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Research article

On a nonlinear mixed-order coupled fractional differential system with new integral boundary conditions

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Abstract: We present the criteria for the existence of solutions for a nonlinear mixed-order coupled fractional differential system equipped with a new set of integral boundary conditions on an arbitrary domain. The modern tools of the fixed point theory are employed to obtain the desired results, which are well-illustrated by numerical examples. A variant problem dealing with the case of nonlinearities depending on the cross-variables (unknown functions) is also briefly described.

Keywords: fractional differential equations; Caputo fractional derivative; systems; existence; fixed point theorems
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1. Introduction

It is well known that the classical boundary conditions cannot describe certain peculiarities of physical, chemical, or other processes occurring within the domain. In order to overcome this situation, the concept of nonlocal conditions was introduced by Bicadze and Samarskiĭ [1]. These conditions are successfully employed to relate the changes happening at nonlocal positions or segments within the given domain to the values of the unknown function at end points or boundary of the domain. For a detailed account of nonlocal boundary value problems, for example, we refer the reader to the articles [2–6] and the references cited therein.

Computational fluid dynamics (CFD) technique directly deals with the boundary data [7]. In case of fluid flow problems, the assumption of circular cross-section is not justifiable for curved structures. The idea of integral boundary conditions serves as an effective tool to describe the boundary data on arbitrary shaped structures. One can find application of integral boundary conditions in the study of thermal conduction, semiconductor, and hydrodynamic problems [8–10]. In fact, there are numerous applications of integral boundary conditions in different disciplines such as chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. [11–13]. Also, integral boundary conditions facilitate to regularize ill-posed parabolic backward problems, for example, mathematical models for bacterial self-regularization [14]. Some recent results on boundary value problems with integral boundary conditions can be found in the articles [15–19] and the references cited therein.

The non-uniformities in form of points or sub-segments on the heat sources can be relaxed by using the integro multi-point boundary conditions, which relate the sum of the values of the unknown function (e.g., temperature) at the nonlocal positions (points and sub-segments) and the value of the unknown function over the given domain. Such conditions also find their utility in the diffraction problems when scattering boundary consists of finitely many sub-strips (finitely many edge-scattering problems). For details and applications in engineering problems, for instance, see [20–23].

The subject of fractional calculus has emerged as an important area of research in view of extensive applications of its tools in scientific and technical disciplines. Examples include neural networks [24, 25], immune systems [26], chaotic synchronization [27, 28], Quasi-synchronization [29, 30], fractional diffusion [31–33], financial economics [34], ecology [35], etc. Inspired by the popularity of this branch of mathematical analysis, many researchers turned to it and contributed to its different aspects. In particular, fractional order boundary value problems received considerable attention. For some recent results on fractional differential equations with multi-point and integral boundary conditions, see [36, 37]. More recently, in [38, 39], the authors analyzed boundary value problems involving Riemann-Liouville and Caputo fractional derivatives respectively. A boundary value problem involving a nonlocal boundary condition characterized by a linear functional was studied in [40]. In a recent paper [41], the existence results for a dual anti-periodic boundary value problem involving nonlinear fractional integro-differential equations were obtained.

On the other hand, fractional differential systems also received considerable attention as such systems appear in the mathematical models associated with physical and engineering processes [42–46]. For theoretical development of such systems, for instance, see the articles [47–52].

Motivated by aforementioned applications of nonlocal integral boundary conditions and fractional differential systems, in this paper, we study a nonlinear mixed-order coupled fractional differential system equipped with a new set of nonlocal multi-point integral boundary conditions on an arbitrary domain given by

$$\begin{cases} {}^cD^{\xi}_{a^+}x(t) = \varphi(t, x(t), y(t)),\ 0 < \xi \le 1,\ \ t \in [a, b], \\ {}^cD^{\zeta}_{a^+}y(t) = \psi(t, x(t), y(t)),\ 1 < \zeta \le 2,\ \ t \in [a, b], \\ px(a) + qy(b) = y_0 + x_0 \displaystyle\int_a^b (x(s) + y(s))ds, \\ y(a) = 0,\ y'(b) = \displaystyle\sum_{i=1}^{m} \delta_i x(\sigma_i) + \lambda \int_\tau^b x(s)ds, \\ a < \sigma_1 < \sigma_2 < \ldots < \sigma_m < \tau < b, \end{cases} \tag{1.1}$$

where ${}^cD^{\chi}$ is Caputo fractional derivative of order $\chi \in \{\xi, \zeta\}, \varphi, \psi : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions, $p, q, \delta_i, x_0, y_0 \in \mathbb{R}, i = 1, 2, \ldots, m.$

Here we emphasize that the novelty of the present work lies in the fact that we introduce a coupled system of fractional differential equations of different orders on an arbitrary domain equipped with coupled nonlocal multi-point integral boundary conditions. It is imperative to notice that much of the work related to the coupled systems of fractional differential equations deals with the fixed domain. Thus our results are more general and contribute significantly to the existing literature on the topic. Moreover, several new results appear as special cases of the work obtained in this paper.

We organize the rest of the paper as follows. In Section 2, we present some basic concepts of fractional calculus and solve the linear version of the problem (1.1). Section 3 contains the main results. Examples illustrating the obtained results are presented in Section 4. Section 5 contains the details of a variant problem. The paper concludes with some interesting observations.

2. Preliminaries

Let us recall some definitions from fractional calculus related to our study [53].

Definition 2.1. *The Riemann–Liouville fractional integral of order $\alpha \in \mathbb{R}$ ($\alpha > 0$) for a locally integrable real-valued function ϱ of order $\alpha \in \mathbb{R}$, denoted by $I^{\alpha}_{a^+}\varrho$, is defined as*

$$I^{\alpha}_{a^+}\varrho(t) = \left(\varrho * \frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\varrho(s)ds, \ -\infty \leq a < t < b \leq +\infty,$$

where Γ denotes the Euler gamma function.

Definition 2.2. *The Riemann–Liouville fractional derivative $D^{\alpha}_{a^+}\varrho$ of order $\alpha \in]m-1, m]$, $m \in \mathbb{N}$ is defined as*

$$D^{\alpha}_{a^+}\varrho(t) = \frac{d^m}{dt^m} I^{1-\alpha}_{a^+}\varrho(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-s)^{m-1-\alpha}\varrho(s)ds, \ -\infty \leq a < t < b \leq +\infty,$$

while the Caputo fractional derivative ${}^cD^{\alpha}_{a^+}u$ is defined as

$${}^cD^{\alpha}_{a^+}\varrho(t) = D^{\alpha}_{a^+}\left[\varrho(t) - \varrho(a) - \varrho'(a)\frac{(t-a)}{1!} - \ldots - \varrho^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!}\right],$$

for $\varrho, \varrho^{(m)} \in L^1[a,b]$.

Remark 2.1. *The Caputo fractional derivative ${}^cD^{\alpha}_{a^+}\varrho$ is also defined as*

$${}^cD^{\alpha}\varrho(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}\varrho^{(m)}(s)ds.$$

In the following lemma, we obtain the integral solution of the linear variant of the problem (1.1).

Lemma 2.1. *Let $\Phi, \Psi \in C([a,b],\mathbb{R})$. Then the unique solution of the system*

$$\begin{cases} {}^cD^{\xi}_{a^+}x(t) = \Phi(t),\ 0<\xi\le 1,\ \ t\in[a,b], \\ {}^cD^{\zeta}_{a^+}y(t) = \Psi(t),\ 1<\zeta\le 2,\ \ t\in[a,b], \\ px(a)+qy(b) = y_0 + x_0\displaystyle\int_a^b (x(s)+y(s))ds, \\ y(a)=0,\ \ y'(b) = \displaystyle\sum_{i=1}^m \delta_i x(\sigma_i) + \lambda\int_\tau^b x(s)ds, \\ \ \ a<\sigma_1<\sigma_2<\ldots<\sigma_m<\tau<b, \end{cases} \tag{2.1}$$

is given by a pair of integral equations

$$\begin{aligned} x(t) &= I^{\xi}_{a^+}\Phi(t) + \frac{1}{\Delta}\Big\{ y_0 + x_0\int_a^b \frac{(b-s)^{\xi}}{\Gamma(\xi+1)}\Phi(s)ds \\ &\quad + \int_a^b \Big(x_0\frac{(b-s)^{\zeta}}{\Gamma(\zeta+1)} + \varepsilon_1\frac{(b-s)^{\zeta-2}}{\Gamma(\zeta-1)} - q\frac{(b-s)^{\xi-1}}{\Gamma(\xi)}\Big)\Psi(s)ds \\ &\quad -\varepsilon_1\sum_{i=1}^m \delta_i\int_a^{\sigma_i}\frac{(\sigma_i-s)^{\xi-1}}{\Gamma(\xi)}\Phi(s)ds - \varepsilon_1\lambda\int_\tau^b\int_a^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}\Phi(u)duds\Big\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} y(t) &= I^{\zeta}_{a^+}\Psi(t) + \frac{(t-a)}{\Delta}\Big\{ \varepsilon_2 y_0 + \varepsilon_2 x_0\int_a^b \frac{(b-s)^{\xi}}{\Gamma(\xi+1)}\Phi(s)ds \\ &\quad + \int_a^b \Big(\varepsilon_2 x_0\frac{(b-s)^{\zeta}}{\Gamma(\zeta+1)} - \varepsilon_2 q\frac{(b-s)^{\xi-1}}{\Gamma(\xi)} - \varepsilon_3\frac{(b-s)^{\zeta-2}}{\Gamma(\zeta-1)}\Big)\Psi(s)ds \\ &\quad +\varepsilon_3\sum_{i=1}^m \delta_i\int_a^{\sigma_i}\frac{(\sigma_i-s)^{\xi-1}}{\Gamma(\xi)}\Phi(s)ds + \varepsilon_3\lambda\int_\tau^b\int_a^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}\Phi(u)duds\Big\}, \end{aligned} \tag{2.3}$$

where

$$\varepsilon_1 = q(b-a) - x_0\frac{(b-a)^2}{2},\ \ \varepsilon_2 = \sum_{i=1}^m \delta_i + \lambda(b-\tau),\ \ \varepsilon_3 = p-(b-a)x_0, \tag{2.4}$$

and it is assumed that

$$\Delta = \varepsilon_3 + \varepsilon_2\varepsilon_1 \neq 0. \tag{2.5}$$

Proof. Applying the integral operators $I^{\xi}_{a^+}$ and $I^{\zeta}_{a^+}$ respectively on the first and second fractional differential equations in (2.1), we obtain

$$x(t) = I^{\xi}_{a^+}\Phi(t) + c_1 \ \text{ and }\ y(t) = I^{\zeta}_{a^+}\Psi(t) + c_2 + c_3(t-a), \tag{2.6}$$

where $c_i\in\mathbb{R}, i=1,2,3$ are arbitrary constants. Using the condition $y(a)=0$ in (2.6), we get $c_2=0$. Making use of the conditions $px(a)+qy(b) = y_0 + x_0\int_a^b(x(s)+y(s))ds$ and $y'(b) = \sum_{i=1}^m \delta_i x(\sigma_i) + \lambda\int_\tau^b x(s)ds$ in (2.6) after inserting $c_2=0$ in it leads to the following system of equations in the unknown constants c_1 and c_3:

$$\Big(p-(b-a)x_0\Big)c_1 + \Big(q(b-a) - x_0\frac{(b-a)^2}{2}\Big)c_3 = y_0 + x_0\int_a^b\frac{(b-r)^{\xi}}{\Gamma(\xi+1)}\Phi(r)dr$$

$$+ x_0 \int_a^b \frac{(b-r)^{\zeta}}{\Gamma(\zeta+1)} \Psi(r) dr - q I_{a^+}^{\zeta} \Psi(b), \tag{2.7}$$

$$\Big(\sum_{i=1}^{m} \delta_i + \lambda(b-\tau)\Big) c_1 - c_3 = I_{a^+}^{\zeta-1} \Psi(b) - \sum_{i=1}^{m} \delta_i I_{a^+}^{\xi} \Phi(\sigma_i) - \lambda \int_\tau^b I_{a+}^{\xi} \Phi(s) ds. \tag{2.8}$$

Solving (2.7) and (2.8) for c_1 and c_3 and using the notation (2.5), we find that

$$\begin{aligned} c_1 &= \frac{1}{\Delta}\Big\{\varepsilon_1\Big(I_{a^+}^{\zeta-1}\Psi(b) - \sum_{i=1}^{m} \delta_i I_{a^+}^{\xi}\Phi(\sigma_i) - \lambda \int_\tau^b I_{a^+}^{\xi}\Phi(s)ds\Big) + y_0 + x_0 \int_a^b \frac{(b-r)^{\xi}}{\Gamma(\xi+1)}\Phi(r)dr \\ &+ x_0 \int_a^b \frac{(b-r)^{\zeta}}{\Gamma(\zeta+1)}\Psi(r)dr - q I_{a^+}^{\xi}\Psi(b)\Big\}, \\ c_3 &= \frac{1}{\Delta}\Big\{\varepsilon_2\Big(y_0 + \int_a^b \frac{(b-r)^{\xi}}{\Gamma(\xi+1)}\Phi(r)dr + x_0 \int_a^b \frac{(b-r)^{\zeta}}{\Gamma(\zeta+1)}\Psi(r)dr - q I_{a^+}^{\xi}\Psi(b)\Big) \\ &- \varepsilon_3\Big(I_{a^+}^{\zeta-1}\Psi(b) - \sum_{i=1}^{m} \delta_i I_{a^+}^{\xi}\Phi(\sigma_i) - \lambda \int_\tau^b I_{a^+}^{\xi}\Phi(s)ds\Big)\Big\}. \end{aligned}$$

Inserting the values of c_1, c_2, and c_3 in (2.6) leads to the solution (2.2) and (2.3). One can obtain the converse of the lemma by direct computation. This completes the proof. □

3. Main results

Let $X = C([a,b],\mathbb{R})$ be a Banach space endowed with the norm $\|x\| = \sup\{|x(t)|, t \in [a,b]\}$.
In view of Lemma 2.1, we define an operator $T : X \times X \to X$ by:

$$T(x(t), y(t)) = (T_1(x(t), y(t)), T_2(x(t), y(t))),$$

where $(X \times X, \|(x,y)\|)$ is a Banach space equipped with norm $\|(x,y)\| = \|x\| + \|y\|$, $x, y \in X$,

$$\begin{aligned} T_1(x,y)(t) &= I_{a^+}^{\xi}\varphi(t, x(t), y(t)) + \frac{1}{\Delta}\Big(y_0 + x_0 \int_a^b \frac{(b-s)^{\xi}}{\Gamma(\xi+1)}\varphi(s, x(s), y(s))ds \\ &+ \int_a^b \rho_1(s)\psi(s, x(s), y(s))ds - \varepsilon_1 \sum_{i=1}^{m} \delta_i \int_a^{\sigma_i} \frac{(\sigma_i - s)^{\xi-1}}{\Gamma(\xi)}\varphi(s, x(s), y(s))ds \\ &- \varepsilon_1 \lambda \int_\tau^b \int_a^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}\varphi(u, x(u), y(u))duds\Big), \end{aligned}$$

$$\begin{aligned} T_2(x,y)(t) &= I_{a^+}^{\zeta}\psi(t, x(t), y(t)) + \frac{(t-a)}{\Delta}\Big(\varepsilon_2 y_0 + \varepsilon_2 x_0 \int_a^b \frac{(b-s)^{\xi}}{\Gamma(\xi+1)}\varphi(s, x(s), y(s))ds \\ &+ \int_a^b \rho_2(s)\psi(s, x(s), y(s))ds + \varepsilon_3 \sum_{i=1}^{m} \delta_i \int_a^{\sigma_i} \frac{(\sigma_i - s)^{\xi-1}}{\Gamma(\xi)}\varphi(s, x(s), y(s))ds \\ &+ \varepsilon_3 \lambda \int_\tau^b \int_a^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}\varphi(u, x(u), y(u))duds\Big), \end{aligned}$$

$$\begin{aligned}
\rho_1(s) &= x_0\frac{(b-s)^{\zeta}}{\Gamma(\zeta+1)}+\varepsilon_1\frac{(b-s)^{\zeta-2}}{\Gamma(\zeta-1)}-q\frac{(b-s)^{\xi-1}}{\Gamma(\xi)},\\
\rho_2(s) &= \varepsilon_2 x_0\frac{(b-s)^{\zeta}}{\Gamma(\zeta+1)}-\varepsilon_2 q\frac{(b-s)^{\xi-1}}{\Gamma(\xi)}-\varepsilon_3\frac{(b-s)^{\zeta-2}}{\Gamma(\zeta-1)}.
\end{aligned}$$

For computational convenience we put:

$$\begin{aligned}
L_1 &= \frac{(b-a)^{\xi}}{\Gamma(\xi+1)}+\frac{1}{|\Delta|}\Bigg(|x_0|\frac{(b-a)^{\xi+1}}{\Gamma(\xi+2)}+|\varepsilon_1|\sum_{i=1}^{m}|\delta_i|\frac{(\sigma_i-a)^{\xi}}{\Gamma(\xi+1)}\\
&\quad+|\varepsilon_1\lambda|\frac{|(b-a)^{\xi+1}-(\tau-a)^{\xi+1}|}{\Gamma(\xi+2)}\Bigg),\\
M_1 &= \frac{1}{|\Delta|}\Bigg(|x_0|\frac{(b-a)^{\zeta+1}}{\Gamma(\zeta+2)}+|\varepsilon_1|\frac{(b-a)^{\zeta-1}}{\Gamma(\zeta)}+|q|\frac{(b-a)^{\xi}}{\Gamma(\xi+1)}\Bigg),\\
L_2 &= \frac{(b-a)}{|\Delta|}\Bigg(|\varepsilon_2 x_0|\frac{(b-a)^{\xi+1}}{\Gamma(\xi+2)}+|\varepsilon_3|\sum_{i=1}^{m}|\delta_i|\frac{(\sigma_i-a)^{\xi}}{\Gamma(\xi+1)}\\
&\quad+|\varepsilon_3\lambda|\frac{|(b-a)^{\xi+1}-(\tau-a)^{\xi+1}|}{\Gamma(\xi+2)}\Bigg),\\
M_2 &= \frac{(b-a)^{\zeta}}{\Gamma(\zeta+1)}+\frac{b-a}{|\Delta|}\Bigg(|\varepsilon_2 x_0|\frac{(b-a)^{\zeta+1}}{\Gamma(\zeta+2)}+|\varepsilon_2 q|\frac{(b-a)^{\xi}}{\Gamma(\xi+1)}+|\varepsilon_3|\frac{(b-a)^{\zeta-1}}{\Gamma(\zeta)}\Bigg).
\end{aligned} \tag{3.1}$$

Our first existence result for the system (1.1) relies on Leray-Schauder alternative [54].

Theorem 3.1. *Assume that:*

(H_1) $\varphi,\psi : [a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are continuous functions and there exist real constants* $k_i,\gamma_i\geq 0$, $(i=1,2)$ *and* $k_0>0,\gamma_0>0$ *such that* $\forall x,y\in\mathbb{R}$,

$$\begin{aligned}
|\varphi(t,x,y)| &\leq k_0+k_1|x|+k_2|y|,\\
|\psi(t,x,y)| &\leq \gamma_0+\gamma_1|x|+\gamma_2|y|.
\end{aligned}$$

Then there exists at least one solution for the system (1.1) on $[a,b]$ *if*

$$(L_1+L_2)k_1+(M_1+M_2)\gamma_1<1 \ \text{ and } \ (L_1+L_2)k_2+(M_1+M_2)\gamma_2<1, \tag{3.2}$$

where $L_i, M_i, i=1,2$ *are given by (3.1).*

Proof. Let us note that continuity of the functions φ and ψ implies that of the operator $T: X\times X\to X\times X$. Next, let $\Omega\subset X\times X$ be bounded such that

$$|\varphi(t,x(t),y(t))|\leq K_1,\quad |\psi(t,x(t),y(t))|\leq K_2,\quad \forall(x,y)\in\Omega,$$

for positive constants K_1 and K_2. Then for any $(x,y)\in\Omega$, we have

$$|T_1(x,y)(t)| \ \leq\ I_{a^+}^{\xi}|\varphi(t,x(t),y(t))|+\frac{1}{|\Delta|}\Bigg(|y_0|+|x_0|\int_a^b\frac{(b-s)^{\xi}}{\Gamma(\xi+1)}|\varphi(s,x(s),y(s))|ds$$

$$
\begin{aligned}
& +\int_a^b |\rho_1(s)||\psi(s,x(s),y(s))|ds \\
& +|\varepsilon_1|\sum_{i=1}^m |\delta_i|\int_a^{\sigma_i}\frac{(\sigma_i-s)^{\xi-1}}{\Gamma(\xi)}|\varphi(s,x(s),y(s))|ds \\
& +|\varepsilon_1\lambda|\int_\tau^b\int_a^s\frac{(s-u)^{\xi-1}}{\Gamma(\xi)}|\varphi(u,x(u),y(u))|duds\Bigg) \\
\le\ & \frac{|y_0|}{|\Delta|}+\Bigg\{\frac{(b-a)^\xi}{\Gamma(\xi+1)}+\frac{1}{|\Delta|}\Bigg(|x_0|\frac{(b-a)^{\xi+1}}{\Gamma(\xi+2)}+|\varepsilon_1|\sum_{i=1}^m|\delta_i|\frac{(\sigma_i-a)^\xi}{\Gamma(\xi+1)} \\
& +|\varepsilon_1\lambda|\frac{|(b-a)^{\xi+1}-(\tau-a)^{\xi+1}|}{\Gamma(\xi+2)}\Bigg)\Bigg\}K_1 \\
& +\Bigg\{\frac{1}{|\Delta|}\Bigg(|x_0|\frac{(b-a)^{\zeta+1}}{\Gamma(\zeta+2)}+|\varepsilon_1|\frac{(b-a)^{\zeta-1}}{\Gamma(\zeta)}+|q|\frac{(b-a)^\xi}{\Gamma(\xi+1)}\Bigg)\Bigg\}K_2 \\
=\ & \frac{|y_0|}{|\Delta|}+L_1K_1+M_1K_2, \qquad (3.3)
\end{aligned}
$$

which implies that

$$\|T_1(x,y)\|\le\frac{|y_0|}{|\Delta|}+L_1K_1+M_1K_2.$$

In a similar manner, one can obtain that

$$\|T_2(x,y)\|\le\frac{|\varepsilon_2y_0|(b-a)}{|\Delta|}+L_2K_1+M_2K2.$$

In consequence, the operator T is uniformly bounded as

$$\|T(x,y)\|\le\frac{|y_0|}{|\Delta|}+\frac{|\varepsilon_2y_0|(b-a)}{|\Delta|}+(L_1+L_2)K_1+(M_1+M_2)K_2.$$

Now we show that T is equicontinuous. Let $t_1,t_2\in[a,b]$ with $t_1<t_2$. Then we have

$$
\begin{aligned}
& |T_1(x(t_2),y(t_2))-T_1(x(t_1),y(t_1))| \\
\le\ & K_1\left|\frac{1}{\Gamma(\xi)}\int_a^{t_2}(t_2-s)^{\xi-1}ds-\frac{1}{\Gamma(\xi)}\int_a^{t_1}(t_1-s)^{\xi-1}ds\right| \\
\le\ & K_1\left\{\frac{1}{\Gamma(\xi)}\int_a^{t_1}[(t_2-s)^{\xi-1}-(t_1-s)^{\xi-1}]ds+\frac{1}{\Gamma(\xi)}\int_{t_1}^{t_2}(t_2-s)^{\xi-1}ds\right\} \\
\le\ & \frac{K_1}{\Gamma(\xi+1)}[2(t_2-t_1)^\xi+|t_2^\xi-t_1^\xi|]. \qquad (3.4)
\end{aligned}
$$

Analogously, we can obtain

$$
\begin{aligned}
& |T_2(x(t_2),y(t_2))-T_2(x(t_1),y(t_1))| \\
\le\ & \frac{K_2}{\Gamma(\zeta+1)}[2(t_2-t_1)^\zeta+|t_2^\zeta-t_1^\zeta|]+\frac{|t_2-t_1|}{|\Delta|}\Bigg\{|\varepsilon_2x_0|\frac{(b-a)^{\xi+1}}{\Gamma(\xi+2)}K_1
\end{aligned}
$$

$$+\Big(|\varepsilon_2 x_0|\frac{(b-a)^{\zeta+1}}{\Gamma(\zeta+2)}+|\varepsilon_2 q|\frac{(b-a)^{\xi}}{\Gamma(\xi+1)}+|\varepsilon_3|\frac{(b-a)^{\zeta-1}}{\Gamma(\zeta)}\Big)K_2$$
$$+|\varepsilon_3|\sum_{i=1}^{m}|\delta_i|\frac{(\sigma_1-b)^{\xi}}{\Gamma(\xi+1)}K_1+|\varepsilon_3\lambda|\frac{|(b-a)^{\xi+1}-(\tau-a)^{\xi+1}|}{\Gamma(\xi+2)}K_1\Big\}.$$

From the preceding inequalities, it follows that the operator $T(x,y)$ is equicontinuous. Thus the operator $T(x,y)$ is completely continuous.

Finally, we consider the set $\mathcal{P}=\{(x,y)\in X\times X:(x,y)=\nu T(x,y),0\le\nu\le 1\}$ and show that it is bounded.

Let $(x,y)\in\mathcal{P}$ with $(x,y)=\nu T(x,y)$. For any $t\in[a,b]$, we have $x(t)=\nu T_1(x,y)(t),y(t)=\nu T_2(x,y)(t)$. Then by (H_1) we have

$$\begin{aligned}|x(t)| &\le \frac{|y_0|}{|\Delta|}+L_1(k_0+k_1|x|+k_2|y|)+M_1(\gamma_0+\gamma_1|x|+\gamma_2|y|)\\ &= \frac{|y_0|}{|\Delta|}+L_1k_0+M_1\gamma_0+(L_1k_1+M_1\gamma_1)|x|+(L_1k_2+M_1\gamma_2)|y|,\end{aligned}$$

and

$$\begin{aligned}|y(t)| &\le \frac{|\varepsilon_2 y_0|(b-a)}{|\Delta|}+L_2(k_0+k_1|x|+k_2|y|)+M_2(\gamma_0+\gamma_1|x|+\gamma_2|y|)\\ &= \frac{|\varepsilon_2 y_0|(b-a)}{|\Delta|}+L_2k_0+M_2\gamma_0+(L_2k_1+M_2\gamma_1)|x|+(L_2k_2+M_2\gamma_2)|y|.\end{aligned}$$

In consequence of the above inequalities, we deduce that

$$\|x\|\le\frac{|y_0|}{|\Delta|}+L_1k_0+M_1\gamma_0+(L_1k_1+M_1\gamma_1)\|x\|+(L_1k_2+M_1\gamma_2)\|y\|,$$

and

$$\|y\|\le\frac{|\varepsilon_2 y_0|(b-a)}{|\Delta|}+L_2k_0+M_2\gamma_0+(L_2k_1+M_2\gamma_1)\|x\|+(L_2k_2+M_2\gamma_2)\|y\|,$$

which imply that

$$\begin{aligned}\|x\|+\|y\| &\le \frac{|y_0|}{|\Delta|}+\frac{|\varepsilon_2 y_0|(b-a)}{|\Delta|}+(L_1+L_2)k_0+(M_1+M_2)\gamma_0\\ &\quad+[(L_1+L_2)k_1+(M_1+M_2)\gamma_1]\|x\|+[(L_1+L_2)k_2+(M_1+M_2)\gamma_2]\|y\|.\end{aligned}$$

Thus

$$\|(x,y)\|\le\frac{1}{M_0}\Big[\frac{|y_0|}{|\Delta|}+\frac{|\varepsilon_2 y_0|(b-a)}{|\Delta|}+(L_1+L_2)k_0+(M_1+M_2)\gamma_0\Big],$$

where $M_0=\min\{1-[(L_1+L_2)k_1+(M_1+M_2)\gamma_1],1-[(L_1+L_2)k_2+(M_1+M_2)\gamma_2]\}$. Hence the set $\mathcal{P}$ is bounded. As the hypothesis of Leray-Schauder alternative [54] is satisfied, we conclude that the operator T has at least one fixed point. Thus the problem (1.1) has at least one solution on $[a,b]$. □

By using Banach's contraction mapping principle we prove in the next theorem the existence of a unique solution of the system (1.1).

Theorem 3.2. *Assume that:*

(H_2) $\varphi, \psi : [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist positive constants* l_1 *and* l_2 *such that for all* $t \in [a,b]$ *and* $x_i, y_i \in \mathbb{R},\ i = 1,2,$ *we have*

$$|\varphi(t,x_1,x_2) - \varphi(t,y_1,y_2)| \le l_1(|x_1 - y_1| + |x_2 - y_2|),$$

$$|\psi(t,x_1,x_2) - \psi(t,y_1,y_2)| \le l_2(|x_1 - y_1| + |x_2 - y_2|).$$

If

$$(L_1 + L_2)l_1 + (M_1 + M_2)l_2 < 1, \tag{3.5}$$

where $L_i, M_i, i = 1,2$ *are given by (3.1) then the system (1.1) has a unique solution on* $[a,b]$.

Proof. Define $\sup_{t\in[a,b]} \varphi(t,0,0) = N_1 < \infty$, $\sup_{t\in[a,b]} \psi(t,0,0) = N_2 < \infty$ and $r > 0$ such that

$$r > \frac{(|y_0|/|\Delta|)(1 + (b-a)|\varepsilon_2|) + (L_1 + L_2)N_1 + (M_1 + M_2)N_2}{1 - (L_1 + L_2)l_1 - (M_1 + M_2)l_2}.$$

Let us first show that $TB_r \subset B_r$, where $B_r = \{(x,y) \in X \times X : \|(x,y)\| \le r\}$. By the assumption (H_2), for $(x,y) \in B_r,\ t \in [a,b]$, we have

$$\begin{aligned} |\varphi(t,x(t),y(t))| &\le |\varphi(t,x(t),y(t)) - \varphi(t,0,0)| + |\varphi(t,0,0)| \\ &\le l_1(|x(t)| + |y(t)|) + N_1 \\ &\le l_1(\|x\| + \|y\|) + N_1 \le l_1 r + N_1. \end{aligned} \tag{3.6}$$

Similarly, we can get

$$|\psi(t,x(t),y(t))| \le l_2(\|x\| + \|y\|) + N_2 \le l_2 r + N_2. \tag{3.7}$$

Using (3.6) and (3.7), we obtain

$$\begin{aligned} |T_1(x,y)(t)| \le\ & I_{a^+}^{\xi}|\varphi(t,x(t),y(t))| + \frac{1}{|\Delta|}\Bigg(|y_0| + |x_0|\int_a^b \frac{(b-s)^{\xi}}{\Gamma(\xi+1)}|\varphi(s,x(s),y(s))|ds \\ & + \int_a^b |\rho_1(s)||\psi(s,x(s),y(s))|ds \\ & + |\varepsilon_1|\sum_{i=1}^{m}|\delta_i|\int_a^{\sigma_i}\frac{(\sigma_i - s)^{\xi-1}}{\Gamma(\xi)}|\varphi(s,x(s),y(s))|ds \\ & + |\varepsilon_1\lambda|\int_\tau^b\int_a^s \frac{(s-u)^{\xi-1}}{\Gamma(\xi)}|\varphi(u,x(u),y(u))|duds\Bigg) \\ \le\ & \frac{|y_0|}{|\Delta|} + \Bigg\{\frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \frac{1}{|\Delta|}\Bigg(|x_0|\frac{(b-a)^{\xi+1}}{\Gamma(\xi+2)} + |\varepsilon_1|\sum_{i=1}^{m}|\delta_i|\frac{(\sigma_i - a)^{\xi}}{\Gamma(\xi+1)} \\ & + |\varepsilon_1\lambda|\frac{|(b-a)^{\xi+1} - (\tau - a)^{\xi+1}|}{\Gamma(\xi+2)}\Bigg)\Bigg\}(l_1 r + N_1) \\ & + \Bigg\{\frac{1}{|\Delta|}\Bigg(|x_0|\frac{(b-a)^{\zeta+1}}{\Gamma(\zeta+2)} + |\varepsilon_1|\frac{(b-a)^{\zeta-1}}{\Gamma(\zeta)} + |q|\frac{(b-a)^{\xi}}{\Gamma(\xi+1)}\Bigg)\Bigg\}(l_2 r + N_2) \end{aligned}$$

$$
\begin{aligned}
&= \frac{|y_0|}{|\Delta|} + L_1(l_1 r + N_1) + M_1(l_2 r + N_2) \\
&= \frac{|y_0|}{|\Delta|} + (L_1 l_1 + M_1 l_2) r + L_1 N_1 + M_1 N_2.
\end{aligned} \tag{3.8}
$$

Taking the norm of (3.8) for $t \in [a, b]$, we get

$$
\|T_1(x, y)\| \le \frac{|y_0|}{|\Delta|} + (L_1 l_1 + M_1 l_2) r + L_1 N_1 + M_1 N_2.
$$

Likewise, we can find that

$$
\|T_2(x, y)\| \le \frac{|\varepsilon_2 y_0|(b-a)}{|\Delta|} + (L_2 l_1 + M_2 l_2) r + L_2 N_1 + M_2 N_2.
$$

Consequently,

$$
\begin{aligned}
\|T(x, y)\| &\le \frac{|y_0|}{|\Delta|} + \frac{|\varepsilon_2 y_0|(b-a)}{|\Delta|} + \Big[(L_1 + L_2) l_1 + (M_1 + M_2) l_2\Big] r \\
&\quad + (L_1 + L_2) N_1 + (M_1 + M_2) N_2 \\
&\le r.
\end{aligned}
$$

Now, for $(x_1, y_1), (x_2, y_2) \in X \times X$ and for any $t \in [a, b]$, we get

$$
\begin{aligned}
& |T_1(x_2, y_2)(t) - T_1(x_1, y_1)(t)| \\
\le\ & \left\{ \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} + \frac{1}{|\Delta|}\left(|x_0| \frac{(b-a)^{\xi+1}}{\Gamma(\xi+2)} + |\varepsilon_1| \sum_{i=1}^{m} |\delta_i| \frac{(\sigma_i - a)^{\xi}}{\Gamma(\xi+1)} \right.\right. \\
& \left.\left. + |\varepsilon_1 \lambda| \frac{|(b-a)^{\xi+1} - (\tau - a)^{\xi+1}|}{\Gamma(\xi+2)} \right) \right\} l_1(\|x_2 - x_1\| + \|y_2 - y_1\|) \\
& + \left\{ \frac{1}{|\Delta|}\left(|x_0| \frac{(b-a)^{\zeta+1}}{\Gamma(\zeta+2)} + |\varepsilon_1| \frac{(b-a)^{\zeta-1}}{\Gamma(\zeta)} + |q| \frac{(b-a)^{\xi}}{\Gamma(\xi+1)} \right) \right\} l_2(\|x_2 - x_1\| + \|y_2 - y_1\|) \\
=\ & (L_1 l_1 + M_1 l_2)(\|x_2 - x_1\| + \|y_2 - y_1\|),
\end{aligned}
$$

which implies that

$$
\|T_1(x_2, y_2) - T_1(x_1, y_1)\| \le (L_1 l_1 + M_1 l_2)(\|x_2 - x_1\| + \|y_2 - y_1\|). \tag{3.9}
$$

Similarly, we find that

$$
\|T_2(x_2, y_2) - T_2(x_1, y_1)\| \le (L_2 l_1 + M_2 l_2)(\|x_2 - x_1\| + \|y_2 - y_1\|). \tag{3.10}
$$

It follows from (3.9) and (3.10) that

$$
\|T(x_2, y_2) - T(x_1, y_1)\| \le [(L_1 + L_2) l_1 + (M_1 + M_2) l_2](\|x_2 - x_1\| + \|y_2 - y_1\|).
$$

From the above inequality, we deduce that T is a contraction. Hence it follows by Banach's fixed point theorem that there exists a unique fixed point for the operator T, which corresponds to a unique solution of problem (1.1) on $[a, b]$. This completes the proof. □

3.1. Example

Consider the following mixed-type coupled fractional differential system

$$\begin{cases} D_{a^+}^{\frac{3}{4}} x(t) = \varphi(t, x(t), y(t)), \ t \in [1, 2], \\ D_{a^+}^{\frac{7}{4}} y(t) = \psi(t, x(t), y(t)), \ t \in [1, 2] \\ \frac{1}{5} x(1) + \frac{1}{10} y(2) = \frac{1}{1000} \int_1^2 (x(s) + y(s)) ds, \\ y(1) = 0, \ \ y'(2) = \sum_{i=1}^{2} \delta_i x(\sigma_i) + \frac{1}{10} \int_{\frac{7}{4}}^{2} x(s) ds, \end{cases} \tag{3.11}$$

where $\xi = 3/4, \zeta = 7/4, p = 1/5, q = 1/10, x_0 = 1/1000, y_0 = 0, \delta_1 = 1/10, \delta_2 = 1/100, \sigma_1 = 5/4, \sigma_2 = 3/2, \tau = 7/4, \lambda = 1/10$. With the given data, it is found that $L_1 \simeq 3.5495 \times 10^{-2}, L_2 \simeq 6.5531 \times 10^{-2}, M_1 \simeq 1.0229, M_2 \simeq 0.90742$.

(1) In order to illustrate Theorem 3.1, we take

$$\begin{aligned} \varphi(t, x, y) &= e^{-2t} + \frac{1}{8} x \ \cos y + \frac{e^{-t}}{3} y \ \sin y, \\ \psi(t, x, y) &= t \sqrt{t^2 + 3} + \frac{e^{-t}}{3\pi} x \ \tan^{-1} y + \frac{1}{\sqrt{48 + t^2}} y. \end{aligned} \tag{3.12}$$

It is easy to check that the condition (H_1) is satisfied with $k_0 = 1/e^2, k_1 = 1/8, k_2 = 1/(3e), \gamma_0 = 2\sqrt{7}, \gamma_1 = 1/(6e), \gamma_2 = 1/7$. Furthermore, $(L_1 + L_2)k_1 + (M_1 + M_2)\gamma_1 \simeq 0.13098 < 1$, and $(L_1 + L_2)k_2 + (M_1 + M_2)\gamma_2 \simeq 0.28815 < 1$. Clearly the hypotheses of Theorem 3.1 are satisfied and hence the conclusion of Theorem 3.1 applies to problem (3.11) with φ and ψ given by (3.12).

(2) In order to illustrate Theorem 3.2, we take

$$\varphi(t, x, y) = \frac{e^{-t}}{\sqrt{3 + t^2}} \cos x + \cos t, \quad \psi(t, x, y) = \frac{1}{5 + t^4} (\sin x + |y|) + e^{-t}, \tag{3.13}$$

which clearly satisfy the condition (H_2) with $l_1 = 1/(2e)$ and $l_2 = 1/6$. Moreover $(L_1 + L_2)l_1 + (M_1 + M_2)l_2 \simeq 0.3403 < 1$. Thus the hypothesis of Theorem 3.2 holds true and consequently there exists a unique solution of the problem (3.11) with φ and ψ given by (3.13) on $[1, 2]$.

4. A variant problem

In this section, we consider a variant of the problem (1.1) in which the nonlinearities φ and ψ do not depend on x and y respectively. In precise terms, we consider the following problem:

$$\begin{cases} {}^c D_{a^+}^{\xi} x(t) = \overline{\varphi}(t, y(t)), \ 0 < \xi \le 1, \ \ t \in [a, b], \\ {}^c D_{a^+}^{\zeta} y(t) = \overline{\psi}(t, x(t)), \ 1 < \zeta \le 2, \ \ t \in [a, b], \\ px(a) + qy(b) = y_0 + x_0 \int_a^b (x(s) + y(s)) ds, \\ y(a) = 0, \ \ y'(b) = \sum_{i=1}^{m} \delta_i x(\sigma_i) + \lambda \int_{\tau}^{b} x(s) ds, \\ a < \sigma_1 < \sigma_2 < \ldots < \sigma_m < \tau < \ldots < b, \end{cases} \tag{4.1}$$

where $\varphi, \psi : [a, b] \times \mathbb{R} \to \mathbb{R}$ are given functions. Now we present the existence and uniqueness results for the problem (4.1). We do not provide the proofs as they are similar to the ones for the problem (1.1).

Theorem 4.1. *Assume that* $\overline{\varphi}, \overline{\psi} : [a, b]\times\mathbb{R} \to \mathbb{R}$ *are continuous functions and there exist real constants* $\overline{k}_i, \overline{\gamma}_i \geq 0,\ (i = 0, 1)$ *and* $\overline{k}_0 > 0, \overline{\gamma}_0 > 0$ *such that,* $\forall x, y \in \mathbb{R}$,

$$|\overline{\varphi}(t, y)| \leq \overline{k}_0 + \overline{k}_1|y|,\ |\overline{\psi}(t, x)| \leq \overline{\gamma}_0 + \overline{\gamma}_1|x|.$$

Then the system (4.1) has at least one solution on $[a, b]$ *provided that* $(M_1+M_2)\overline{\gamma}_1 < 1$ *and* $(L_1+L_2)\overline{k}_1 < 1$, *where* L_1, M_1 *and* L_2, M_2 *are given by (3.1).*

Theorem 4.2. *Let* $\overline{\varphi}, \overline{\psi} : [a, b] \times \mathbb{R} \to \mathbb{R}$ *be continuous functions and there exist positive constants* $\bar{l}_1$ *and* $\bar{l}_2$ *such that, for all* $t \in [a, b]$ *and* $x_i, y_i \in \mathbb{R},\ i = 1, 2$,

$$|\overline{\varphi}(t, x_1) - \overline{\varphi}(t, y_1)| \leq \bar{l}_1|x_1 - y_1|, \qquad |\overline{\psi}(t, x_1) - \overline{\psi}(t, y_1)| \leq \bar{l}_2|x_1 - y_1|.$$

If $(L_1 + L_2)\bar{l}_1 + (M_1 + M_2)\bar{l}_2 < 1$, *where* L_1, M_1 *and* L_2, M_2 *are given by (3.1) then the system (4.1) has a unique solution on* $[a, b]$.

5. Conclusions

We studied the solvability of a coupled system of nonlinear fractional differential equations of different orders supplemented with a new set of nonlocal multi-point integral boundary conditions on an arbitrary domain by applying the tools of modern functional analysis. We also presented the existence results for a variant of the given problem containing the nonlinearities depending on the cross-variables (unknown functions). Our results are new not only in the given configuration but also yield some new results by specializing the parameters involved in the problems at hand. For example, by taking $\delta_i = 0, i = 1, 2, \ldots, m$ in the obtained results, we obtain the ones associated with the coupled systems of fractional differential equations in (1.1) and (4.1) subject to the boundary conditions:

$$px(a) + qy(b) = y_0 + x_0 \int_a^b (x(s) + y(s))ds,\ \ y(a) = 0,\ \ y'(b) = \lambda \int_\tau^b x(s)ds.$$

For $\lambda = 0$, our results correspond to the boundary conditions of the form:

$$px(a) + qy(b) = y_0 + x_0 \int_a^b (x(s) + y(s))ds,\ y(a) = 0,\ y'(b) = \sum_{i=1}^m \delta_i x(\sigma_i). \tag{5.1}$$

Furthermore, the methods employed in this paper can be used to solve the systems involving fractional integro-differential equations and multi-term fractional differential equations complemented with the boundary conditions considered in the problem (1.1).

Conflict of interest

All authors declare no conflicts of interest in this paper.

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