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Research article

The Lawson-Simons' theorem on warped product submanifolds with geometric information

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Abstract: The main objective of this paper is to investigate topological properties from the view point of compact warped product submanifolds of a space form with the vanishing constant sectional curvature. That is, we prove the non-existence of stable integral p-currents in a compact oriented warped product pointwise semi-slant submanifold M^n in the Euclidean space $\mathbb{R}^{p+2q}$ which satisfies an operative condition involving the Laplacian of a warped function and a pointwise slant function, and show that their homology groups are zero on this operative condition. Moreover, under the assumption of extrinsic conditions, we derive new topological sphere theorems on a warped product submanifold M^n, and prove that M^n is homeomorphic to $\mathbb{S}^n$ if $n = 4$, and M^n is homotopic to $\mathbb{S}^n$ if $n = 3$. Furthermore, the same results are generalized for CR-warped products and our results recovered [17].

Keywords: warped product manifolds; integral p-currents; sphere thoerems; CR-warped products; Ricci-flat
Mathematics Subject Classification: 53B50, 53C20, 53C40

1. Introduction

The theory of integral currents was introduced by Federer-Fleming [8]. The notion of an integral current plays an important role in providing tooplogical information by combining the geometric structure of differentiable manifolds and homology groups with integral coefficients. In 1970 s, Lawson-Simons [11] provided the optimization for the nonexistence of stable currents in submanifolds of the sphere S^n with pinched the second fundamental form.

On the other hand, vanishing homology and non-existence of stable integral currents in warped product submanifolds in an odd dimensional sphere were constructed in [15]. By taking the help of Lawson-Simons' results [11], they proved that the homology groups were trivial and did not exist stable currents of a contact CR-warped product submanifold in an odd dimentional sphere (see [15] for more detail). Later on, F. Şahin [17, 18] proved similar results for CR-warped product submanifold in Euclidean spaces and in the near Kaehler six sphere $\mathbb{S}^6$. Motivated by previous development, Ali et al. [1] optimized the warped function and pointwise slant functions for a warped product submanifold in the unit sphere with trivial homology groups on the pointwise slant fiber. Inspired by the results [7] on a hyperbolic space for negative constant sectional curvature, Ali et al. [3] derived the results for non-existence stable currents and vanishing homology groups for CR-warped product in complex hyperbolic spaces. As applications, some topological sphere theorems have been proved in [3]. The same study has been extended for Lagrangian warped product submanifolds of the near Kaehler six sphere $\mathbb{S}^6$ in [2], and similar results in a generalized complex space can be found in [13]. By imposing certain conditions on the second fundamental form in ([12, 14, 20–22]), several mathematicians obtained many results for topological and differentiable structures of submanifolds.

In this work, we provide these results in non-trivial warped product pointwise semi-slant submanifold in Kaehler manifold with zero holomorphic sectional curvature, which generalizes a CR-warped product submanifold class, proved by B. Sahin [16]. Moreover, we show that our results are an extension of F. Sahin in [17].

2. Preliminaries

A Kaehler manifold $\widetilde{M}$ of dimension $2m$ with an almost complex structure J and a Riemannian metric g satisfies

$$\begin{aligned} J^2 &= -I \\ g(JU, JV) &= g(U, V) \\ (\widetilde{\nabla}_U J)V &= 0 \end{aligned} \tag{2.1}$$

for all vector fields $U, V \in \mathfrak{X}(T\widetilde{M})$, where $\mathfrak{X}(T\widetilde{M})$ denotes the collection of all sections on the tangent bundle $T\widetilde{M}$ on $\widetilde{M}$ and $\widetilde{\nabla}$ denotes the covariant differential operator on $\widetilde{M}$ with respect to the Riemannian metric g [23].

If ∇ and $\nabla^\perp$ are the induced Riemannian connections on the tangent bundle TM and the normal bundle $T^\perp M$ of a submanifold M, respectively, then the following formulas are recognized as Gauss and Weingarten

$$\widetilde{\nabla}_U V = \nabla_U V + h(U, V), \tag{2.2}$$

$$\widetilde{\nabla}_U N = -A_N U + \nabla^\perp_U N \tag{2.3}$$

for $U, V \in \mathfrak{X}(TM)$ and $N \in \mathfrak{X}(T^\perp M)$. If the tangential and normal components of $JU(JN)$ are represented by $PU(tN)$ and $FU(fN)$ respectively, then they are related as:

$$\text{(i) } JU = PU + FU, \quad \text{(ii) } JN = tN + fN \tag{2.4}$$

Here, a totally real submanifold M provides P is identically zero and a holomorphic submanifold characterizes F is identically zero. If the curvature tensors of $\widetilde{M}$ and M are symbolized $\widetilde{R}$ and R, respectively, the Gauss equation for a submanifold M is given as:

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g\big(h(X, Z), h(Y, W)\big) - g\big(h(X, W), h(Y, Z)\big) \tag{2.5}$$

for $X, Y, Z, W \in \mathfrak{X}(TM)$. Several classes of a submanifold M of Kaehler manifold $\widetilde{M}$ are categorised as in the following by the behaviour of almost complex structure J:

(i) M is holomorphic if $J(T_xM) \subseteq T_xM$ and totally real if $J(T_xM) \subseteq T^{\perp}M$ at each $x \in M$ [5].

(ii) A submanifold M is a CR-submanifold if the tangent space TM of M is expressed as $TM = \mathcal{D}^T \oplus \mathcal{D}^{\perp}$ for a totally real distribution $\mathcal{D}^{\perp}$, i.e., $J(\mathcal{D}^{\perp}) \subseteq (T^{\perp}M)$ and a holomorphic distribution $\mathcal{D}^T$, that is, $J(\mathcal{D}^T) \subseteq \mathcal{D}^T$ [5].

(iii) If the Wirtinger angle $\theta(X)$ between JX and T_xM for any nonzero vector $X \in T_xM$ and $x \in M$ provides $\theta : T^*M \to \mathbb{R}$ is a real-valued function, then M is called a pointwise slant submanifold [6]. If the angle $\theta(X)$ is globally constant then M is slant submanifold. The necessary and sufficent condition for M to be pointwise slant if the tangential endomorphism P is satisfied the following

$$P^2 = -\lambda I. \tag{2.6}$$

for $\lambda \in [0, 1]$ such that $\lambda = \cos^2\theta$. The following formulas can be constructed by using (2.4) and (2.6) as:

$$g(PU, PV) = \cos^2\theta g(U, V), \tag{2.7}$$

$$g(FU, FV) = \sin^2\theta g(U, V) \tag{2.8}$$

for any $U, V \in \mathfrak{X}(T\mathcal{D}^{\theta})$.

(iv) If the tangent space TM of M is disintegrated as $TM = \mathcal{D}^T \oplus \mathcal{D}^{\theta}$ for the poinwise slant distribution $\mathcal{D}^{\theta}$, then it is called a pointwise semi-slant submanifold [16]. Moreover, if μ is an *invariant* subspace under J of the normal bundle $T^{\perp}M$, then the normal bundle $T^{\perp}M$ can be decomposed as $T^{\perp}M = F\mathcal{D}^{\theta} \oplus \mu$.

If p and q are the ranks of the complex distribution $\mathcal{D}^T$ and the pointwise slant distribution $\mathcal{D}^{\theta}$ of a pointwise semi-slant submanifold in a Kaehler manifold $\widetilde{M}$, respectively, then the following remarks hold:

Remark 1. M is invariant if $q = 0$ and pointwise slant if $p = 0$.

Remark 2. If we consider the slant function $\theta : M \to \mathbb{R}$ is globally constant on M and $\theta = \frac{\pi}{2}$, then M is a CR-submanifold.

Remark 3. If the slant function $\theta : M \to (0, \frac{\pi}{2})$, then M is called a proper pointwise semi-slant submanifold.

Moreover, there are some examples and related problems of a pointwise semi-slant submanifold in a Kaehler manifold in [16]. Furthermore, the initial concept of a warped product manifold is given by Bishop and O'Neill [4]. The product manifold of the form $N_1^p \times_f N_2^q$ with the metric $g = g_1 + f^2 g_2$ of two Riemannian manifolds N_1^p and N_2^q is called a warped product manifold and f denotes the warping function on the base N_1^p. Now we have

$$\nabla_Z X = \nabla_X Z = (X \ln f)Z \tag{2.9}$$

for any $X \in \Gamma(TN_1)$ and $Z \in \Gamma(TN_2)$, where the gradient $\nabla(\ln f)$ of $\ln f$ is given by

$$g(\nabla \ln f, X) = X(\ln f). \tag{2.10}$$

The following relation is proved in [4]

$$R(X,Z)Y = \frac{\mathcal{H}^f(X,Y)}{f}Z \tag{2.11}$$

where $\mathcal{H}^f$ is a Hessian tensor of f.

Remark 4. A warped product manifold $M^n = N_1^p \times_f N_2^q$ is trivial if and only if f is constant along N_1^p.

For the Laplacian $\Delta(\ln f)$ of the warping function f, we have

$$\Delta(\ln f) = -div\Big(\frac{\nabla f}{f}\Big) = -g(\nabla\frac{1}{f}, \nabla f) - \frac{1}{f}div(\nabla f) = \|\nabla \ln f\|^2 + \frac{\nabla f}{f}. \tag{2.12}$$

From (2.12), we get

$$\frac{\Delta f}{f} = \Delta(\ln f) - \|\nabla(\ln f)\|^2. \tag{2.13}$$

3. Main results

To prove our main results, we need the following which was obtained by Lawson and Simons [11], we summarize these result as follows;

Lemma 3.1. *[11, 20] For the second fundamental form h and any positive integers p and q with $p + q = n$, if the inequality*

$$\sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{n}\Big(2\|h(e_\alpha, e_\beta)\|^2 - g\big(h(e_\alpha, e_\alpha), h(e_\beta, e_\beta)\big)\Big) < pqc \tag{3.1}$$

is satisfied for an n-dimensional compact submanifold M^n in a space form $\widetilde{M}(c)$ of constant curvature $c \geq 0$, then there is no stable p-currents in M^n and $H_p(M^n, \mathbb{Z}) = H_q(M^n, \mathbb{Z}) = 0$, where $H_\alpha(M^n, \mathbb{Z})$ is the α-th homology group of M^n with integer coefficients and $\{e_\alpha\}_{1\leq\alpha\leq n}$ are orthonormal basis of M^n.

From the generalized Poincare conjecture, it was proved in [11] that if the second fundamental form bounded above of an n-dimensional compact submanifold M^n in the unit sphere $\mathbb{S}^{n+p}$, then M^n is either homeomorphic to a sphere $\mathbb{S}^n$ for $n \neq 3$ or homotopic to a sphere $\mathbb{S}^n$ for $n = 3$. In [19], if an n-dimensional compact oriented submanifold M^n of the $(n + k)$-dimensional Euclidean space $\mathbb{E}^{n+k}$ satisfies the pinching condition $Ric(X) > \delta_1(n)g(A_H X, X)$, where the Ricci curvature Ric and the shape operator A_H with respect to the mean curvature H, and $\delta_1(n)$ is constant given by $\delta_1(n) = \frac{n(n-3)}{n-1}$ if n is odd and $\delta_1(n) = n-2$ if n is even, then there are no stable currents in M^n, and also M^n is homeomorphic to $\mathbb{S}^n$.

In the case study of the present paper, we consider the only non-trvial warped product pointwise semi-slant submanifold of the form $N_T^p \times_f N_\theta^q$ in a Kaehler manifold because other warped product submanifolds are trivial (see [16] for detail). Before proceeding to the main theorem, we recall that the following result [16]:

Lemma 3.2. *Let $M^n = N_T^p \times_f N_\theta^q$ be a warped product pointwise semi-slant submanifold of a Kaehler manifold $\widetilde{M}$. Then*

$$g(h(X,Z), FPZ) = -(X \ln f)\cos^2\theta \|Z\|^2 \tag{3.2}$$

$$g(h(Z,JX), FZ) = (X \ln f)\|Z\|^2 \tag{3.3}$$

for $X, Y \in \mathfrak{X}(TN_T)$ and $Z \in \mathfrak{X}(TN_\theta)$.

Let $\mathbb{C}^q$ be a complex vector space, which is identified with a real vector space $\mathbb{R}^{2q}$, expressed as $(z^1, \ldots, z^q) \to (x^1 \ldots, x^q, y^1 \ldots y^q)$ with $z^\beta = x^\beta + iy^\beta$, for $\beta = 1, \ldots q$. With the standard metric in $\mathbb{R}^{2q}$, it becomes the Euclidean $2q-$space, denoted by $\mathbb{R}^{2q}$. Therefore, with the standard flat Kaehlerian metric, an almost complex structure J on $\mathbb{R}^{2q}$ is said to be compatible if $(\mathbb{R}^{2q}, J)$ is analytically isometric to the complex number space $\mathbb{C}^q$. From now on, we denote the Euclidean $2q-$space with compatible almost complex structure J by $\mathbb{R}^{2q}$. Thus, $(\mathbb{R}^{2q}, J)$ is a Kaehler manifold with zero constant sectional curvature. Therefore, we have nonexistence stable integral $p-$currents theorem for a warped product pointwise semi-slant submanifold in a flat space or the Euclidean space $\mathbb{R}^{p+2q}$ as follows;

Theorem 3.3. *Let $M^n = N_T^p \times_f N_\theta^q$ be a compact, oriented warped product pointwise semi-slant submanifold in the Euclidean space $\mathbb{R}^{p+2q}$ with $n = p + q$. If the following condition is satisfied*

$$\Delta(\ln f) > \Big(1 - q + \csc^2\theta + \cot^2\theta\Big)\|\nabla(\ln f)\|^2 \tag{3.4}$$

then there does not exist stable integral p-currents in M^n and

$$H_p(M^n, \mathbb{Z}) = H_q(M^n, \mathbb{Z}) = 0,$$

where $H_i(M^n, \mathbb{Z})$ is the i-th homology group of M^n with integer coefficients.

Proof. Let $\dim(N_T) = p = 2\alpha$ and $\dim(N_\theta) = q = 2\beta$, where N_θ and N_T are integral manifolds of $\mathcal{D}^\theta$ and $\mathcal{D}$, respectively.

Consider $\{e_1, e_2, \cdots, e_\alpha, e_{\alpha+1} = Je_1, \cdots, e_{2\alpha} = Je_\alpha\}$ and $\{e_{2\alpha+1} = e_1^*, \cdots, e_{2\alpha+\beta} = e_\beta^*, e_{2\alpha+\beta+1} = e_{\beta+1}^* = \sec\theta Pe_1^*, \cdots, e_{p+q} = e_q^* = \sec\theta Pe_\beta^*\}$ to be orthonormal frames of TN_T and TN_θ, respectively. Thus the orthonormal frames of the normal subbundles $F\mathcal{D}^\theta$ is $\{e_{n+1} = \bar{e}_1 = \csc\theta Fe_1^*, \cdots, e_{n+\beta} = \bar{e}_\beta = \csc\theta Fe_1^*, e_{n+\beta+1} = \bar{e}_{\beta+1} = \csc\theta\sec\theta FPe_1^*, \cdots, e_{n+2\beta} = \bar{e}_{2\beta} = \csc\theta\sec\theta FPe_\beta^*\}$. Thus, we have

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=p+1}^{n}&\Big\{2\|h(e_i,e_j)\|^2 - g\big(h(e_i,e_i), h(e_j,e_j)\big)\Big\}\\ &= \sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\Big(h_{ij}^r\Big)^2\\ &\quad + \sum_{i=1}^{p}\sum_{j=p+1}^{n}\Big\{\|h(e_i,e_j)\|^2 - g\big(h(e_i,e_i), h(e_j,e_j)\big)\Big\}.\end{aligned} \tag{3.5}$$

Then from Gauss Eq (2.5) for the Euclidean space $\mathbb{R}^{p+2q}$, we get

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\Big\{2\|h(e_i,e_j)\|^2 - g\big(h(e_i,e_i), h(e_j,e_j)\big)\Big\} &= \sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\Big(h_{ij}^r\Big)^2\\ &\quad + \sum_{i=1}^{p}\sum_{j=1}^{q} g(R(e_i,e_j)e_i, e_j).\end{aligned} \tag{3.6}$$

From $R(e_i, e_j)e_i = \frac{\mathcal{H}^f(e_i,e_i)}{f} e_j$ in (2.11), we derive

$$\sum_{i=1}^{p}\sum_{j=1}^{q} g(R(e_i, e_j)e_i, e_j) = \frac{q}{f}\sum_{i=1}^{p} g(\nabla_{e_i}\nabla f, e_i). \tag{3.7}$$

Combining (3.6) and (3.7), we derive

$$\begin{aligned}\sum_{i=1}^{p}\sum_{j=p+1}^{n}&\Big\{2\|h(e_i, e_j)\|^2 - g\big(h(e_i, e_i), h(e_j, e_j)\big)\Big\}\\ &= \frac{q}{f}\sum_{i=1}^{p} g(\nabla_{e_i}\nabla f, e_i) + \sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}\left(h_{ij}^r\right)^2.\end{aligned} \tag{3.8}$$

First we compute the term Δf, that is the Laplacian of f, one derives

$$\begin{aligned}\Delta f &= -\sum_{k=1}^{n} g(\nabla_{e_k}(\nabla f), e_k)\\ &= -\sum_{i=1}^{p} g(\nabla_{e_i}(\nabla f), e_i) - \sum_{j=1}^{q} g(\nabla_{e^*{}_j}(\nabla f), e_j^*).\end{aligned}$$

The above equation can be expressed as components of N_θ^q from adapted orthonormal frame, one obtains

$$\begin{aligned}\Delta f = &-\sum_{i=1}^{p} g(\nabla_{e_i}(\nabla f), e_i) - \sum_{j=1}^{\beta} g(\nabla_{e^*{}_j}(\nabla f), e_j^*)\\ &- \sec^2\theta\sum_{j=1}^{\beta} g(\nabla_{Pe_j^*}(\nabla f), Pe_j^*).\end{aligned}$$

Since N_T^p is totally geodesic in M^n, $grad\ln f \in \mathfrak{X}(TN_T)$, we obtain

$$\begin{aligned}\Delta f = &-\frac{1}{f}\sum_{j=1}^{\beta}\left(g(e_j^*, e_j^*)\|\nabla(f)\|^2 + \sec^2\theta g(Pe_j^*, Pe_j^*)\|\nabla f\|^2\right)\\ &-\sum_{i=1}^{p} g(\nabla_{e_i}(\nabla f), e_i).\end{aligned}$$

Multiply to above equation by $\frac{1}{f}$, and from (2.7), we get:

$$\frac{\Delta f}{f} = -\frac{1}{f}\sum_{i=1}^{p} g(\nabla_{e_i}(\nabla f), e_i) - q\|\nabla(\ln f)\|^2.$$

Making use of (2.13), we find that

$$\frac{1}{f}\sum_{i=1}^{p} g(\nabla_{e_i}(\nabla f), e_i) = -\Delta(\ln f) + (1-q)\|\nabla\ln f\|^2. \tag{3.9}$$

Thus from (3.8) and (3.9), we compute that

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\Big\{2\|h(e_i,e_j)\|^2 - g\big(h(e_i,e_i),h(e_j,e_j)\big)\Big\} \\ = q(1-q)\|\nabla(\ln f)\|^2 - q\Delta(\ln f) + \sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h^r_{ij})^2. \tag{3.10}$$

Now, let $X = e_\alpha\,(1 \le \alpha \le p)$ and $Z = e^*_\beta\,(1 \le \beta \le q)$, we have

$$\sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h^r_{ij})^2 = \sum_{r=1}^{2\beta}\sum_{i=1}^{p}\sum_{j=1}^{q} g\big(h(e_i,e^*_j),\bar{e}_r\big)^2 \\ = \sum_{i=1}^{p}\sum_{j,r=1}^{\beta}\Big\{g\big(h(e_i,e^*_j),\csc\theta F e^*_r\big)^2 + g\big(h(e_i,e^*_j),\csc\theta\sec\theta FPe^*_r\big)^2\Big\}.$$

At the above equation, the term in the right hand side is $F\mathcal{D}^\theta$-component and for pointwise semi-slant, the fact $P\mathcal{D}^\theta \subseteq \mathcal{D}^\theta$ and $t(T^\perp M) = \mathcal{D}^\theta$. Then using (3.2) and (3.3) of Lemma 3.2 in the above equations, after summation over the vector fields on N_T and N_θ, we derive

$$\sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h^r_{ij})^2 = 2\Big(\csc^2\theta + \cot^2\theta\Big)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(e_i\ln f))^2 g(e^*_j,e^*_j)^2 \\ +2\Big(\csc^2\theta + \cot^2\theta\Big)\sum_{i=1}^{\alpha}\sum_{j=1}^{\beta}(Je_i\ln f))^2 g(e^*_j,e^*_j)^2$$

The last equation can be expressed as:

$$\sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h^r_{ij})^2 = 2\Big(\csc^2\theta + \cot^2\theta\Big)\sum_{i=1}^{p}(e_i\ln f)^2\sum_{j=1}^{\beta} g(e^*_j,e^*_j)^2,$$

which implies

$$\sum_{r=n+1}^{p+2q}\sum_{i=1}^{p}\sum_{j=p+1}^{n}(h^r_{ij})^2 = q(\csc^2\theta + \cot^2\theta)\|\nabla\ln f\|^2. \tag{3.11}$$

Following from (3.10) and (3.11), we arrive at

$$-q\Delta(\ln f) + q(1-q)\|\nabla(\ln f)\|^2 + q(\csc^2\theta + \cot^2\theta)\|\nabla(\ln f)\|^2 \\ = \sum_{i=1}^{p}\sum_{j=p+1}^{n}\Big\{2\|h(e_i,e_j)\|^2 - g\big(h(e_i,e_i),h(e_j,e_j)\big)\Big\}. \tag{3.12}$$

If (3.4) in Theorem 3.3 is satisfied, then from above equation, we get

$$\sum_{i=1}^{p}\sum_{j=p+1}^{n}\Big\{2\|h(e_i,e_j)\|^2 - g\big(h(e_i,e_i),h(e_j,e_j)\big)\Big\} < 0 \tag{3.13}$$

Applying Lemma 3.1, for Euclidean space $c = 0$, we get the final conclusion of our theorem. □

In particular, when the slant function is constant with $\theta = \frac{\pi}{2}$ in Theorem 3.3, a warped product pointwise semi-slant submanifold $M^n = N_T^p \times_f N_\theta^q$ becomes a CR-warped product in a Kaehler manifold such that N_T^p and $N_\theta^q = N_\perp^q$ are holomorphic and totally real submanifolds, respectively. Therefore, due to the studied of Chen [5], we prove the nonexistence of a stable integrable p-currents for CR-warped products in Euclidean space $\mathbb{R}^{p+2q}$ as a consequence of Theorem 3.3.

Theorem 3.4. *[17] Let $M^n = N_T^p \times_f N_\perp^q$ be a compact CR-warped product submanifold in Euclidean space $\mathbb{R}^{p+2q}$ with $n = p + q$. If the following condition is satisfied*

$$\Delta (\ln f) > (2 - q)\|\nabla (\ln f)\|^2, \tag{3.14}$$

then there does not exist a stable integral p-currents in M^n and

$$H_p(M^n, \mathbb{Z}) = H_q(M^n, \mathbb{Z}) = 0,$$

where $H_i(M^n, \mathbb{Z})$ is the i-th homology group of M^n with integer coefficients.

Then we state next sphere theorem as follows;

Theorem 3.5. *Let $M^n = N_T^p \times_f N_\theta^q$ be a compact oriented warped product pointwise semi-slant submanifold in $\mathbb{R}^{p+2q}$ with $n = p + q$, if the following condition is satisfied*

$$\Delta (\ln f) > \Big(1 - q + \csc^2\theta + \cot^2\theta\Big)\|\nabla (\ln f)\|^2, \tag{3.15}$$

Then M^{p+q} is homeomorphic to $\mathbb{S}^n$ when $p + q = 4$ and if $p + q = 3$, then M^{p+q} is homotopic to $\mathbb{S}^n$.

Proof. From Theorem 3.3, there does not exist stable integral p-currents in a warped product pointwise semi-slant submanifold M and their homology group zero for all positive integer p, q such that $p + q = n \neq 3$, that is, $H_q(M, \mathbb{Z}) = H_q(M, \mathbb{Z}) = 0$. Therefore, M is a homology sphere. By choosing a Riemannian universal covering of ambient manifold $\widetilde{M}$ of M together with compactness of M and (3.12), $\widetilde{M}$ is compact due to Myer's theorem. Thus the above justification can be imposed to the Riemannian universal covering $\widetilde{M}$ of M to show that M is a homology sphere. As $\widetilde{M}$ is a homology sphere with fundamental group $\pi_1(\widetilde{M}) = 0$ and it is also a homotopy sphere. Therefore, applying the generalized *Poincarẽ's conjecture* (Smale $n \geq 5$ [14], Freedman $n = 4$ [9]), we get that M is homotopy to the Euclidean sphere $\mathbb{S}^n$.

On other hand, if $n = 3$, using the result of R. S. Hamilton [10], we conclude that M^3 is a three dimensional spherical space form and hence $\pi_1(M) = 0$, it is homotopic to a Euclidean sphere $\mathbb{S}^3$. The proof of completed. □

From Theorem 3.4, there is a consequence of Theorem 3.5 as the sphere theorem for compact CR-warped products in $\mathbb{R}^{p+2q}$;

Theorem 3.6. *[17] Let $M^n = N_T^p \times_f N_\perp^q$ be a compact,oriented CR-warped product submanifold in $\mathbb{R}^{p+2q}$ with $n = p + q$, if the following is satisfied*

$$\Delta (\ln f) > (2 - q)\|\nabla (\ln f)\|^2, \tag{3.16}$$

then M^{p+q} is homeomorphic to $\mathbb{S}^n$ when $p + q = 4$ and if $p + q = 3$, then M^{p+q} is homotopic to $\mathbb{S}^n$.

Remark 5. Theorems 3.5 and 3.6 are topological sphere theorems for warped product submanifolds without the assumption that M^n is simply connected. Moreover, our results become more significant due to involving the new pinching conditions in terms of a pointwise slant function and the Laplacian of a warping function.

Remark 6. It noted that our results are generalized all results which were proved in [17].

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Conflict of interest

The authors declare that they have no competing interests.

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