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Research article

Convolution properties of meromorphically harmonic functions defined by a generalized convolution q-derivative operator

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Abstract: The goal of this article is to define, explore and analyze two new families of meromorphically harmonic functions by applying the concept of a certain generalized convolution q-operator along with the idea of convolution. We investigate convolution properties and sufficiency criteria for these families of meromorphically harmonic functions. Some of the interesting consequences of our investigation are also included.

Keywords: harmonic analytic functions; meromorphically harmonic functions; meromorphically harmonic starlike functions; meromorphically harmonic convex functions; generalized convolution q-derivative operator; functions of the Janowski type; convolution operator; coefficent estimates; inclusion relations
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1. Introduction and definitions

A new research area, which was initially developed by Clunie and Sheil-Small [14], is the subject of univalent harmonic functions (see also [31, 38]). The significance of such functions is attributed to their usages in the analysis of minimal surfaces as well as in problems relevant to applied mathematics. Hengartner and Schober [21] introduced and analyzed some specific types of harmonic functions which are given in the region $\widetilde{\mathbb{U}}$ in the complex z-plane $\mathbb{C}$, which is given by

$$\widetilde{\mathbb{U}} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| > 1\}.$$

Hengartner and Schober [21] proved, among other things, that a harmonic complex-valued and sense preserving univalent mapping f, defined in $\widetilde{\mathbb{U}}$ and such that $f(\infty) = \infty$, must satisfy the following representation:

$$f(z) = \mathfrak{L}_1(z) + \overline{\mathfrak{L}_2(z)} + Q \log|z|, \tag{1.1}$$

with $Q \in \mathbb{C}$ and, for $0 \leqq |\eta_2| < |\eta_1|$,

$$\mathfrak{L}_1(z) = \eta_1 z + \sum_{n=1}^{\infty} a_n z^{-n} \quad \text{and} \quad \mathfrak{L}_2(z) = \eta_2 \bar{z} + \sum_{n=1}^{\infty} b_n \bar{z}^{-n}.$$

In the year 1999, Jahangiri and Silverman [27] gave adequate coefficient criteria for which functions of the type (1.1) are univalent. They also provided necessary and sufficient coefficient criteria within certain constraints for functions to be harmonic and starlike, also see [39]. Later, Jahangiri [25] and Murugusundaramoorthy [33, 34] analyzed the families of meromorphically harmonic function in $\widetilde{\mathbb{U}}$. The authors in [12, 13] used the technique developed by Zou and his co-authors in [47] to examine the nature of meromorphically harmonic starlike functions with respect to symmetrical conjugate points in the punctured unit disk $\mathbb{U}^*$ given by

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}.$$

Particularly, in [13], a sharp approximation of the coefficient and a structural description of these functions were also determined. To understand the basics in a more clear way, let us represent the symbol $\mathcal{H}$ by the family of harmonic functions f which have the series form given by

$$f(z) = \lambda(z) + \overline{\mu(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} (a_n z^n + b_n \bar{z}^n) \qquad (z \in \mathbb{U}^*), \tag{1.2}$$

where the functions λ and μ are holomorphic in $\mathbb{U}^*$ and $\mathbb{U}$, respectively, with the following series forms:

$$\lambda(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (z \in \mathbb{U}^*) \qquad \text{and} \qquad \mu(z) = \sum_{n=1}^{\infty} b_n z^n \quad (z \in \mathbb{U}). \tag{1.3}$$

and

$$|a_n| \geq 1, \qquad |b_n| \geq 1, \qquad (n = 2, 3, \cdots).$$

We denote the family of all complex-valued functions $f \in \mathcal{H}$ by $\mathcal{M}_{\mathcal{H}}$, which are sense-preserving and univalent in $\mathbb{U}^*$. Indeed, if

$$\mu(z) \equiv 0 \qquad (z \in \mathbb{U}),$$

then

$$f(z) = \lambda(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (z \in \mathbb{U}^*).$$

In the view of this point, the function class $\mathcal{M}_{\mathcal{H}}$ coincides with the class $\mathcal{M}$ of normalized holomorphic univalent functions in $\mathbb{U}^*$. The above-mentioned papers have obviously opened up a new door for researchers to investigate further inputs in this area of geometric function theory. In this regard, we consider the collections of meromorphically harmonic-starlike and meromorphically harmonic-convex functions in $\mathbb{U}^*$ given, respectively, by

$$\mathcal{MS}^*_{\mathcal{H}} = \left\{ f : f \in \mathcal{M}_{\mathcal{H}} \quad \text{and} \quad \Re\left(\frac{\mathcal{D}_{\mathcal{H}} f(z)}{f(z)}\right) < 0 \qquad (z \in \mathbb{U}^*) \right\}$$

and

$$\mathcal{MS}^c_{\mathcal{H}} = \left\{ f : f \in \mathcal{M}_{\mathcal{H}} \quad \text{and} \quad \Re\left(\frac{\mathcal{D}_{\mathcal{H}}(\mathcal{D}_{\mathcal{H}} f(z))}{\mathcal{D}_{\mathcal{H}} f(z)}\right) < 0 \qquad (z \in \mathbb{U}^*) \right\},$$

where

$$\mathcal{D}_{\mathcal{H}} f(z) = z\lambda'(z) - \overline{z\mu'(z)}.$$

Now we give the definition of weak subordination in $\mathbb{U}$. For some details about subordinations for hamonic mappings, we refer [15, 16].

Acomplex valued function f in $\mathbb{U}$ is said to be a weakly subordinate to a complex valued function g in $\mathbb{U}$ and written as $f(z) \preceq g(z)$ or simply as $f \preceq g$ if $f(\infty) = g(\infty)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. If $f \preceq g$ and g is univalent in $\mathbb{U}$, then we consider a complex valued function $w(z) = g^{-1}(f(z))$, $z \in \mathbb{U}$ which maps $\mathbb{U}$ onto itself with $w(\infty) = \infty$. Conversely if $w(z) = g^{-1}(f(z))$ in $\mathbb{U}$ and maps $\mathbb{U}$ onto itself with $w(\infty) = \infty$, then $f \preceq g$.

This can be written in the following equivalence.

Lemma 1.1. *[17] A complex valued function f in $\mathbb{U}$ is weakly subordinate to a complex valued function g in $\mathbb{U}$ if and only if there exists a complex valued function w which maps $\mathbb{U}$ onto itself with $w(\infty) = \infty$ such that $f(z) = g(w(z))$, $z \in \mathbb{U}$.*

Many sub-families of meromorphically harmonic functions have also been introduced and investigated by several earlier researchers (see, for example, the works of Bostanci [10], Bostanci and Öztürk [11], Öztürk and Bostanci [35], Wang *et al.* [46], Al-Dweby and Darus [3], Al-Shaqsi and Darus [4], Ponnusamy and Rajasekaran [37], Ahuja and Jahangiri [2], Al-Zkeri and Al-Oboudi [5], Jahangiri *et al.* [26] and Stephen *et al.* [45]).

2. The generalized q-derivative operator

The theory of the Hadamard product (or convolution) is incredibly essential in the solution of many function-theoretic problems and, as a result of these realities, this technique becomes a significant part of the area of our study. The objective of this section is to examine the properties and consequences of the convolution for the two newly-defined families of meromorphically harmonic functions. For two functions $g_1,\ g_2 \in \mathcal{M}$ with their series expansions as follows:

$$g_1(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad \text{and} \qquad g_2(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n,$$

the Hadamard product (or convolution), which is denoted by $(g_1 * g_2)(z)$, is defined by

$$(g_1 * g_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n = (g_2 * g_1)(z) \qquad (z \in \mathbb{U}^*).$$

The potentially useful properties, which are mentioned below, can only be true if $g \in \mathcal{M}$:

$$g(z) * \frac{1}{z(1-z)} = g(z) \qquad \text{and} \qquad g(z) * \frac{1-2z}{z(1-z)^2} = -zg'(z). \tag{2.1}$$

We now consider a function $\mathfrak{R}_q$ (with $q \in \mathbb{C}$) of the following form:

$$\mathfrak{R}_q(z) = \frac{1}{z} - \frac{z}{(1-qz)(1-z)} = \frac{1}{z} - \sum_{n=1}^{\infty} [n]_q \; z^n \qquad (z \in \mathbb{U}^*),$$

where, as usual, the q-number $[\nu]_q$ $(\nu \in \mathbb{C})$ is given by

$$[\nu]_q = \begin{cases} \frac{1-q^\nu}{1-q} & (\nu, q \in \mathbb{C}) \\ \\ \sum\limits_{k=0}^{n-1} qk & (\nu = n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \end{cases}$$

where $\mathbb{N}$ denotes the set of natural numbers. Then, clearly, the function $\mathfrak{R}_q$ is meromorphically starlike for all complex numbers q such that $|q| < 1$. Also, one can easily see that, if $q \to 1-$, then the $\mathfrak{R}_q$ reduces as follows:

$$\lim_{q \to 1-} \mathfrak{R}_q(z) = \frac{1}{z} - \frac{z}{(1-z)^2} = \frac{1}{z} - \sum_{n=1}^{\infty} nz^n \qquad (z \in \mathbb{U}^*).$$

Moreover, the q-derivative (or q-difference) operator D_q of a function f defined on a subset of the complex space $\mathbb{C}$ is given by (see [23, 24])

$$\left(D_q f\right)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{2.2}$$

provided that the first-order derivative of the function $f(z)$ at $z = 0$ exists.

Now, by using the function $\mathfrak{R}_q$, we define the generalized convolution q-derivative operator $\mathfrak{D}_q$ for meromorphically functions $f \in \mathcal{M}$ by

$$\mathfrak{D}_q f(z) = -\frac{1}{z}\left[f(z) * \left(\frac{1}{z} - \frac{z}{(1-qz)(1-z)}\right)\right] = -\frac{1}{z}\left[f(z) * \mathfrak{R}_q(z)\right]. \tag{2.3}$$

We observe that, by taking $q = 1$ in the above q-derivative operator $\mathfrak{D}_q$, we achieve the ordinary derivative operator $\frac{d}{dz}$. Also, for $0 < q < 1$, we attain Jackson's q-derivative (or q-difference) D_q of the function f which is defined above by (2.2).

The q-derivative (or q-difference) operator D_q has fascinated and inspired many researchers due mainly to its use in various areas of the mathematical and physical sciences. Although the first article in which a link was established between geometric nature of analytic functions associate with the q-derivative operator D_q was initiated in [22] in 1990, yet the usage of q-calculus in geometric function theory as well as a solid and comprehensive foundation was given in 1989 in a book chapter by Srivastava [41]. After this development, many researchers introduced and studied various useful operators in q-calculus together with the applications of the associated convolution concepts. For example, Kanas and Răducanu [28] studied the q-derivative operator D_q and examined its behavior in geometric function theory. The operator D_q was generalized for multivalent analytic functions by Arif *et al* [9]. Analogous to these q-derivative operators, Arif *et al.* [8] and Khan *et al.* [30] contributed by introducing the q-integral operators for analytic and multivalent functions. Similarly, in the authors in [6] developed and analyzed some analogues of the q-derivative operators for meromorphically functions. Very recently, a survay-cum-expository review article on the subject of quantum (or q-) calculus and its various applications in geometric function theory was published by Srivastava [40] (see also [1, 7, 18–20, 29, 32, 36, 42–44]).

We next define an operator for the function $f \in \mathcal{M}_{\mathcal{H}}$ as follows. Let $\mathcal{D}_{\mathcal{H}}^{q,\tau} : \mathcal{M}_{\mathcal{H}} \to \mathcal{M}_{\mathcal{H}}$ be a linear operator defined for a function $f = \lambda + \overline{\mu} \in \mathcal{M}_{\mathcal{H}}$ by

$$\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z) = z\mathfrak{D}_q\lambda(z) + \tau\overline{z\mathfrak{D}_q\mu(z)} \qquad (|\tau| = 1), \tag{2.4}$$

where the convolution q-derivative operator $\mathfrak{D}_q$ is given by (2.3).

Making use of the operator $\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)$, we now introduce two families $\mathcal{MS}_{\mathcal{H}}[q, L, M]$ and $\mathcal{MK}_{\mathcal{H}}[q, L, M]$ of functions of the Janowski type for $|q| \leqq 1$ and $-1 \leqq M \leqq L \leqq 1$, which are defined below:

$$\mathcal{MS}_{\mathcal{H}}[q, L, M] = \left\{ f : f \in \mathcal{M}_{\mathcal{H}} \quad \text{and} \quad -\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)}{f(z)} \preceq \frac{1 + Lz}{1 + Mz} \qquad (z \in \mathbb{U}^*) \right\} \tag{2.5}$$

and

$$\mathcal{MK}_{\mathcal{H}}[q, L, M] = \left\{ f : f \in \mathcal{M}_{\mathcal{H}} \quad \text{and} \quad -\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau}\left(\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)\right)}{\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)} \preceq \frac{1 + Lz}{1 + Mz} \qquad (z \in \mathbb{U}^*) \right\}, \tag{2.6}$$

respectively. We note the following special cases:

(i) Taking $L = 1 - 2\xi$ and $M = -1$ in $\mathcal{MS}_{\mathcal{H}}[q, L, M]$ and $\mathcal{MK}_{\mathcal{H}}[q, L, M]$, we have

$$\mathcal{MS}_{\mathcal{H}}[q, 1 - 2\xi, -1] = \mathcal{MS}_{\mathcal{H}}(q, \xi)$$

and

$$\mathcal{MK}_{\mathcal{H}}[q, 1 - 2\xi, -1] = \mathcal{MK}_{\mathcal{H}}(q, \xi),$$

where

$$\mathcal{MS}_{\mathcal{H}}(q, \xi) = \left\{ f : f \in \mathcal{M}_{\mathcal{H}} \quad \text{and} \quad -\Re\left(\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)}{f(z)}\right) > \xi \qquad (0 \leqq \xi < 1;\ z \in \mathbb{U}^*) \right\} \tag{2.7}$$

and

$$\mathcal{MK}_{\mathcal{H}}(q,\xi)=\left\{f: f\in\mathcal{M}_{\mathcal{H}} \quad \text{and} \quad -\Re\left(\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau}\left(\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)\right)}{\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)}\right)>\xi \qquad (0\leqq\xi<1;\ z\in\mathbb{U}^{*})\right\}. \tag{2.8}$$

(ii). By setting $L=(1-2\xi)\beta$ and $M=-\beta$ in $\mathcal{MS}_{\mathcal{H}}[q,L,M]$ and $\mathcal{MK}_{\mathcal{H}}[q,L,M]$ with $0\leqq\xi<1$ and $0\leqq\beta<1$, we get the following function classes:

$$\mathcal{MS}_{\mathcal{H}}[q,(1-2\xi)\beta,-\beta]=\mathcal{MS}_{\mathcal{H}}(q,\xi,\beta)$$

and

$$\mathcal{MK}_{\mathcal{H}}[q,(1-2\xi)\beta,-\beta]=\mathcal{MK}_{\mathcal{H}}(q,\xi,\beta),$$

where

$$\mathcal{MS}_{\mathcal{H}}(q,\xi,\beta)=\left\{f: f\in\mathcal{M}_{\mathcal{H}} \quad \text{and} \quad \left|\frac{\left(\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)}{f(z)}\right)-1}{\left(\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)}{f(z)}\right)+1-2\xi}\right|<\beta \qquad (z\in\mathbb{U}^{*})\right\} \tag{2.9}$$

and

$$\mathcal{MK}_{\mathcal{H}}(q,\xi,\beta)=\left\{f: f\in\mathcal{M}_{\mathcal{H}} \quad \text{and} \quad \left|\frac{\left(\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau}\left(\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)\right)}{\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)}\right)-1}{\left(\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau}\left(\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)\right)}{\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)}\right)+1-2\xi}\right|<\beta \qquad (z\in\mathbb{U}^{*})\right\}. \tag{2.10}$$

We observe also that

$$\mathcal{MS}_{\mathcal{H}}[L,M]=\lim_{q\to 1-}\mathcal{MS}_{\mathcal{H}}[q,L,M]$$

and

$$\mathcal{MK}_{\mathcal{H}}[L,M]=\lim_{q\to 1-}\mathcal{MK}_{\mathcal{H}}[q,L,M],$$

where

$$\mathcal{MS}_{\mathcal{H}}[L,M]=\left\{f: f\in\mathcal{M}_{\mathcal{H}} \quad \text{and} \quad -\frac{\mathcal{D}_{\mathcal{H}}^{\tau}f(z)}{f(z)}\prec\frac{1+Lz}{1+Mz} \qquad (z\in\mathbb{U}^{*})\right\}$$

and

$$\mathcal{MK}_{\mathcal{H}}[L,M]=\left\{f: f\in\mathcal{M}_{\mathcal{H}} \quad \text{and} \quad -\frac{\mathcal{D}_{\mathcal{H}}^{\tau}\left(\mathcal{D}_{\mathcal{H}}^{\tau}f(z)\right)}{\mathcal{D}_{\mathcal{H}}^{\tau}f(z)}\prec\frac{1+Lz}{1+Mz} \qquad (z\in\mathbb{U}^{*})\right\}.$$

From the definitions of the newly-introduced classes, we have

$$f\in\mathcal{MK}_{\mathcal{H}}[q,L,M]\Leftrightarrow -z\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)\in\mathcal{MS}_{\mathcal{H}}[q,L,M]. \tag{2.11}$$

In this paper, we investigate a number of convolution properties and several coefficient estimates for functions in the classes $\mathcal{MS}_{\mathcal{H}}[q,L,M]$ and $\mathcal{MK}_{\mathcal{H}}[q,L,M]$, which are associated with the generalized convolution q-derivative operator $\mathcal{D}_{\mathcal{H}}^{q,\tau}$.

Unless otherwise mentioned, we assume throughout this paper that

$$-1\leqq M<L\leqq 1, \quad 0<q<1, \quad |\tau|=1, \quad \rho\geqq 0 \qquad \text{and} \qquad \beta<1.$$

3. Convolution properties

Our first result in this section, which is asserted by Theorem 3.1 below, provides a necessary and sufficient condition for a given function to be in the class $\mathcal{MS}_{\mathcal{H}}[q, L, M]$.

Theorem 3.1. *A function f defined by* (1.2) *is in the class $\mathcal{MS}_{\mathcal{H}}[q, L, M]$ if and only if*

$$\left[z\left\{f(z) * \left(\frac{1+\rho(1+(1-q)z)z-qz}{z(1-qz)(1-z)} + \tau\frac{1+\rho(1+(1-q)\bar{z})\bar{z}-q\bar{z}}{\bar{z}(1-q\bar{z})(1-\bar{z})}\right)\right\}\right] \neq 0 \tag{3.1}$$

for all ρ given by

$$\rho = \rho_\zeta := \frac{\zeta^{-1}+M}{L-M} \qquad (|\zeta| = 1)$$

and also for $\rho = 0$.

Proof. Let $f \in \mathcal{MS}_{\mathcal{H}}[q, L, M]$ have the series form (1.2). Then, from the concept of weak subordination, a function u exists, with the restrictions that $u(\infty) = \infty$ and $|u(z)| < 1$, such that

$$-\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)}{f(z)} = \frac{1+Lu(z)}{1+Mu(z)}, \tag{3.2}$$

which is equivalent to the following assertion:

$$-\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)}{f(z)} \neq \frac{1+L\zeta}{1+M\zeta} \qquad (z \in \mathbb{U};\ |\zeta| = 1) \tag{3.3}$$

or

$$z\left[-\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)(1+M\zeta) - f(z)(1+L\zeta)\right] \neq 0. \tag{3.4}$$

Since

$$\begin{aligned}\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z) &= z\mathcal{D}_q\lambda(z) + \tau\overline{z\mathcal{D}_q\mu(z)} \\ &= \left(\lambda(z) * \frac{(1-z)(1-qz)-z^2}{z(1-z)(1-qz)}\right) + \tau\left(\overline{\mu(z)} * \frac{(1-\bar{z})(1-q\bar{z})-\bar{z}^2}{\bar{z}(1-\bar{z})(1-q\bar{z})}\right),\end{aligned}$$

together with following identities:

$$\lambda(z) * \frac{1}{z(1-z)} = \lambda(z)$$

and

$$\overline{\mu(z)} * \frac{1}{\bar{z}(1-\bar{z})} = \overline{\mu(z)},$$

We find from (3.4) that

$$
\begin{aligned}
0 &\neq z\left[-\partial_{\mathcal{H}}^{q,\tau} f(z)(1+M\zeta) - f(z)(1+L\zeta)\right] \\
&= z\left[\left\{\left(\lambda(z) * \frac{(1-z)(1-qz)-z^2}{z(1-z)(1-qz)}\right)(1+M\zeta) - \left(\lambda(z) * \frac{1}{z(1-z)}\right)(1+L\zeta)\right\}\right. \\
&\quad \left. +\tau\left(\overline{\mu(z)} * \frac{(1-\overline{z})(1-q\overline{z})-\overline{z}^2}{\overline{z}(1-\overline{z})(1-q\overline{z})}\right)(1+M\zeta) - \left(\overline{\mu(z)} * \frac{1}{\overline{z}(1-\overline{z})}\right)(1+L\zeta)\right] \\
&= z\left[\lambda(z) * \left\{\frac{\left((1-z)(1-qz)-z^2\right)(1+M\zeta)-(1-qz)(1+L\zeta)}{z(1-z)(1-qz)}\right\} + \right. \\
&\quad \left. \overline{\mu(z)} * \left\{\tau\frac{\left((1-\overline{z})(1-q\overline{z})-\overline{z}^2\right)(1+M\zeta)-(1-q\overline{z})(1+L\zeta)}{\overline{z}(1-\overline{z})(1-q\overline{z})}\right\}\right] \\
&= z\zeta(M-L)\left[\lambda(z) * \left\{\frac{(1-qz)+z(1+(1-q)z)\rho}{z(1-z)(1-qz)}\right\}\right. \\
&\quad \left. +\tau\overline{\mu(z)} * \left\{\frac{(1-q\overline{z})+\overline{z}(1+(1-q)\overline{z})\rho}{\overline{z}(1-\overline{z})(1-q\overline{z})}\right\}\right] \\
&= \zeta(M-L)z\left[f(z) * \left\{\frac{(1-qz)+z(1+(1-q)z)\rho}{z(1-z)(1-qz)}\right.\right. \\
&\quad \left.\left. +\frac{(1-q\overline{z})+\overline{z}(1+(1-q)\overline{z})\rho}{\overline{z}(1-\overline{z})(1-q\overline{z})}\right\}\right],
\end{aligned}
$$

which leads to (3.1) and the necessary part of the proof of Theorem 3.1 is completed.

Conversely, we suppose that (3.1) holds true for $\rho = 0$. Then it follows that $zf(z) \neq 0$ for all $z \in \mathbb{U}$. Consequently, the function $\Phi(z)$ given by

$$\Phi(z) = -\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)}{f(z)}$$

is regular at $z_0 = 0$, with $\Phi(0) = 1$. It was shown that the assumption (3.1) is equivalent to (3.3), so we obtain

$$-\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau} f(z)}{f(z)} \neq \frac{1+L\zeta}{1+M\zeta} \qquad (z \in \mathbb{U};\ |\zeta| = 1). \tag{3.5}$$

Let we now put

$$\Psi(z) = \frac{1+Lz}{1+Mz} \qquad (z \in \mathbb{U}).$$

Then the relation (3.5) shows that $\Phi(\mathbb{U}) \cap \Psi(\partial\mathbb{U}) = \emptyset$. Thus, clearly, $\Phi(\mathbb{U})$ is connected to $\mathbb{C}\backslash\Psi(\partial\mathbb{U})$. Hence we have

$$\Phi(0) = \Psi(0),$$

which, together with the univalence of Ψ, shows that $\Phi(z) \preceq \Psi(z)$ in terms of the subordination in (3.2), that is, $f \in \mathcal{MS}_{\mathcal{H}}^{q}[L, M]$. This completes the proof of Theorem 3.1. □

Upon letting $q \to 1-$ in Theorem 3.1, we obtain the following result.

Corollary 3.2. *A function f defined by* (1.2) *is in the class $\mathcal{MS}_{\mathcal{H}}[L, M]$ if and only if*

$$z\left[f(z) * \left(\frac{1+(\rho-1)z}{z(1-z)^2} + \tau\frac{1+(\rho-1)\overline{z}}{\overline{z}(1-\overline{z})^2}\right)\right] \neq 0,$$

for all ρ given by

$$\rho = \frac{\zeta^{-1}+M}{L-M} \qquad (|\zeta| = 1)$$

and also for $\rho = 0$.

Putting $L = 1-2\xi\,(0 \leqq \xi < 1)$ and $M = -1$ in Theorem 3.1, we deduce the following result.

Corollary 3.3. *A function f defined by* (1.2) *is in the class $\mathcal{MS}_{\mathcal{H}}(q, \xi)$ if and only if*

$$z\left[f(z) * \left(\frac{1+\vartheta(1+(1-q)z)z-qz}{z(1-qz)(1-z)} + \tau\frac{1+\vartheta(1+(1-q)\overline{z})\overline{z}-q\overline{z}}{\overline{z}(1-q\overline{z})(1-\overline{z})}\right)\right] \neq 0$$

for all ϑ given by

$$\vartheta = \frac{\zeta^{-1}-1}{2(1-\xi)} \qquad (|\zeta| = 1;\ 0 \leqq \xi < 1)$$

and also for $\vartheta = 0$.

By letting $q \to 1-$ in Corollary 3.3, we have the following result.

Corollary 3.4. *A function f defined by* (1.2) *is in the class $\mathcal{MS}_{\mathcal{H}}(\xi)$ if and only if*

$$z\left[f(z) * \left(\frac{1+(\vartheta-1)z}{z(1-z)^2} + \tau\frac{1+(\vartheta-1)\overline{z}}{\overline{z}(1-\overline{z})^2}\right)\right] \neq 0$$

for all ϑ given by

$$\vartheta = \frac{\zeta^{-1}-1}{2(1-\xi)} \qquad (|\zeta| = 1;\ 0 \leqq \xi < 1)$$

and also for $\vartheta = 0$.

Our next result in this section, which is asserted by Theorem 3.5 below, provides a necessary and sufficient condition for a given function to be in the class $\mathcal{MK}_{\mathcal{H}}[q, L, M]$.

Theorem 3.5. *A function f defined by* (1.2) *is in the class $\mathcal{MK}_{\mathcal{H}}[q, L, M]$ if and only if*

$$z\left[f(z) * \left(\frac{\left(1-q^2z\right)(1-(q+1)z)-\rho q\left(2-q^2+qz\right)z^2}{qz(1-z)(1-qz)(1-q^2z)}\right.\right.$$
$$\left.\left.+\tau\,\frac{\left(1-q^2\overline{z}\right)(1-(q+1)\overline{z})-\rho q\left(2-q^2+q\overline{z}\right)\overline{z}^2}{q\overline{z}(1-\overline{z})(1-q\overline{z})(1-q^2\overline{z})}\right)\right] \neq 0$$

for all ρ given by

$$\rho = \frac{\zeta^{-1}+M}{L-M} \qquad (|\zeta| = 1)$$

and also for $\rho = 0$.

Proof. First of all, we suppose that the function f defined by (1.2) is in the class $\mathcal{MK}_{\mathcal{H}}[q,L,M]$ if it satisfies the condition (2.6) or, equivalently,

$$-\frac{\mathcal{D}_{\mathcal{H}}^{q,\tau}\left(\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)\right)}{\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)} \neq \frac{1+L\zeta}{1+M\zeta}.$$

By setting

$$\chi(z) = \frac{1+\rho(1+(1-q)z)z-qz}{z(1-qz)(1-z)},$$

we note that

$$-z\mathcal{D}_q\chi(z) = \frac{\left(1-q^2z\right)(1-(q+1)z)-\rho q\left(2-q^2+qz\right)z^2}{qz(1-z)(1-qz)(1-q^2z)}.$$

We also recall the following identity:

$$\left[-z\mathcal{D}_q\lambda(z)\right]*\chi(z) = \lambda(z)*\left[-z\mathfrak{D}_q\chi(z)\right]$$

and the fact that

$$f(z)\in\mathcal{MK}_{\mathcal{H}}[q,L,M] \Longleftrightarrow -z\mathcal{D}_{\mathcal{H}}^{q,\tau}f(z)\in\mathcal{MS}_{\mathcal{H}}[q,L,M].$$

Hence, clearly, the result asserted by Theorem 3.1, would follow by using the above relations in conjunction with Theorem 3.1. □

If we let $q\to 1-$ in Theorem 3.5, we obtain the following result.

Corollary 3.6. *A function f defined by* (1.2) *is in the class $\mathcal{MK}_{\mathcal{H}}[L,M]$ if and only if*

$$z\left[f(z)*\left(\frac{1-(1+\rho(1+z)z)z}{z(1-z)^3}+\tau\frac{1-(1+\rho(1+\bar{z})\bar{z})\bar{z}}{\bar{z}(1-\bar{z})^3}\right)\right]\neq 0$$

for all ρ given by

$$\rho = \frac{\zeta^{-1}+M}{L-M} \qquad (|\zeta|=1)$$

and also for $\rho = 0$.

Putting $L = 1-2\xi$ $(0\leqq\xi<1)$ and $M=-1$ in Theorem 3.5, we obtain the following corollary.

Corollary 3.7. *A function f defined by* (1.2) *is in the class $\mathcal{MK}_{\mathcal{H}}(q,\xi)$ $(0\leqq\xi<1)$ if and only if*

$$z\left[f(z)*\left(\frac{\left(1-q^2z\right)(1-(q+1)z)-\vartheta q\left(2-q^2+qz\right)z^2}{qz(1-z)(1-qz)(1-q^2z)}\right.\right.$$
$$\left.\left.+\tau\frac{\left(1-q^2\bar{z}\right)(1-(q+1)\bar{z})-\rho q\left(2-q^2+q\bar{z}\right)\bar{z}^2}{q\bar{z}(1-\bar{z})(1-q\bar{z})(1-q^2\bar{z})}\right)\right]\neq 0$$

for all ϑ given by

$$\vartheta = \frac{\zeta^{-1}-1}{2(1-\xi)} \qquad (|\zeta|=1;\ 0\leqq\xi<1)$$

and also for $\vartheta = 0$.

Letting $q \to 1-$ in Corollary 3.7, we obtain the following result.

Corollary 3.8. *A function f defined by* (1.2) *is in the class $\mathcal{MK}_{\mathcal{H}}(\xi)$* $(0 \leqq \xi < 1)$ *if and only if*

$$z\left[f(z) * \left(\frac{1-((1+\vartheta(1+z)z))z}{z(1-z)^3} + \tau\frac{1-((1+\vartheta(1+\overline{z})\overline{z}))\overline{z}}{\overline{z}(1-\overline{z})^3}\right)\right] \neq 0$$

for all ϑ given by

$$\vartheta = \frac{\zeta^{-1}-1}{2(1-\xi)} \qquad (|\zeta| = 1;\ 0 \leqq \xi < 1)$$

and also for $\vartheta = 0$.

Theorem 3.9 below provides a necessary and sufficient condition for a given function to be in the class $\mathcal{MS}_{\mathcal{H}}[q, L, M]$.

Theorem 3.9. *A function f defined by* (1.2) *is in the class $\mathcal{MS}_{\mathcal{H}}[q, L, M]$ if and only if*

$$1 - \sum_{n=1}^{\infty} \frac{[n]_q\left(\zeta^{-1}+M\right)+\zeta^{-1}+L}{M-L}\left(a_n z^{n+1} + \tau\overline{b_n z^{n+1}}\right) \neq 0 \qquad (z \in \mathbb{U}). \tag{3.6}$$

Proof. From Theorem 3.1, we know that $f \in \mathcal{MS}_{\mathcal{H}}[q, L, M]$ if and only if

$$z\left[f(z) * \left(\frac{1+\rho\left((1-qz)z+z^2\right)-qz}{z(1-qz)(1-z)} + \tau\frac{1+\rho\left((1-q\overline{z})\overline{z}+\overline{z}^2\right)-q\overline{z}}{\overline{z}(1-q\overline{z})(1-\overline{z})}\right)\right] \neq 0 \tag{3.7}$$

for all ρ given by

$$\rho = \rho_\zeta := \frac{\zeta^{-1}+M}{L-M} \qquad (|\zeta| = 1)$$

and also for $\rho = 0$.

The left-hand side of (3.7) can be written as follows:

$$\begin{aligned}
&z\left[\lambda(z) * \frac{1-qz+\rho\left((1-qz)z+z^2\right)}{z(1-qz)(1-z)} + \tau\overline{\mu(z)} * \frac{1-q\overline{z}+\rho\left((1-q\overline{z})\overline{z}+\overline{z}^2\right)}{\overline{z}(1-q\overline{z})(1-\overline{z})}\right] \\
&= z\left[\lambda(z) * \left(\frac{1}{z(1-z)} + \rho\left(\frac{z}{(1-qz)(1-z)} + \frac{1}{(1-z)}\right)\right)\right. \\
&\qquad \left. + \tau\overline{\mu(z)} * \left(\frac{1}{\overline{z}(1-\overline{z})} + \rho\left(\frac{\overline{z}}{(1-q\overline{z})(1-\overline{z})} + \frac{1}{(1-\overline{z})}\right)\right)\right] \\
&= z\left[\lambda(z) + \rho\left(z\mathcal{D}_q\lambda(z) + \lambda(z)\right) + \tau\left(\overline{\mu(z)} + \rho\left(\overline{z}\mathcal{D}_q\overline{\mu(z)} + \overline{\mu(z)}\right)\right)\right] \\
&= z\left[\frac{1}{z} + \sum_{n=1}^{\infty}\left(1+\rho\left([n]_q+1\right)\right)a_n z^n + \tau\sum_{n=1}^{\infty}\left(1+\rho\left([n]_q+1\right)\right)\overline{b_n z^n}\right] \\
&= 1 + \sum_{n=1}^{\infty}\left(1+\rho\left([n]_q+1\right)\right)a_n z^{n+1} + \tau\sum_{n=1}^{\infty}\left(1+\rho\left([n]_q+1\right)\right)\overline{b_n z^{n+1}}.
\end{aligned}$$

This evidently completes our proof of the result asserted by Theorem 3.9. □

Upon letting $q \to 1-$ in Theorem 3.9, we obtain the following result.

Corollary 3.10. *A function f defined by* (1.2) *is in the class $\mathcal{MS}_{\mathcal{H}}[L, M]$ if and only if*

$$1 - \sum_{n=1}^{\infty} \frac{n\left(\zeta^{-1} + M\right) + \zeta^{-1} + L}{M - L}\left(a_n z^{n+1} + \tau \overline{b_n z^{n+1}}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

Putting $L = 1 - 2\xi$ $(0 \leqq \xi < 1)$ and $M = -1$ in Theorem 3.9, we deduce the following result.

Corollary 3.11. *A function f defined by* (1.2) *is in the class $\mathcal{MS}_{\mathcal{H}}(q, \xi)$ if and only if*

$$1 + \sum_{n=1}^{\infty} \frac{[n]_q\left(\zeta^{-1} - 1\right) + \zeta^{-1} + 1 - 2\xi}{2(1 - \xi)}\left(a_n z^{n+1} + \tau \overline{b_n z^{n+1}}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

Taking $q \to 1-$ in Corollary 3.11, we obtain the following result.

Corollary 3.12. *A function f defined by* (1.2) *is in the class $\mathcal{MS}_{\mathcal{H}}(\xi)$ if and only if*

$$1 + \sum_{n=1}^{\infty} \frac{n\left(\zeta^{-1} - 1\right) + \zeta^{-1} + 1 - 2\xi}{2(1 - \xi)}\left(a_n z^{n+1} + \tau \overline{b_n z^{n+1}}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

Theorem 3.13. *A function f defined by* (1.2) *is in the class $\mathcal{MK}_{\mathcal{H}}[q, L, M]$ if and only if*

$$1 - \sum_{n=1}^{\infty} [n]_q \frac{[n]_q\left(\zeta^{-1} + M\right) + \zeta^{-1} + L}{M - L}\left(a_n z^{n+1} + \tau \overline{b_n z^{n+1}}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

Taking $q \to 1-$ in Theorem 3.13, we obtain the following result.

Corollary 3.14. *A function f defined by* (1.2)*, is in the class $\mathcal{MK}_{\mathcal{H}}[L, M]$ if and only if*

$$1 - \sum_{n=1}^{\infty} n \frac{n\left(\zeta^{-1} + M\right) + \zeta^{-1} + L}{M - L}\left(a_n z^{n+1} + \tau \overline{b_n z^{n+1}}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

Putting $L = 1 - 2\xi$, $(0 \leqq \xi < 1)$ and $M = -1$ in Theorem 3.13, we obtain the following result.

Corollary 3.15. *A function f defined by* (1.2) *is in the class $\mathcal{MK}_{\mathcal{H}}(q, \xi)$ if and only if*

$$1 + \sum_{n=1}^{\infty} [n]_q \frac{[n]_q\left(\zeta^{-1} - 1\right) + \zeta^{-1} + 1 - 2\xi}{2(1 - \xi)}\left(a_n z^{n+1} + \tau \overline{b_n z^{n+1}}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

Letting $q \to 1-$ in Corollary 3.15, we obtain the following result.

Corollary 3.16. *A function f defined by* (1.2) *is in the class $\mathcal{MK}_{\mathcal{H}}(\xi)$ if and only if*

$$1 + \sum_{n=1}^{\infty} n \frac{n\left(\zeta^{-1} - 1\right) + \zeta^{-1} + 1 - 2\xi}{2(1 - \xi)}\left(a_n z^{n+1} + \tau \overline{b_n z^{n+1}}\right) \neq 0 \qquad (z \in \mathbb{U}).$$

4. Coefficient estimates and inclusion relations

We now determine the coefficient estimates and inclusion relations for functions belonging to the classes $\mathcal{MS}_{\mathcal{H}}[q,L,M]$ and $\mathcal{MK}_{\mathcal{H}}[q,L,M]$.

Theorem 4.1. *If a function f defined by* (1.2) *satisfies the following inequality*:

$$\sum_{n=1}^{\infty}\left([n]_q(1+|M|)+1+L\right)(|a_n|+|b_n|) \leqq L-M,$$

then $f \in \mathcal{MS}_{\mathcal{H}}[q,L,M]$.

Proof. From (3.6), we have

$$\begin{aligned}
&\left|1-\sum_{n=1}^{\infty}\frac{[n]_q\left(\zeta^{-1}+M\right)+\zeta^{-1}+L}{M-L}\left(a_n z^{n+1}+\tau\overline{b_n z^{n+1}}\right)\right| \\
&\quad > 1-\sum_{n=1}^{\infty}\frac{[n]_q\left|\left(\zeta^{-1}+M\right)+\zeta^{-1}+L\right|}{L-M}(|a_n|+|b_n|) \\
&\quad > 1-\sum_{n=1}^{\infty}\frac{[n]_q(1+|M|)+1+L}{L-M}(|a_n|+|b_n|) \geqq 0,
\end{aligned}$$

which completes the proof of Theorem 4.1. □

By letting $q \to 1-$ in Theorem 4.1, we obtain the following result.

Corollary 4.2. *If a function f defined by* (1.2) *satisfies the following inequality*:

$$\sum_{n=1}^{\infty}(n(1+|M|)+1+L)(|a_n|+|b_n|) \leqq L-M,$$

then $f \in \mathcal{MS}_{\mathcal{H}}[L,M]$.

Putting $L = 1-2\xi$ $(0 \leqq \xi < 1)$ and $M = -1$ in Theorem 4.1, we obtain the following result.

Corollary 4.3. *If the function f defined by* (1.2) *satisfies the following inequality*:

$$\sum_{n=1}^{\infty}\left([n]_q+1-\xi\right)(|a_n|+|b_n|) \leqq 1-\xi,$$

then $f \in \mathcal{MS}_{\mathcal{H}}(q,\xi)$.

Letting $q \to 1-$ in Corollary 4.3, we obtain the following result.

Corollary 4.4. *If a function f defined by* (1.2) *satisfies the following inequality*:

$$\sum_{n=1}^{\infty}(n+1-\xi)(|a_n|+|b_n|) \leqq 1-\xi,$$

then $f \in \mathcal{MS}_{\mathcal{H}}(\xi)$.

Similarly, we can prove the next result (Theorem 4.5 below).

Theorem 4.5. *If the function f defined by* (1.2) *satisfies the following inequality*:

$$\sum_{n=1}^{\infty} [n]_q \left([n]_q (1 + |M|) + 1 + L\right) (|a_n| + |b_n|) \leqq L - M,$$

then $f \in \mathcal{MK_H}[q, L, M]$.

Upon letting $q \to 1-$ in Theorem 4.5, we obtain the following result.

Corollary 4.6. *If the function f defined by* (1.2) *satisfies the following inequality*:

$$\sum_{n=1}^{\infty} n (n (1 + |M|) + 1 + L) (|a_n| + |b_n|) \leqq L - M,$$

then $f \in \mathcal{MK_H}[L, M]$.

Putting $L = 1 - 2\xi$ $(0 \leqq \xi < 1)$ and $M = -1$ in Theorem 4.5, we obtain the following corollary.

Corollary 4.7. *If the function f defined by* (1.2) *satisfies the following inequality*:

$$\sum_{n=1}^{\infty} [n]_q \left([n]_q + 1 - \xi\right) (|a_n| + |b_n|) \leqq 1 - \xi,$$

then $f \in \mathcal{MK_H}(q, \xi)$.

If we let $q \to 1-$ in Corollary 4.7, we obtain the following result.

Corollary 4.8. *If the function f defined by* (1.2) *satisfies the following inequality*:

$$\sum_{n=1}^{\infty} n (n + 1 - \xi) (|a_n| + |b_n|) \leqq 1 - \xi,$$

then $f \in \mathcal{MK_H}(\xi)$.

5. Conclusions

In this paper, we have introduced the generalized convolution q-derivative operator $\mathfrak{D}_q$, which is defined by

$$\mathfrak{D}_q f(z) = -\frac{1}{z}\left[f(z) * \left(\frac{1}{z} - \frac{z}{(1 - qz)(1 - z)}\right)\right],$$

where $q \in \mathbb{C}$ and $|q| \leqq 1$. By letting $q \to 1-$, this generalized convolution q-operator $\mathfrak{D}_q$ takes the form of the ordinary derivative. Also, for $0 < q < 1$, we obtain the q-analog of derivative operator. By applying this operator, we have defined a corresponding operator for meromorphically harmonic functions and introduced some subclasses of meromorphically harmonic starlike and meromorphically harmonic convex functions and have studied a number of properties and results for functions belonging to each of these function classes.

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [41, pp. 351–352] and [40, p. 328]). Moreover, in this recently-published survey-cum-expository review article by Srivastava [40], the so-called (p, q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant (see, for details, [40, p. 340]). This observation by Srivastava [40] will indeed apply also to any attempt to produce the rather straightforward (p, q)-variations of the results which we have presented in this paper.

Conflicts of interest

The authors declare no conflicts of interest.

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