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Research article

Determination of three parameters in a time-space fractional diffusion equation

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Abstract: In this paper, we consider a nonlinear inverse problem of recovering two fractional orders and a diffusion coefficient in a one-dimensional time-space fractional diffusion equation. The uniqueness of fractional orders and the diffusion coefficient, characterizing slow diffusion, can be obtained from the accessible boundary data. Two computational methods, Tikhonov method and Levenberg-Marquardt method, are proposed to solving this problem. Finally, an example is presented to illustrate the efficiency of the two numerical algorithm.

Keywords: ill-posedness; regularization; uniqueness; fractional diffusion equation
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1. Introduction

The fractional diffusion equation (FDE), which is obtained by replacing the first-order time derivative and/or second-order space derivative in the standard diffusion equation by a generalized derivative of fractional order respectively, were successfully used for modelling relevant physical processes, see [1, 5, 14, 15, 21]. Recently the research on inverse problems connected with fractional derivatives becomes more and more popular. Since Cheng in [2] studied an inverse problem on fractional diffusion equation, many topics are well discussed [11, 12, 23, 26, 27]. For the problem of fractional numerical differentiation, in [17, 18], the authors give different regularization methods. In [13, 26, 30], some inverse source problems for fractional diffusion equations are considered. In [25, 28], Liu et al consider the backward time-fractional diffusion problem. In [4, 8, 10, 16], many results on inverse coefficient problems are established. For more reference on inverse problems for fractional diffusion equations, please consult the survey paper [9]. However, in some situation of anomalous diffusion, the diffusion indexes and the diffusion coefficient are unknown. This leads to determination of coefficients which is a classical inverse problem. It should be mentioned that most of the existing literature investigate the determination of only one unknown parameter or functions.

However, in many practical situations, one wishes to simultaneously reconstruct more than one physical parameters. To the authors' knowledge, there are few works on this aspect. For examples, in [4], the fractional order α in ${}_tD_*^\alpha u(x,t) = \triangle u(x,t)$ is determined by an analytic method. In [24], the fractional orders α, β in ${}_tD_*^\alpha u(x,t) = -(-\triangle)^{\frac{\beta}{2}}u(x,t)$ are reconstructed by the classical Levenberg-Marquardt method based on disrete least squares functional. The simultaneous inversion for the fractional order α and the space-dependent diffusion coefficient has been considered in [2, 10]. In this paper, we reconstruct three important parameters in the time-space fractional diffusion equation from only one boundary measurement.

Let us consider the time and space-symmetric fractional diffusion equation in one-dimensional space

$$
{}_tD_*^\alpha u(x,t) = -\kappa(-\triangle)^{\frac{\beta}{2}}u(x,t),\ 0 \le t \le T,\ 0 \le x \le L, \tag{1.1}
$$

subject to homogeneous Neumann boundary conditions

$$
\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \tag{1.2}
$$

and the initial condition

$$
u(x,0) = f(x), \tag{1.3}
$$

where u is a solute concentration, $\kappa > 0$ represents the diffusion coefficient. ${}_tD_*^\alpha$ is the Caputo time fractional derivative of order α ($0 < \alpha < 1$) with the starting point at $t = 0$ defined as follows [19]:

$$
{}_tD_*^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\eta)}{\partial \eta} \frac{d\eta}{(t-\eta)^\alpha}.
$$

The symmetric-space fractional derivative $(-\triangle)^{\frac{\beta}{2}}$ of order β ($1 < \beta \le 2$) is defined by [3, 6, 7]. For readability, we reproduce the following definition for $(-\triangle)^{\frac{\beta}{2}}$, $1 < \beta \le 2$:

Definition 1. [29] *Suppose the Laplace operator $-\triangle$ has a complete set of orthonormal eigenfunctions φ_n corresponding to eigenvalues λ_n^2 on a bounded domain D, i.e., $(-\triangle)\varphi_n = \lambda_n^2 \varphi_n$ on a bounded domain D, $\mathbf{B}(\varphi) = 0$ on ∂D is one of the standard three homogeneous boundary conditions. Let*

$$
\mathbb{G}_\gamma = \left\{ f = \sum_{n=1}^\infty c_n\varphi_n,\ c_n = (f,\varphi_n),\ \sum_{n=1}^\infty |c_n|^2|\varphi_n|^\gamma < \infty,\ \gamma = \max\{\beta, 0\} \right\},
$$

then for any $f \in \mathbb{G}_\gamma$: $(-\triangle)^{\frac{\beta}{2}}$ is defined by

$$
(-\triangle)^{\frac{\beta}{2}}f = \sum_{n=1}^\infty c_n(\lambda_n^2)^{\frac{\beta}{2}}\varphi_n. \tag{1.4}
$$

In the case of $\alpha = 1$, $\beta = 2$, Eq (1.1) reduces to the classical diffusion equation. For $0 < \alpha < 1$, $\beta = 2$, Eq (1.1) models subdiffusion due to particles having long-tailed resting times. For $\alpha = 1$, $1 < \beta < 2$, Eq (1.1) corresponds to the Lévy process. Hence the solution of (1.1) is important for describing the competition between these two anomalous diffusion processes.

If $\alpha, \beta, \kappa, f(x)$ are given, the solution for the direct problem of Eqs (1.1)–(1.3) can be obtained analytically by the method of separation of variables: Setting $u(x,t) = X(x)T(t)$ and substituting into (1.1) yields

$$
{}_tD_*^\alpha X(x)T(t) + \kappa(-\triangle)^{\frac{\beta}{2}}X(x)T(t) = 0.
$$

Letting ω is the separation constant, we obtain two fractional ordinary linear differential equations for $X(x)$ and $T(t)$ as

$$(-\triangle)^{\frac{\beta}{2}} X(x) - \omega X(x) = 0; \tag{1.5}$$

$$_t D_*^\alpha T(t) + \kappa \omega T(t) = 0, \tag{1.6}$$

respectively. Following the definition of fractional Laplacian $(-\triangle)^{\frac{\beta}{2}}$ defined on a bounded domain, Eq (1.5) can be expressed as

$$\sum_{n=1}^{\infty} c_n (\lambda_n^2)^{\frac{\beta}{2}} x_n + \omega_n \sum_{n=1}^{\infty} c_n x_n = 0. \tag{1.7}$$

Hence under homogeneous Neumann conditions, the eigenvalues are $\omega_n = \lambda_n^\beta = (\frac{n\pi}{L})^\beta (1 < \beta \le 2)$ for $n = 0, 1, 2, \cdots$ and the corresponding eigenfunctions are $x_n = \cos(\frac{n\pi x}{L})$. Finally, the analytical solution of Eqs (1.1)–(1.3) is

$$
\begin{aligned}
u(x, t) &= \sum_{n=0}^{\infty} T_n(t) \cos(\frac{n\pi x}{L}) \\
&= \frac{1}{2} f_0 + \sum_{n=1}^{\infty} E_{\alpha,1}(-\kappa(\frac{n\pi}{L})^\beta t^\alpha) f_n \cos(\frac{n\pi x}{L}),
\end{aligned} \tag{1.8}
$$

where

$$f_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi x}{L}) dx, \quad n = 0, 1, 2, \cdots.$$

Here we have used the result $E_{\alpha,1}(0) = 1$, where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ z \in \mathbb{C}. \tag{1.9}$$

By a similar method to [22], we can prove that (1.8) certainly gives the weak solution to Eqs (1.1)–(1.3).

Consider the following inverse problem:

Given $g(t) := u(0, t)$ (or $\tilde{g}(t) := u(L, t)$) with unknown α, β, κ, we want to recover the orders α, β and the coefficient κ from the data $g(t)$ (or $\tilde{g}(t)$).

Usually $g(t)$ is measured and only available data on $g(t)$ is its perturbation $g^\delta(t)$, we assume that there exists a known noise level δ such that

$$\|g(\cdot) - g^\delta(\cdot)\| \le \delta,$$

where the norm $\| \cdot \|$ denotes L^2-norm.

In this paper, our main work is to give the uniqueness result on determination of α, β, κ from the data $g(t)$ and two numerical methods for solving the inverse problems. Although in the paper [24], the authors give a uniqueness result on a similar problem, the result holds only for $0 < \alpha < 1/2$ and sufficiently large T . Our result do not require this restriction and holds for $0 < \alpha < 1$ and a finite T. This is done by adding some more smoothness assumption on the initial data $f(x)$.

Throughout this paper, sometimes we denote the solution of the problem as $u(x, t) = u(\alpha, \beta, \kappa, x, t)$ to show its dependence on α, β, κ.

2. Inverse problem and its uniqueness

Now from (1.8), we have the relationship:

$$g(t) := u(0, t) = \frac{1}{2}f_0 + \sum_{n=1}^{\infty} E_{\alpha,1}(-\kappa(\frac{n\pi}{L})^\beta t^\alpha) f_n. \tag{2.1}$$

The uniqueness of the inverse problem is stated as follows:

Theorem 1. *Suppose that $u_1(\alpha_1, \beta_1, \kappa_1; x, t)$ and $u_2(\alpha_2, \beta_2, \kappa_2; x, t)$ represent the solutions of the inverse problem with $\alpha = \alpha_1$, $\beta = \beta_1$, $\kappa = \kappa_1$ and $\alpha = \alpha_2$, $\beta = \beta_2$, $\kappa = \kappa_2$ respectively. We assume that the initial data satisfies*

$$f \in H^4(0, L), \ f''(0) < 0, \ f'(0) = f'(L) = 0 \ (Comapatible \ condition), \tag{2.2}$$

$$f_n > 0, \quad n = 0, 1, 2, \cdots. \tag{2.3}$$

If $u_1(\alpha_1, \beta_1, \kappa_1; 0, t) = u_2(\alpha_2, \beta_2, \kappa_2; 0, t)$ $(0 < t < T)$, then

$$\alpha_1 = \alpha_2, \ \beta_1 = \beta_2, \ \kappa_1 = \kappa_2.$$

Proof. From Eq (2.1), we have

$$u_1(\alpha_1, \beta_1, \kappa_1; 0, t) = \frac{1}{2}f_0 + \sum_{n=1}^{\infty} E_{\alpha_1,1}(-\kappa(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}) f_n,$$

$$u_2(\alpha_2, \beta_2, \kappa_2; 0, t) = \frac{1}{2}f_0 + \sum_{n=1}^{\infty} E_{\alpha_2,1}(-\kappa(\frac{n\pi}{L})^{\beta_2} t^{\alpha_2}) f_n.$$

By assumption $u_1(\alpha_1, \beta_1, \kappa_1; 0, t) = u_2(\alpha_2, \beta_2, \kappa_2; 0, t)$, it yields that

$$\sum_{n=1}^{\infty} E_{\alpha_1,1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}) f_n = \sum_{n=1}^{\infty} E_{\alpha_2,1}(-\kappa_2(\frac{n\pi}{L})^{\beta_2} t^{\alpha_2}) f_n, \tag{2.4}$$

for $0 < t < T$.

As an initial step, we will prove that $\alpha_1 = \alpha_2$.

By the definition of Mittag-Leffler function, we have

$$E_{\alpha_1,1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}) = 1 + \frac{1}{\Gamma(\alpha_1 + 1)}\left(-\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}\right) + \sum_{k=2}^{\infty} \frac{(-\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1})^k}{\Gamma(\alpha_1 k + 1)}$$

$$= 1 - \frac{1}{\Gamma(\alpha_1 + 1)}\left(\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}\right) + t^{2\alpha_1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 \sum_{k=2}^{\infty} \frac{(-\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1})^{(k-2)}}{\Gamma(\alpha_1(k-2) + 2\alpha_1 + 1)}$$

$$= 1 - \frac{1}{\Gamma(\alpha_1 + 1)}\left(\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}\right) + t^{2\alpha_1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 E_{\alpha_1, 2\alpha_1+1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}). \tag{2.5}$$

And by the same argument, we obtain

$$E_{\alpha_2,1}(-\kappa_2(\frac{n\pi}{L})^{\beta_2} t^{\alpha_2})$$

$$=1 - \frac{1}{\Gamma(\alpha_2 + 1)}\left(\kappa_2(\frac{n\pi}{L})^{\beta_2}t^{\alpha_2}\right) + t^{2\alpha_2}(-\kappa_2(\frac{n\pi}{L})^{\beta_2})^2 E_{\alpha_2,2\alpha_2+1}(-\kappa_2(\frac{n\pi}{L})^{\beta_2}t^{\alpha_2}). \tag{2.6}$$

From Eq (2.4), the following result

$$\sum_{n=1}^{\infty} f_n - \frac{t^{\alpha_1}}{\Gamma(\alpha_1 + 1)}\sum_{n=1}^{\infty}\kappa_1(\frac{n\pi}{L})^{\beta_1}f_n + t^{2\alpha_1}\sum_{n=1}^{\infty}(\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 E_{\alpha_1,2\alpha_1+1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1}t^{\alpha_1})f_n$$

$$= \sum_{n=1}^{\infty} f_n - \frac{t^{\alpha_2}}{\Gamma(\alpha_2 + 1)}\sum_{n=1}^{\infty}\kappa_2(\frac{n\pi}{L})^{\beta_2}f_n + t^{2\alpha_2}\sum_{n=1}^{\infty}(\kappa_2(\frac{n\pi}{L})^{\beta_2})^2 E_{\alpha_2,2\alpha_2+1}(-\kappa_2(\frac{n\pi}{L})^{\beta_2}t^{\alpha_2})f_n, \tag{2.7}$$

holds true. Now, we need to estimate the terms:

$$t^{2\alpha_1}\sum_{n=1}^{\infty}(\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 E_{\alpha_1,2\alpha_1+1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1}t^{\alpha_1})f_n$$

and

$$t^{2\alpha_2}\sum_{n=1}^{\infty}(\kappa_2(\frac{n\pi}{L})^{\beta_2})^2 E_{\alpha_2,2\alpha_2+1}(-\kappa_2(\frac{n\pi}{L})^{\beta_2}t^{\alpha_2})f_n.$$

According to the inequality

$$|E_{\alpha,\beta}(-\eta)| \le \frac{C_0}{1 + |\eta|}, \quad \eta \in \mathbb{C}, |\arg \eta| < \frac{\pi}{2} + \delta_0, 0 < \alpha < 2, \beta > 0, \tag{2.8}$$

where δ_0, C_0 are some positive constants, we get

$$|t^{2\alpha_1}\sum_{n=1}^{\infty}(\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 E_{\alpha_1,2\alpha_1+1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1}t^{\alpha_1})f_n|$$

$$\le t^{2\alpha_1}\sum_{n=1}^{\infty}(\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 |E_{\alpha_1,2\alpha_1+1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1}t^{\alpha_1})||f_n|$$

$$\le C_0 t^{2\alpha_1}\sum_{n=1}^{\infty}(\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 \frac{1}{1 + \kappa_1(\frac{n\pi}{L})^{\beta_1}t^{\alpha_1}}|f_n|. \tag{2.9}$$

Furthermore, we recall that $f(x) \in H^4(0, L)$, then

$$\|f^{(4)}(x)\|_{L^2(0,L)} = \|\sum_{n=0}^{\infty} f_n(\frac{n\pi}{L})^4 \cos(\frac{n\pi x}{L})\|_{L^2(0,L)}$$

$$= C_1\left(\sum_{n=0}^{\infty}|f_n|^2(\frac{n\pi}{L})^8\right)^{\frac{1}{2}} < \infty, \tag{2.10}$$

where C_1 is a constant independent on n.

For the term on the right-side hand of Eq (2.9),

$$t^{2\alpha_1}\sum_{n=1}^{\infty}(\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 \frac{1}{1 + \kappa_1(\frac{n\pi}{L})^{\beta_1}t^{\alpha_1}}|f_n|$$

$$= t^{\alpha_1+\varepsilon} \sum_{n=1}^{\infty} \frac{(\kappa_1(\frac{n\pi}{L})^{\beta_1})^2}{\kappa_1(\frac{n\pi}{L})^{\beta_1-\epsilon_0}} \frac{\kappa_1(\frac{n\pi}{L})^{\beta_1-\epsilon_0} t^{\alpha_1-\varepsilon}}{1 + \kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}} |f_n|, \tag{2.11}$$

holds, when $0 < \epsilon_0 < 3/2$ and $0 < \varepsilon < \alpha_1$. Since

$$\sup_{n\in\mathbb{N}, t\geq 0} \frac{\kappa_1(\frac{n\pi}{L})^{\beta_1-\epsilon_0} t^{\alpha_1-\varepsilon}}{1 + \kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}} := C_2 < \infty,$$

then

$$t^{2\alpha_1} \sum_{n=1}^{\infty} (\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 \frac{1}{1 + \kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}} |f_n|$$

$$\leq C_2 t^{\alpha_1+\varepsilon} \sum_{n=1}^{\infty} \kappa_1(\frac{n\pi}{L})^{\beta_1+\epsilon_0} |f_n|. \tag{2.12}$$

On the other hand, by Cauchy-Schwarz inequality

$$\sum_{n=1}^{\infty} \kappa_1(\frac{n\pi}{L})^{\beta_1+\epsilon_0} |f_n| = \sum_{n=1}^{\infty} \kappa_1(\frac{n\pi}{L})^{\beta_1+\epsilon_0-4}(\frac{n\pi}{L})^4 |f_n|$$

$$\leq \left(\sum_{n=1}^{\infty} (\frac{n\pi}{L})^8 |f_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} \kappa_1^2 (\frac{n\pi}{L})^{2(\beta_1+\epsilon_0-4)} \right)^{\frac{1}{2}}. \tag{2.13}$$

Noting that $\epsilon_0 < 3/2$ and $1 < \beta_1 < 2$, then $2(1 + \epsilon_0 - 4) < 2(\beta_1 + \epsilon_0 - 4) < -1$. Combining (2.10), we can conclude that

$$\sum_{n=1}^{\infty} \kappa_1(\frac{n\pi}{L})^{\beta_1+\epsilon_0} |f_n| < \infty. \tag{2.14}$$

Therefore,

$$|t^{2\alpha_1} \sum_{n=1}^{\infty} (\kappa_1(\frac{n\pi}{L})^{\beta_1})^2 E_{\alpha_1,2\alpha_1+1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1} t^{\alpha_1}) f_n| = O(t^{\alpha_1+\varepsilon}), \quad \text{for } t \to 0 \tag{2.15}$$

with small $\varepsilon > 0$. Similarly, we can verify that

$$|t^{2\alpha_2} \sum_{n=1}^{\infty} (\kappa_2(\frac{n\pi}{L})^{\beta_2})^2 E_{\alpha_2,2\alpha_2+1}(-\kappa_2(\frac{n\pi}{L})^{\beta_2} t^{\alpha_2}) f_n| = O(t^{\alpha_2+\varepsilon}), \text{for } t \to 0 \tag{2.16}$$

with small $\varepsilon > 0$.

Inserting the above two equalities into (2.7), we have

$$-\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} \sum_{n=1}^{\infty} \kappa_1(\frac{n\pi}{L})^{\beta_1} f_n + O(t^{\alpha_1+\varepsilon}) = -\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} \sum_{n=1}^{\infty} \kappa_2(\frac{n\pi}{L})^{\beta_2} f_n + O(t^{\alpha_2+\varepsilon}). \tag{2.17}$$

If $\alpha_2 < \alpha_1$, then

$$-\frac{t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1+1)}\sum_{n=1}^{\infty}\kappa_1(\frac{n\pi}{L})^{\beta_1}f_n + O(t^{\alpha_1-\alpha_2+\varepsilon}) = -\frac{1}{\Gamma(\alpha_2+1)}\sum_{n=1}^{\infty}\kappa_2(\frac{n\pi}{L})^{\beta_2}f_n + O(t^{\varepsilon}),$$

holds for $t \to 0$. That means

$$-\frac{1}{\Gamma(\alpha_2+1)}\sum_{n=1}^{\infty}\kappa_2(\frac{n\pi}{L})^{\beta_2}f_n \to 0. \tag{2.18}$$

for $t \to 0$. However, this is impossible. Because if $\sum_{n=1}^{\infty}\kappa_2(\frac{n\pi}{L})^{\beta_2}f_n = 0$, then due to $f_n > 0, 1 < \beta_2 \leq 2$,

$$0 = \sum_{n=1}^{\infty}\kappa_2(\frac{n\pi}{L})^{\beta_2}f_n = \sum_{n=1}^{\infty}\kappa_2(\frac{n\pi}{L})^2(\frac{n\pi}{L})^{\beta_2-2}f_n$$

$$\leq (\frac{\pi}{L})^{\beta_2-2}\sum_{n=1}^{\infty}\kappa_2(\frac{n\pi}{L})^2 f_n = \kappa_2(\frac{\pi}{L})^{\beta_2-2}f''(0) < 0. \tag{2.19}$$

This is a contradiction. Hence $\alpha_2 < \alpha_1$ does not hold. Similarly, $\alpha_2 > \alpha_1$ does not hold, too. As a result, we get $\alpha_1 = \alpha_2$.

Next, we will prove that $\beta_1 = \beta_2$, $\kappa_1 = \kappa_2$.

Now we have the following equation from (2.4)

$$\sum_{n=1}^{\infty}E_{\alpha,1}(-\kappa_1(\frac{n\pi}{L})^{\beta_1}t^{\alpha})f_n = \sum_{n=1}^{\infty}E_{\alpha,1}(-\kappa_2(\frac{n\pi}{L})^{\beta_2}t^{\alpha})f_n, \tag{2.20}$$

for $\alpha_1 = \alpha_2 = \alpha$. By the analytic proposition of $E_{\alpha,1}(t)$, the above equation holds for $t > 0$. First for $0 < \alpha < 1$, we have the Laplace transform of Mittag-Leffler function:

$$\int_0^{\infty}e^{-zt}E_{\alpha,1}(-\kappa(\frac{n\pi}{L})^{\beta}t^{\alpha})dt = \frac{z^{\alpha-1}}{z^{\alpha}+\kappa(\frac{n\pi}{L})^{\beta}}, \quad \Re z > 0. \tag{2.21}$$

Taking Laplace transform on both sides of Eq (2.20), we have

$$\sum_{n=1}^{\infty}f_n\frac{z^{\alpha-1}}{z^{\alpha}+\kappa_1(\frac{n\pi}{L})^{\beta_1}} = \sum_{n=1}^{\infty}f_n\frac{z^{\alpha-1}}{z^{\alpha}+\kappa_2(\frac{n\pi}{L})^{\beta_2}}, \quad \Re z > 0, \tag{2.22}$$

i.e.,

$$\sum_{n=1}^{\infty}\frac{f_n}{z^{\alpha}+\kappa_1(\frac{n\pi}{L})^{\beta_1}} = \sum_{n=1}^{\infty}\frac{f_n}{z^{\alpha}+\kappa_2(\frac{n\pi}{L})^{\beta_2}}, \quad \Re z > 0. \tag{2.23}$$

That is

$$\sum_{n=1}^{\infty}\frac{f_n}{\eta+\kappa_1(\frac{n\pi}{L})^{\beta_1}} = \sum_{n=1}^{\infty}\frac{f_n}{\eta+\kappa_2(\frac{n\pi}{L})^{\beta_2}}, \quad \Re\eta > 0. \tag{2.24}$$

We can analytically continue both sides of (2.24) in η. So (2.24) holds for

$$\eta \in \mathbb{C}/(\{-\kappa_1(\frac{n\pi}{L})^{\beta_1}\}_{n\geq 1} \cup \{-\kappa_2(\frac{n\pi}{L})^{\beta_2}\}_{n\geq 1}).$$

Now we deduce that $\kappa_1(\frac{n\pi}{L})^{\beta_1} = \kappa_2(\frac{n\pi}{L})^{\beta_2}$. From (2.24), W. L. O. G., we assume that $\kappa_1(\frac{n\pi}{L})^{\beta_1} < \kappa_2(\frac{n\pi}{L})^{\beta_2}$. Then we can take a suitable disc R_1 which includes $-\kappa_1(\frac{n\pi}{L})^{\beta_1}|_{n=1}$ but does not include $\{-\kappa_1(\frac{n\pi}{L})^{\beta_1}\}_{n\geq 2} \cup \{-\kappa_2(\frac{n\pi}{L})^{\beta_2}\}_{n\geq 1}$. According to (2.24) and the analyticity of both sides of (2.24), we have

$$\sum_{n=1}^{\infty} \int_{R_1} \frac{f_n}{\eta + \kappa_1(\frac{n\pi}{L})^{\beta_1}} d\eta = \sum_{n=1}^{\infty} \int_{R_1} \frac{f_n}{\eta + \kappa_2(\frac{n\pi}{L})^{\beta_2}} d\eta, \tag{2.25}$$

and hence, the Cauchy integral formula and Cauchy integral theorem yield

$$2\pi i f_1 = 0. \tag{2.26}$$

However, $f_1 \neq 0$, i.e., $2\pi i f_1 \neq 0$. Therefore $\kappa_1(\frac{n\pi}{L})^{\beta_1} < \kappa_2(\frac{n\pi}{L})^{\beta_2}$ does not hold.

By the same argument, $\kappa_1(\frac{n\pi}{L})^{\beta_1} > \kappa_2(\frac{n\pi}{L})^{\beta_2}$ does not hold, either. Therefore, there holds $\kappa_1(\frac{n\pi}{L})^{\beta_1} = \kappa_2(\frac{n\pi}{L})^{\beta_2}$. Now, we are in the position to prove $\kappa_1 = \kappa_2, \beta_1 = \beta_2$ from $\kappa_1(\frac{n\pi}{L})^{\beta_1} = \kappa_2(\frac{n\pi}{L})^{\beta_2}$. It is easy to see that if $\kappa_1 \neq \kappa_2$ or $\beta_1 \neq \beta_2$ then there must exist at least a constant n such that $\kappa_1(\frac{n\pi}{L})^{\beta_1} \neq \kappa_2(\frac{n\pi}{L})^{\beta_2}$. By analysis of its contrapositivity, we easily have $\kappa_1 = \kappa_2, \beta_1 = \beta_2$. □

In general, the conditions (2.2) is not easy for one to verify. Therefore we give the weak conditions for the uniqueness:

Theorem 2. *Suppose that $u_1(\alpha_1, \beta_1, \kappa_1; x, t)$ and $u_2(\alpha_2, \beta_2, \kappa_2; x, t)$ represent the solutions of the inverse problem with $\alpha = \alpha_1, \beta = \beta_1, \kappa = \kappa_1$ and $\alpha = \alpha_2, \beta = \beta_2, \kappa = \kappa_2$ respectively. We assume that the initial data satisfies*

$$f \in H^2(0, L),\ f'(0) = f'(L) = 0\ (Comapatible\ condition), \tag{2.27}$$
$$(-\triangle)^{\beta_k/2} f(0) \neq 0, k = 1, 2. \tag{2.28}$$

If $u_1(\alpha_1, \beta_1, \kappa_1; 0, t) = u_2(\alpha_2, \beta_2, \kappa_2; 0, t)$ for $0 < t < T$, then
$\alpha_1 = \alpha_2, \beta_1 = \beta_2, \kappa_1 = \kappa_2$.

Proof. W. L. O. G, let $L = 1, \varphi_n(x) = \sqrt{2} \cos n\pi x$ for $n \in \mathbf{N}$ and $\varphi_0(x) = 1$. Then we consider

$$\begin{cases} {}_tD_*^\alpha u(x, t) = -\kappa(-\triangle)^{\beta/2} u(x, t), & 0 < x < 1, 0 < t < T, \\ u_x(0, t) = u_x(1, t), & t > 0, \\ u(x, 0) = f(x), & 0 < x < 1. \end{cases}$$

Then

$$u(x, t) = \sum_{n=0}^{\infty} E_{\alpha,1}(-k(n\pi)^\beta t^\alpha)(f, \varphi_n)\varphi_n(x).$$

Setting $f_n = (f, \varphi_n)\sqrt{2}$ for $n \in \mathbf{N}$ and $f_0 = (f, 1)$, we have

$$u(0, t) = f_0 + \sum_{n=1}^{\infty} E_{\alpha,1}(-k(n\pi)^\beta t^\alpha) f_n.$$

Assume

$$u_{\alpha_1, \beta_1, k_1}(0, t) = u_{\alpha_2, \beta_2, k_2}(0, t), \quad t > 0.$$

Then due to Podlubny [19] (p.34),

$$-\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}\sum_{n=1}^{\infty}k_1(n\pi)^{\beta_1}(f,\varphi_n)\varphi_n(0)+O(t^{2\alpha_1})$$

$$=-\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)}\sum_{n=1}^{\infty}k_2(n\pi)^{\beta_2}(f,\varphi_n)\varphi_n(0)+O(t^{2\alpha_2}),\quad\text{as }t\to 0.$$

Since $f\in D((-\triangle)^{\beta_k/2})$ and $(-\triangle)^{\beta_k/2}$ is self-adjoint for $k=1,2$, we have

$$\sum_{n=1}^{\infty}(n\pi)^{\beta_1}(f,\varphi_n)\varphi_n(0)=\sum_{n=0}^{\infty}(n\pi)^{\beta_1}(f,\varphi_n)\varphi_n(0)$$

$$=\sum_{n=0}^{\infty}(f,(n\pi)^{\beta_1}\varphi_n)\varphi_n(0)=\sum_{n=0}^{\infty}(f,(-\triangle)^{\beta_1/2}\varphi_n)\varphi_n(0)$$

$$=\sum_{n=0}^{\infty}((-\triangle)^{\beta_1/2}f,\varphi_n)\varphi_n(0)=(-\triangle)^{\beta_1/2}f(0).$$

Hence we have

$$-\frac{\kappa_1 t^{\alpha_1}}{\Gamma(\alpha_1+1)}(-\triangle)^{\beta_1/2}f(0)+O(t^{2\alpha_1})=-\frac{\kappa_2 t^{\alpha_2}}{\Gamma(\alpha_2+1)}(-\triangle)^{\beta_2/2}f(0)+O(t^{2\alpha_2}).\tag{2.29}$$

Now we can prove that $\alpha_1=\alpha_2$.

Indeed we assume that $\alpha_1<\alpha_2$. Dividing (2.29) by t^{α_1}, we obtain

$$-\frac{k_1}{\Gamma(\alpha_1+1)}(-\triangle)^{\beta_1/2}f(0)+O(t^{\alpha_1})=-\frac{k_2}{\Gamma(\alpha_2+1)}(-\triangle)^{\beta_2/2}f(0)t^{\alpha_2-\alpha_1}+O(t^{2\alpha_2-\alpha_1}).$$

By $\alpha_1<\alpha_2$, letting $t\to 0$, we obtain $\frac{k_1}{\Gamma(\alpha_1+1)}(-\triangle)^{\beta_1/2}f(0)=0$, which contradicts (2.29).

The uniqueness proof on β and κ is the same as that in **Theorem 1**. □

Remark on (2.29). We can satisfy (2.29) by a generous condition by the comparison principle of $(-\triangle)^{\beta/2}$.

3. Two numerical methods

In this section, we propose two numerical methods for solving this problem based least squares functional. The first is based on Tikhonov method in the function space. The second method is based on the classical Levenberg-Marquardt optimization method in the discrete Euclid space.

3.1. Tikhonov method combined with the gradient flow

Denote $a=\begin{pmatrix}\alpha\\\beta\\\kappa\end{pmatrix}\in\mathbb{R}^3$, let $u(x,t;a):=u(\alpha,\beta,\kappa)(x,t)$ be the unique solution of forward problem.

A feasible way to numerical computation for the unknown a is to solve the following minimization problem.

$$\min_{a\in\mathbb{R}^3}J(a):=\min_{a\in\mathbb{R}^3}\frac{1}{2}\left(\|u(a;0,t)-g^{\delta}(t)\|_{L^2(0,T)}^2+\lambda\|a\|_{\mathbb{R}^3}^2\right).\tag{3.1}$$

The gradient $\nabla J(a)$ of the functional $J(a)$ is given by

$$\left[\begin{array}{c} \int_0^T \frac{\partial u(\alpha,\beta,\kappa,0,t)}{\partial \alpha}(u(\alpha,\beta,\kappa,0,t) - g^\delta(t))dt + \lambda\alpha \\ \int_0^T \frac{\partial u(\alpha,\beta,\kappa,0,t)}{\partial \beta}(u(\alpha,\beta,\kappa,0,t) - g^\delta(t))dt + \lambda\beta \\ \int_0^T \frac{\partial u(\alpha,\beta,\kappa,0,t)}{\partial \kappa}(u(\alpha,\beta,\kappa,0,t) - g^\delta(t))dt + \lambda\kappa \end{array} \right].$$

If let $\nabla J(a)=0$, then we can get the Euler equation for the minimizer. However, it is a nonlinear equation and is not easily be solved directly. Here we turn to the approximate solution by the iterative method. Using the gradient flow method with an initial value a_0, we get

$$\frac{da}{dt} = -\nabla J(a), \tag{3.2}$$

where t is the artificial time. Using a simple method, i.e. the explicit Euler method, we arrives the following iteration schemes with a time step size τ:

$$a^{j+1} = a^j - \tau\nabla J(a^j), \tag{3.3}$$

i.e.,

$$\alpha^{j+1} = \alpha^j - \tau_\alpha \left(\int_0^T \frac{\partial u(\alpha,\beta,\kappa,0,t)}{\partial \alpha}\Big|_{\alpha=\alpha^j,\,\beta=\beta^j,\,\kappa=\kappa^j}(u(\alpha^j,\beta^j,\kappa^j,0,t) - g^\delta(t))dt + \lambda\alpha^j \right);$$

$$\beta^{j+1} = \beta^j - \tau_\beta \left(\int_0^T \frac{\partial u(\alpha,\beta,\kappa,0,t)}{\partial \beta}\Big|_{\alpha=\alpha^j,\,\beta=\beta^j,\,\kappa=\kappa^j}(u(\alpha^j,\beta^j,\kappa^j,0,t) - g^\delta(t))dt + \lambda\beta^j \right);$$

$$\kappa^{j+1} = \kappa^j - \tau_\kappa \left(\int_0^T \frac{\partial u(\alpha,\beta,\kappa,0,t)}{\partial \kappa}\Big|_{\alpha=\alpha^j,\,\beta=\beta^j,\,\kappa=\kappa^j}(u(\alpha^j,\beta^j,\kappa^j,0,t) - g^\delta(t))dt + \lambda\kappa^j \right). \tag{3.4}$$

3.2. Levenberg-Marquardt method

Because in most of the practical applications, the data are measured at discrete times. Assume the measured data is given by $g^\delta(t_i)$, $i = 0, 1, \cdots, q$. Let us consider the minimization problem in discrete case:

$$\|u(a; 0, t_i) - g^\delta(t_i)\|_{\mathbb{R}^q}^2,$$

where $u(a; 0, t_i)$ is the computed data from the forward problem with a given a, which is used to fit the measured data. A standard method for solving this least squares problem is the Levenberg-Marquardt method with a damped parameter $\tilde{\lambda}$ which plays the same role as the regularization parameter λ in Tikhonov method. For readability, we give the details of this algorithm:

A updated sequences is given by

$$a^{j+1} = a^j + \Delta a^j, \ j = 1, 2, \cdots, \tag{3.5}$$

where Δa^j is the updated stepsize of a^j in each iteration step j. We consider the the minimization problem about Δa^j at each iteration step j:

$$F(\Delta a^j) := \|u(a^j + \Delta a^j; 0, t_i) - g^\delta(t_i)\|_{\mathbb{R}^q}^2. \tag{3.6}$$

Make the Taylor expansion for $u(a^j + \Delta a^j; 0, t_i)$ at a^j and take a linear approximation, we have

$$u(a^j + \Delta a^j; 0, t_i) \approx u(a^j; 0, t_i) + \nabla_a^{tr} u(a; 0, t_i) \cdot a^j.$$

Plus this into (3.6), we get

$$F(\Delta a^j) := \|\nabla_a^{tr} u(a; 0, t_i) \cdot a^j - (g^\delta(t_i) - u(a^j; 0, t_i))\|_{\mathbb{R}^q}^2. \tag{3.7}$$

However, this least square problem is ill-posed due to the original problem, therefore we consider the Tikhonov method:

$$F(\Delta a^j) := \|\nabla_a^{tr} u(a; 0, t_i) \cdot a^j - (g^\delta(t_i) - u(a^j; 0, t_i))\|_{\mathbb{R}^q}^2 + \tilde{\lambda}\|\Delta a^j\|_{\mathbb{R}^3}, \tag{3.8}$$

where $\nabla_a^{tr} u(a; 0, t_i) \cdot a^j$ is computed by finite difference method and is given by $\nabla_a^{tr} u(a; 0, t_i) \cdot a^j \approx \sum_{k=1}^{3} \frac{u(a_k^j + h; 0, t_i) - u(a_k^j; 0, t_i)}{h} \Delta a_k^j$. Now the minimization problem (3.8) is a linear problem and can be easily solved for the updated stepsize Δa^j with a regularization $\tilde{\lambda}$.

3.3. Numerical test

In this section, we consider a simple example to show the effectiveness of the aforementioned two algorithms, i.e., Tikhonov method and Levenberg-Marquardt method. We want to determine the parameters (α, β, κ) in the following problem

$$_t D_*^\alpha u(x, t) = -\kappa(-\triangle)^{\frac{\beta}{2}} u(x, t), \ 0 \le t \le T, \ 0 \le x \le \pi, \tag{3.9}$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0, \tag{3.10}$$

$$u(x, 0) := f(x) = -x^2(3\pi/2 - x). \tag{3.11}$$

The exact solution is given by

$$u(x, t) = -\frac{\pi^3}{4} + \sum_{n=1}^{\infty} E_{\alpha,1}(-\kappa n^\beta t^\alpha)\left[\frac{12(1 + (-1)^{n+1})}{\pi n^4}\right] \cos(nx). \tag{3.12}$$

Now the input data $g(t) := u(0, t)$ is obtained and the noisy data $g^\delta(t)$ is generated in the following way:

$$g^\delta(t) = g(t) + \sigma(t), \tag{3.13}$$

where $\sigma(t) = \theta r(t)$ and $r(t)$ is a random number between $[0, 1]$ and $\theta = \max\{g(t)\} * \eta\%$ is the noise level. In the numerical experiment, we fix the parameters $T = 1, \eta = 1$. The algorithm of calculating Mittag Leffler function is given in [20]. First we plot the solution $u(x, t = 1)$ of direct problem when $\alpha = 0.5, \beta = 1.2, \kappa = 0.10$, which is shown in Figure 1. The input exact data $g(t)$ is displayed in Figure 2.
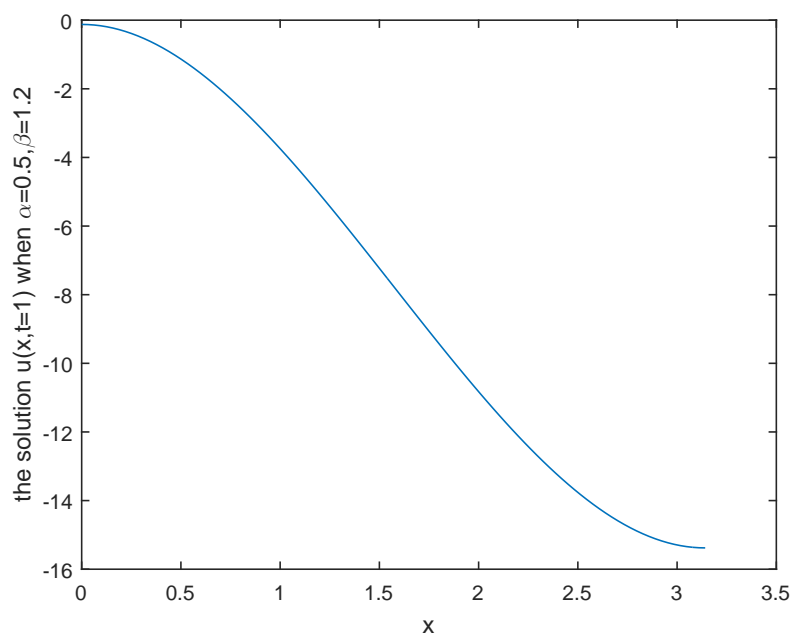
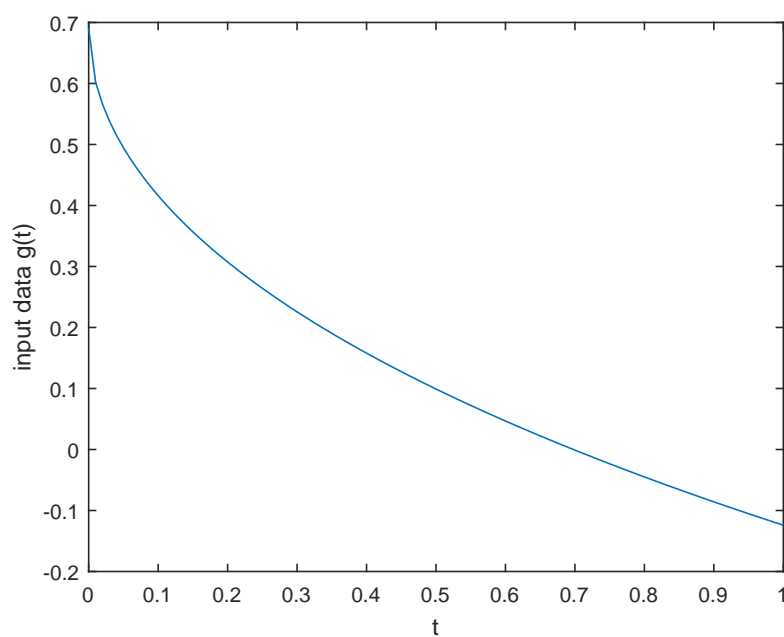**Figure 1.** The solution $u(x, t)$ at the time $t = 1$ for the direct problem.

Figure 2. The input data for reconstruction.

The exact fractional orders and diffusion coefficient are $\alpha = \mathbf{0.5}, \beta = \mathbf{1.2}, \kappa = \mathbf{0.10}$.

In the numerical test for the Tikhonov method, the parameters and their values in the computation are listed below:

1. Summing idex of $u(x, t)$ in (3.12): $n = 20$ is taken.

2. The number of $t_i = i/q \in [0, 1], (i = 0, \cdots, q)$ and $q = 20$ in two methods (q is the total points for trapezoidal rule of the numerical integral in (3.4)).

3. Stepsizes $\tau_s(s = \alpha, \beta, \kappa)$ in (3.4) for coputing the derivatives $\frac{\partial u}{\partial \alpha}$, $\frac{\partial u}{\partial \beta}$ and $\frac{\partial u}{\partial \kappa}$ by finite difference method: $\tau_\alpha = \tau_\beta = \tau_\kappa = 0.01$.

4. The initial guess for $(\alpha, \beta, \kappa) = (0.4, 1.0, 0.05)$.

5. The regularization parameter $\lambda = 0.001$.

6. The iterative step size is $\tau = (0.07, 0.01, 0.02)$.

6. Stop criterion: when $\|a^{j+1} - a^j\| \leq 0.01$.

Finally we obtain the approximate value $(\alpha, \beta, \kappa) = (0.5289, 1.1836, 0.0943)$. Keep the same parameters with the damped parameter $\tilde{\lambda} = 0.001$, we use the Levenberg-Marquardt method to get the approximate value where we don't use the Matlab optimization toolbox on Levenberg-Marquardt method.

$$(\alpha, \beta, \kappa) = (0.4933, 1.1968, 0.0998).$$

This result shows the numerical methods are effective. Here we list more results using the above parameters for the Levenberg-Marquardt method. First we fix the initial guess for $\alpha = 0.1, \kappa = 0.05$ and let β range from 1 to 1.4. The numerical results are displayed in Table 1.

Table 1. Numerical results for different initial guesses with fixed $\alpha = 0.1, \kappa = 0.05$.

β	Approximation for (α, β, κ)
1	(0.4992;1.2012;0.1001)
1.1	(0.5015;1.2016;0.1002)
1.2	(0.49995;1.2014;0.1002)
1.3	(0.5011;1.19999;0.09953)
1.4	(0.4981;1.202;0.09999)

The numerical results show that the methods are stable.

4. Conclusions

In this paper, we give the proof of uniqueness for determining three parameters in a time-space fractional diffusion equation by means of observation data from accessible boundary. By our uniqueness result a Tikhonov method and the Levenberg-Marquardt method are tested preliminarily. Some further research on the stability of the proposed methods will be investigated in the future.

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Conflict of interest

There is no conflict of interest.

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