1\le i\le 2}\left\{\frac{\alpha_i(1-w_i)}{(1-\alpha_i w_i)}\right\}\max_{1\le i\le 2}\|u^{i,0}-u_h^{i,0}\|_{L^\infty(\Omega)}.$$

Thus,

$$\|u^{1,1}-u_h^{1,1}\|_{L^\infty(\Omega)} \le (\gamma+\rho)\max_{n\ge 0}\max_{1\le i\le 2}\|u^{i,n}-w_h^{i,n}\|_{L^\infty(\Omega)}. \quad (5.15)$$

- In a similar way, that is by following the same steps as for $u^{1,1}$ and $u_h^{1,1}$, $u^{2,1}$ and $u_h^{2,1}$ satisfy

$$\|u^{2,1}-u_h^{2,1}\|_{L^\infty(\Omega)} \le \|u^{2,1}-w_h^{2,1}\|_{L^\infty(\Omega)} + \|w_h^{2,1}-u_h^{2,1}\|_{L^\infty(\Omega)}.$$

So,

$$\|u^{2,1}-u_h^{2,1}\|_{L^\infty(\Omega)} \le \|u^{2,1}-w_h^{2,1}\|_{L^\infty(\Omega)} + \max\begin{Bmatrix}\alpha_2 w_2\|u^{2,1}-u_h^{2,1}\|_{L^\infty(\Omega)} + \alpha_2(1-w_2)\|u^{2,0}-u_h^{2,0}\|_{L^\infty(\Omega)}; \\ \|u^{1,1}-u_h^{1,1}\|_{L^\infty(\Omega)}\end{Bmatrix}. \quad (5.16)$$

We distinguish also two cases

$$\max \begin{Bmatrix} \alpha_2 w_2 \left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} + \alpha_2(1-w_2) \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)}; \\ \left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)} \end{Bmatrix} \tag{5.17}$$
$$= \alpha_2 w_2 \left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} + \alpha_2(1-w_2) \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)}$$

or

$$\max \begin{Bmatrix} \alpha_2 w_2 \left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} + \alpha_2(1-w_2) \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)}; \\ \left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)} \end{Bmatrix} = \left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)}. \tag{5.18}$$

(5.16) in conjunction with case (5.17); we get

$$\left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} \le \left\| u^{2,1} - w_h^{2,1} \right\|_{L^\infty(\Omega)} + \alpha_2 w_2 \left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)}$$
$$+ \alpha_2(1-w_2) \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)}$$

with

$$\left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)} \le \alpha_2 w_2 \left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} + \alpha_2(1-w_2) \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)}. \tag{5.19}$$

So,

$$\left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} \le \frac{1}{(1-\alpha_2 w_2)} \left\| u^{2,1} - w_h^{2,1} \right\|_{L^\infty(\Omega)} + \frac{\alpha_2(1-w_2)}{(1-\alpha_2 w_2)} \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)} \tag{5.20}$$

with, according to (5.20)

$$\left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)} \le \frac{1}{(1-\alpha_2 w_2)} \left\| u^{2,1} - w_h^{2,1} \right\|_{L^\infty(\Omega)} + \frac{\alpha_2(1-w_2)}{(1-\alpha_2 w_2)} \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)}.$$

Then,

$$\left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)}$$
$$\le \max_{1\le i\le 2} \left\{ \frac{1}{(1-\alpha_i w_i)} \right\} \left\| u^{i,1} - w_h^{i,1} \right\|_{L^\infty(\Omega)} + \max_{1\le i\le 2} \left\{ \frac{\alpha_i(1-w_i)}{(1-\alpha_i w_i)} \right\} \left\| u^{i,0} - u_h^{i,0} \right\|_{L^\infty(\Omega)}.$$

Therefore,

$$\left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)} \le (\gamma + \rho) \max_{n\ge 0} \max_{1\le i\le 2} \left\| u^{i,n} - w_h^{i,n} \right\|_{L^\infty(\Omega)},$$

which coincides with (5.15). The conjunction of (5.16) with case (5.18), implies

$$\left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} \le \left\| u^{2,1} - w_h^{2,1} \right\|_{L^\infty(\Omega)} + \left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)} \tag{5.21}$$

with

$$\alpha_2 w_2 \left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} + \alpha_2(1-w_2) \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)} \le \left\| u^{1,1} - u_h^{1,1} \right\|_{L^\infty(\Omega)}.$$

Then, by multiplying (5.21) by $\alpha_2 w_2$ and adding $\alpha_2(1-w_2)\left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)}$, we obtain that the term $\alpha_2 w_2 \left\| u^{2,1} - u_h^{2,1} \right\|_{L^\infty(\Omega)} + \alpha_2(1-w_2) \left\| u^{2,0} - u_h^{2,0} \right\|_{L^\infty(\Omega)}$ is bounded by both

$$\alpha_2 w_2\left\|u^{2,1}-w_h^{2,1}\right\|_{L^\infty(\Omega)}+\alpha_2 w_2\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}+\alpha_2(1-w_2)\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}$$

and

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}.$$

So, we distinguish again, the two following alternatives

$$\frac{\alpha_2 w_2}{(1-\alpha_2 w_2)}\left\|u^{2,1}-w_h^{2,1}\right\|_{L^\infty(\Omega)}+\frac{\alpha_2(1-w_2)}{(1-\alpha_2 w_2)}\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}\leq\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}$$
$$\leq(\gamma+\rho)\max_{n\geq 0}\max_{1\leq i\leq 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}$$

or

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}\leq\frac{\alpha_2 w_2}{(1-\alpha_2 w_2)}\left\|u^{2,1}-w_h^{2,1}\right\|_{L^\infty(\Omega)}+\frac{\alpha_2(1-w_2)}{(1-\alpha_2 w_2)}\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}$$
$$\leq(\gamma+\rho)\max_{n\geq 0}\max_{1\leq i\leq 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}.$$

We remark that both alternatives coincide with (5.15), which implies that case (5.18) is possible if and only if

$$\left\|u^{1,1}-u_h^{1,1}\right\|_{L^\infty(\Omega)}=\frac{\alpha_2 w_2}{(1-\alpha_2 w_2)}\left\|u^{2,1}-w_h^{2,1}\right\|_{L^\infty(\Omega)}+\frac{\alpha_2(1-w_2)}{(1-\alpha_2 w_2)}\left\|u^{2,0}-u_h^{2,0}\right\|_{L^\infty(\Omega)}. \quad (5.22)$$

By substituting (5.22) in (5.21), we get (5.20). Hence, in both cases (5.17) and (5.18), we get

$$\left\|u^{2,1}-u_h^{2,1}\right\|_{L^\infty(\Omega)}$$
$$\leq\max_{1\leq i\leq 2}\left\{\frac{1}{(1-\alpha_i w_i)}\right\}\max_{1\leq i\leq 2}\left\|u^{i,1}-w_h^{i,1}\right\|_{L^\infty(\Omega)}$$
$$+\max_{1\leq i\leq 2}\left\{\frac{\alpha_i(1-w_i)}{(1-\alpha_i w_i)}\right\}\max_{1\leq i\leq 2}\left\|u^{i,0}-u_h^{i,0}\right\|_{L^\infty(\Omega)}.$$

Thus,

$$\left\|u^{2,1}-u_h^{2,1}\right\|_{L^\infty(\Omega)}\leq(\gamma+\rho)\max_{n\geq 0}\max_{1\leq i\leq 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}. \quad (5.23)$$

(5.15) and (5.23) imply

$$\|U^1-U_h^1\|_\infty\leq(\gamma+\rho)\max_{n\geq 0}\|U^n-W_h^n\|_\infty.$$

- Let us assume that for $n\geq 0$ and $i=1,2$

$$\left\|u^{i,n}-u_h^{i,n}\right\|_{L^\infty(\Omega)}\leq(\gamma(1+\rho+\cdots+\rho^{n-1})+\rho^n)\max_{n\geq 0}\max_{1\leq i\leq 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}. \quad (5.24)$$

- And prove for $i=1,2$

$$\left\|u^{i,n+1}-u_h^{i,n+1}\right\|_{L^\infty(\Omega)}\leq(\gamma(1+\rho+\cdots+\rho^n)+\rho^{n+1})\max_{n\geq 0}\max_{1\leq i\leq 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}. \quad (5.25)$$

We operate in the same way as in iterate $n = 0$. Let us begin with case $i = 1$ in (5.25)

$$\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} \le \left\|u^{1,n+1} - w_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \left\|w_h^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)}.$$

So, by applying (2.17), we get

$$\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} \le \left\|u^{1,n+1} - w_h^{1,n+1}\right\|_{L^\infty(\Omega)}$$

$$+\max\left\{\begin{matrix}\alpha_1 w_1\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,n} - u_h^{1,n}\right\|_{L^\infty(\Omega)}; \\ \left\|u^{2,n} - u_h^{2,n}\right\|_{L^\infty(\Omega)}\end{matrix}\right\} \tag{5.26}$$

We distinguish again two cases

$$\max\left\{\begin{matrix}\alpha_1 w_1\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,n} - u_h^{1,n}\right\|_{L^\infty(\Omega)}; \\ \left\|u^{2,n} - u_h^{2,n}\right\|_{L^\infty(\Omega)}\end{matrix}\right\}$$

$$= \alpha_1 w_1\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,n} - u_h^{1,n}\right\|_{L^\infty(\Omega)} \tag{5.27}$$

or

$$\max\left\{\begin{matrix}\alpha_1 w_1\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,n} - u_h^{1,n}\right\|_{L^\infty(\Omega)}; \\ \left\|u^{2,n} - u_h^{2,n}\right\|_{L^\infty(\Omega)}\end{matrix}\right\}$$

$$= \left\|u^{2,n} - u_h^{2,n}\right\|_{L^\infty(\Omega)}. \tag{5.28}$$

(5.26) in conjunction with case (5.27) implies

$$\begin{aligned}&\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} \\ &\le \left\|u^{1,n+1} - w_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \alpha_1 w_1\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} \\ &+ \alpha_1(1-w_1)\left\|u^{1,n} - u_h^{1,n}\right\|_{L^\infty(\Omega)}\end{aligned}$$

and

$$\left\|u^{2,n} - u_h^{2,n}\right\|_{L^\infty(\Omega)} \le \alpha_1 w_1\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \alpha_1(1-w_1)\left\|u^{1,n} - u_h^{1,n}\right\|_{L^\infty(\Omega)}.$$

Then,

$$\left\|u^{1,n+1} - u_h^{1,n+1}\right\|_{L^\infty(\Omega)} \le \frac{1}{(1-\alpha_1 w_1)}\left\|u^{1,n+1} - w_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\left\|u^{1,n} - u_h^{1,n}\right\|_{L^\infty(\Omega)} \tag{5.29}$$

with, according to (5.29)

$$\left\|u^{2,n} - u_h^{2,n}\right\|_{L^\infty(\Omega)} \le \frac{\alpha_1 w_1}{(1-\alpha_1 w_1)}\left\|u^{1,n+1} - w_h^{1,n+1}\right\|_{L^\infty(\Omega)} + \frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\left\|u^{1,n} - u_h^{1,n}\right\|_{L^\infty(\Omega)}.$$

(5.24) implies

$$\left\|u^{1,n+1}-u_h^{1,n+1}\right\|_{L^\infty(\Omega)}$$
$$\le \frac{1}{(1-\alpha_1 w_1)}\left\|u^{1,n+1}-w_h^{1,n+1}\right\|_{L^\infty(\Omega)}$$
$$+\frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\left(\left(\gamma(1+\rho+\cdots+\rho^{n-1})+\rho^n\right)\max_{n\ge 0}\max_{1\le i\le 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}\right)$$

with

$$\left\|u^{2,n}-u_h^{2,n}\right\|_{L^\infty(\Omega)}\le \frac{\alpha_1 w_1}{(1-\alpha_1 w_1)}\left\|u^{1,n+1}-w_h^{1,n+1}\right\|_{L^\infty(\Omega)}$$
$$+\frac{\alpha_1(1-w_1)}{(1-\alpha_1 w_1)}\left(\left(\gamma(1+\rho+\cdots+\rho^{n-1})+\rho^n\right)\max_{n\ge 0}\max_{1\le i\le 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}\right).$$

Thus,

$$\left\|u^{1,n+1}-u_h^{1,n+1}\right\|_{L^\infty(\Omega)}$$
$$\le \gamma\left\|u^{1,n+1}-w_h^{1,n+1}\right\|_{L^\infty(\Omega)}$$
$$+\rho\left(\left(\gamma(1+\rho+\cdots+\rho^{n-1})+\rho^n\right)\max_{n\ge 0}\max_{1\le i\le 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}\right)$$

and as $\alpha_1 w_1 < 1$

$$\left\|u^{2,n}-u_h^{2,n}\right\|_{L^\infty(\Omega)}$$
$$\le \gamma\left\|u^{1,n+1}-w_h^{1,n+1}\right\|_{L^\infty(\Omega)}+\rho\left(\left(\gamma(1+\rho+\cdots+\rho^{n-1})+\rho^n\right)\max_{n\ge 0}\max_{1\le i\le 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}\right).$$

Hence,

$$\left\|u^{1,n+1}-u_h^{1,n+1}\right\|_{L^\infty(\Omega)}\le \left(\gamma(1+\rho+\cdots+\rho^n)+\rho^{n+1}\right)\max_{n\ge 0}\max_{1\le i\le 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}$$

and

$$\left\|u^{2,n}-u_h^{2,n}\right\|_{L^\infty(\Omega)}\le \left(\gamma(1+\rho+\cdots+\rho^n)+\rho^{n+1}\right)\max_{n\ge 0}\max_{1\le i\le 2}\left\|u^{i,n}-w_h^{i,n}\right\|_{L^\infty(\Omega)}.$$

which corresponds with (5.24) for $i = 2$*:* Inequality (5.26) with (5.28) imply

$$\left\|u^{1,n+1}-u_h^{1,n+1}\right\|_{L^\infty(\Omega)}\le \left\|u^{1,n+1}-w_h^{1,n+1}\right\|_{L^\infty(\Omega)}+\left\|u^{2,n}-u_h^{2,n}\right\|_{L^\infty(\Omega)} \tag{5.30}$$

and

$$\alpha_1 w_1\left\|u^{1,n+1}-u_h^{1,n+1}\right\|_{L^\infty(\Omega)}+\alpha_1(1-w_1)\left\|u^{1,n}-u_h^{1,n}\right\|_{L^\infty(\Omega)}\le \left\|u^{2,n}-u_h^{2,n}\right\|_{L^\infty(\Omega)}.$$

By multiplying (5.30) by $\alpha_1 w_1$ and adding the term $\alpha_1(1-w_1)\left\|u^{1,n}-u_h^{1,n}\right\|_{L^\infty(\Omega)}$, we get that the term

$$\alpha_1 w_1\left\|u^{1,n+1}-u_h^{1,n+1}\right\|_{L^\infty(\Omega)}+\alpha_1(1-w_1)\left\|u^{1,n}-u_h^{1,n}\right\|_{L^\infty(\Omega)}$$

is bounded by the following two terms

$$\alpha_1 w_1 \left\| u^{1,n+1} - w_h^{1,n+1} \right\|_{L^\infty(\Omega)} + \alpha_1 w_1 \left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)} + \alpha_1 (1 - w_1) \left\| u^{1,n} - u_h^{1,n} \right\|_{L^\infty(\Omega)}$$

and

$$\left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)}.$$

So, we need to distinguish the followings possibilities

$$\alpha_1 w_1 \left\| u^{1,n+1} - w_h^{1,n+1} \right\|_{L^\infty(\Omega)} + \alpha_1 w_1 \left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)} + \alpha_1 (1 - w_1) \left\| u^{1,n} - u_h^{1,n} \right\|_{L^\infty(\Omega)} \leq \left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)}$$

or

$$\left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)} \leq \alpha_1 w_1 \left\| u^{1,n+1} - w_h^{1,n+1} \right\|_{L^\infty(\Omega)} + \alpha_1 w_1 \left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)} + \alpha_1 (1 - w_1) \left\| u^{1,n} - u_h^{1,n} \right\|_{L^\infty(\Omega)},$$

which implies

$$\frac{\alpha_1 w_1}{(1 - \alpha_1 w_1)} \left\| u^{1,n+1} - w_h^{1,n+1} \right\|_{L^\infty(\Omega)} + \frac{\alpha_1 (1 - w_1)}{(1 - \alpha_1 w_1)} \left\| u^{1,n} - u_h^{1,n} \right\|_{L^\infty(\Omega)} \leq \left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)}$$

or

$$\left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)} \leq \frac{\alpha_1 w_1}{(1 - \alpha_1 w_1)} \left\| u^{1,n+1} - w_h^{1,n+1} \right\|_{L^\infty(\Omega)} + \frac{\alpha_1 (1 - w_1)}{(1 - \alpha_1 w_1)} \left\| u^{1,n} - u_h^{1,n} \right\|_{L^\infty(\Omega)}.$$

By using (5.24), we can write

$$\frac{\alpha_1 w_1}{(1 - \alpha_1 w_1)} \left\| u^{1,n+1} - w_h^{1,n+1} \right\|_{L^\infty(\Omega)} + \frac{\alpha_1 (1 - w_1)}{(1 - \alpha_1 w_1)} \left\| u^{1,n} - u_h^{1,n} \right\|_{L^\infty(\Omega)} \leq \left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)} \leq \left(\gamma (1 + \rho + \cdots + \rho^{n-1}) + \rho^n \right) \underset{n \geq 0}{max} \underset{1 \leq i \leq 2}{max} \left\| u^{i,n} - w_h^{i,n} \right\|_{L^\infty(\Omega)},$$

or

$$\left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)} \leq \frac{\alpha_1 w_1}{(1 - \alpha_1 w_1)} \left\| u^{1,n+1} - w_h^{1,n+1} \right\|_{L^\infty(\Omega)} + \frac{\alpha_1 (1 - w_1)}{(1 - \alpha_1 w_1)} \left\| u^{1,n} - u_h^{1,n} \right\|_{L^\infty(\Omega)} \leq \left(\gamma (1 + \rho + \cdots + \rho^{n-1}) + \rho^n \right) \underset{n \geq 0}{max} \underset{1 \leq i \leq 2}{max} \left\| u^{i,n} - w_h^{i,n} \right\|_{L^\infty(\Omega)}.$$

Only the last alternative is true because it matches with (5.24) for $i = 2$. So, in (5.28) we get

$$\left\| u^{2,n} - u_h^{2,n} \right\|_{L^\infty(\Omega)} \leq \frac{\alpha_1 w_1}{(1 - \alpha_1 w_1)} \left\| u^{1,n+1} - w_h^{1,n+1} \right\|_{L^\infty(\Omega)} + \frac{\alpha_1 (1 - w_1)}{(1 - \alpha_1 w_1)} \left\| u^{1,n} - u_h^{1,n} \right\|_{L^\infty(\Omega)}. \quad (5.31)$$

By replacing (5.31) in (5.30)*;* we get (5.29). Hence, in both cases (5.27) and (5.28), we obtain

$$\|u^{1,n+1} - u_h^{1,n+1}\|_{L^\infty(\Omega)}$$
$$\le \max_{1\le i\le 2}\left\{\frac{1}{(1-\alpha_i w_i)}\right\} \max_{1\le i\le 2}\|u^{i,n+1} - w_h^{i,n+1}\|_{L^\infty(\Omega)}$$
$$+ \max_{1\le i\le 2}\left\{\frac{\alpha_i(1-w_i)}{(1-\alpha_i w_i)}\right\} \max_{1\le i\le 2}\|u^{i,n} - u_h^{i,n}\|_{L^\infty(\Omega)}.$$

So,

$$\|u^{1,n+1} - u_h^{1,n+1}\|_{L^\infty(\Omega)}$$
$$\le \gamma \max_{n\ge 0} \max_{1\le i\le 2}\|u^{i,n} - w_h^{i,n}\|_{L^\infty(\Omega)}$$
$$+ \rho(\gamma(1+\rho+\cdots+\rho^{n-1}) + \rho^n) \max_{n\ge 0}\max_{1\le i\le 2}\|u^{i,n} - w_h^{i,n}\|_{L^\infty(\Omega)}.$$

Therefore,

$$\|u^{1,n+1} - u_h^{1,n+1}\|_{L^\infty(\Omega)} \le (\gamma(1+\rho+\cdots+\rho^{n-1}+\rho^n) + \rho^{n+1}) \max_{n\ge 0}\max_{1\le i\le 2}\|u^{i,n} - w_h^{i,n}\|_{L^\infty(\Omega)}. \quad (5.32)$$

By using the last inequality (5.32) and by adopting the same reasoning we prove (5.25); $i = 2$, therefore, we get (5.6).

5.2. The main result

Theorem 2 *Let U and U_h be the solution of systems* (1.1) *and* (4.8)*, respectively. Then, there exists a constant C independent of h such that*

$$\|U - U_h\|_\infty \le \frac{\gamma}{1-\rho} h^2|logh|^2. \quad (5.33)$$

Proof. Making use of (3.6), (5.6) and (4.8), we have

$$\|U - U_h\|_\infty \le \|U - U^{n+1}\|_\infty + \|U^{n+1} - U_h^{n+1}\|_\infty + \|U_h^{n+1} - U_h\|_\infty$$
$$\le \rho^{n+1}\|U - U^0\|_\infty + \left(\gamma\left(\frac{1-\rho^{n+1}}{1-\rho}\right) + \rho^{n+1}\right)\max_{n\ge 0} \|U^n - W_h^n\|_\infty + \rho^{n+1}\|U_h - U_h^0\|_\infty.$$

As $n \to +\infty$ and by using (5.5) we get (5.33).

6. Conclusions

In this work an optimal convergence order is derived for a class of system of two elliptic quasi-variational inequalities where terms sources and obstacles depend upon the solution, where the continuous and discrete Lipschitz dependence with respect to the terms sources, boundary condition and obstacles' played a leading role in obtaining the main result of this paper. As (1.1) plays a key role in solving Hamilton-Jacobi-Bellman equation the results obtained in this paper can give an optimal error estimate for HJB equation also even for $J \ge 2$. The approach used and the results obtained in this paper (optimal convergence order) remain valid when we deal with systems of $J \ge 2$ quasi-variational inequalities with terms sources depends on solution and the obstacles ^{i} independent of the solution, that is systems of the form; Find a vector $U = (u^1, \dots, u^J) \in (H_0^1(\Omega))^J$ satisfying

$$\begin{cases} a^i(u^i, v - u^i) \geq (f^i(u^i), v - u^i);\ v \in H_0^1(\Omega) \\ \quad v, u^i \leq \psi^i;\ u^i \geq 0 \ and\ i = 1, \dots, J. \end{cases}$$

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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