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Research article

Input-to-state stability of delayed reaction-diffusion neural networks with multiple impulses

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Abstract: This paper concerns the input-to-state stability problem of delayed reaction-diffusion neural networks with multiple impulses. After reformulating the neural-network model in term of an abstract impulsive functional differential equation, the criteria of input-to-state stability are established by the direct estimate of mild solution and an integral inequality with infinite distributed delay. It shows that the input-to-state stability of the continuous dynamics can be retained under certain multiple impulsive disturbance and the unstable continuous dynamics can be stabilised by the multiple impulsive control, if the intervals between the multiple impulses are bounded. The numerical simulation of two examples is given to show the effectiveness of theoretical results.

Keywords: input-to-state stability; multiple impulses; reaction-diffusion; delayed neural networks **Mathematics Subject Classification:** 34K20, 34K45, 35R12

1. Introduction

Over the past few decades, neural networks have been widely applied in the field of smart grid, secure communication, machine learning, among many others [1–6]. Such applications heavily depend on their dynamical behaviors such as stability, synchronization, periodicity, passivity, to name just a few. Among them, the input-to-state stability (ISS), which is originally developed by E. D. Sontag in the late 1980s [7] and measures the influence of external input to stability, is of comparable significance in the dynamical analysis of nonlinear systems including neural networks. Extensions of ISS have also been proposed for various kinds of nonlinear systems and ignited plenty of valuable works [8–15].

In the ISS analysis of neural networks, the time delays are usually involved in neural networks, since the finite speed of signal transmission and amplifier switching inevitably causes the hysteresis of neural networks. There are usually two kinds of time delays considered in neural networks: the

time-varying delays and the infinite distributed delays [16–21]. For instance, the exponential ISS of recurrent neural networks with multiple time-varying delays was studied by the Lyapunov method in [18], whose results were further extended to the stochastic case with both time-varying delays and infinite distributed delays [19]. In [20], the *p*th moment exponential ISS of stochastic recurrent neural networks with time-varying delay was investigated by the vector Lyapunov function to reduce the conservatism caused by the scalar Lyapunov function used in previous literature.

Recently, the reaction-diffusion is introduced in neural-network models because the electrons sometimes have the diffusive shift trajectory in nonuniform electromagnetic field. Different from the delayed neural networks (DNNs) without reaction-diffusion, the delayed reaction-diffusion neural networks (DRDNNs) are described by PDEs because their dynamics depends on both the spatial derivative and the time derivative. Therefore, the dynamical analysis of DRDNNs has attracted the interest of plenty of researchers and the extension of ISS has also been carried out from DNNs to DRDNNs [22–30]. In addition, the ISS is used not only for DRDNNs, but is in fact a key concept in robust control of infinite-dimensional systems, with the expectation that ISS will enable similar advances in the control theory of infinite-dimensional systems as it has for finite-dimensional systems. See [31,32] and the references therein.

On the other hand, the impulsive effects may occur in the hardware implementation of neural networks since the nodes may be shocked by defective connections, sudden attacks, and abrupt changes [24], so the impulsive delayed neural networks (IDNNs) have been massively studied [33–37] where the impulses are classified into two kinds: the stabilising impulses which force the trajectory of neural network into desirable pattern, and the destabilising impulses which bring fluctuation to neural networks. However, the multiple impulses containing both stabilising impulses and destabilising impulses are more elegant to model the instantaneous shocks of the neural networks. Even though some ISS properties of impulsive nonlinear system with multiple impulses are unveiled in recent literature [38, 39], the ISS of DNNs with multiple impulses, is rarely investigated, not to mention the ISS of DRDNNs with multiple impulses, because the multiple impulses are difficult to handle with the infinite distributed delays included in neural networks.

Motivated by the above discussion, the aim of this paper is to establish the ISS criteria of DRDNNs with multiple impulses. The contributions lie in the following aspects: (1) The multiple impulses, infinite distributed delays, and reaction-diffusion are considered simultaneously in neural-network model; (2) The ISS conditions of the DRDNNs with multiple impulses are obtained by the direct estimate of mild solution and an integral inequality; (3) It show that the ISS property of continuous dynamics can be retained under certain multiple impulsive disturbance and the unstable continuous dynamics can be stabilised by multiple impulsive control, if the intervals between the multiple impulses are bounded.

The remainder of this paper is organized as follows. Section 2 introduces the neural-network model and preliminaries. Section 3 gives the sufficient conditions for ISS of the DRDNNs with multiple impulses. Section 4 presents the numerical simulation of two examples. Finally, the conclusions are drawn in Section 5.

2. Model description and preliminaries

In this paper, unless otherwise specified, the following notations are used. $\bar{n} = \{1, 2, \dots, n\}$ and $\mathbb{N} = \{1, 2, 3, \cdots\}$. For $a, b \in \mathbb{R}$, $a \wedge b$ denotes the minimum of a and b. $\mathcal{L} \triangleq (L^2(O))^n$ and $L^2(O)$ is a Hilbert space with inner product $\langle z_1, z_2 \rangle = \int_O z_1(x) z_2(x) dx$ and norm $||z||^2 = \langle z, z \rangle$, where $O = \{x | x = (x_1, \dots, x_w)^T, |x_j| \le \rho_j, \rho_j \in \mathbb{R}_+, j \in \overline{w}\}$. Here, we also use the same symbol $|| \cdot ||$ to denote the usual norm of linear bounded operators from \mathcal{L} to \mathcal{L} . $\mathcal{H} \triangleq (H)^n$ where $H = \{z \in L^2(\mathcal{O}) :$ $(\partial z)/(\partial x_i), (\partial^2 z)/(\partial x_i \partial x_j) \in L^2(O), \ z(t, x)|_{x \in \partial O} = 0, \ i, j = 1, 2, \cdots, w$. Let $\mathcal{F}_0 = \{t_1, t_2, t_3, \cdots\}$ be the sequences of impulse times satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ to prevent the occurrence of accumulation points. For $\dot{\eta} > 0$ and $\dot{\eta} > 0$, $\mathcal{F}^+(\dot{\eta})$, $\mathcal{F}_-(\dot{\eta})$, and $\mathcal{F}(\dot{\eta}, \dot{\eta})$ denote the sets of admissible sequences of impulse times in \mathcal{F}_0 satisfying $0 < t_k - t_{k-1} \leq \dot{\eta}, \dot{\eta} \leq t_k - t_{k-1} < \infty$, and $\dot{\eta} \leq t_k - t_{k-1} \leq \dot{\eta}$ for any $k \in \mathbb{N}$, respectively. PC(R, J) represents the space of functions $f : R \to J$ which are continuous on (t_{k-1}, t_k) for $k \in \mathbb{N}$ and $f(t^+) = f(t)$ for $\forall t \in R$ where $R = \mathbb{R}$ or $R = \mathbb{R}_+$ and J is an Euclidean space or a Hilbert space. $\mathcal{U} \triangleq PC(\mathbb{R}_+, \mathcal{H} \cap \mathcal{L})$. $\mathcal{P}C$ represents the space of functions $f: (-\infty, 0] \to \mathcal{L} \cap \mathcal{H}$ which have at most a finite number of jump discontinuities on $(-\infty, 0]$ and $f(t^+) = f(t)$ for $\forall t \in (-\infty, 0]$. $\mathcal{P}C^b = \{f | f \in \mathcal{P}C \text{ and } f(t) \text{ is bounded on } (-\infty, 0]\} \text{ with norm } ||f||_{\mathcal{P}C^b} = \sup_{-\infty < t \le 0} ||f(t)||.$ A function z(t, x) is said to be piecewise continuous if z(t, x) is piecewise continuous for all $x \in O$. K represents the class of continuous strictly increasing function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ with $\kappa(0) = 0$. \mathcal{K}_{∞} is the subset of \mathcal{K} functions that are unbounded. A function β is said to belong to the class of \mathcal{KL} , if $\beta(\cdot, t)$ is of class \mathcal{K} for each fixed t > 0 and $\beta(s, t)$ decreases to 0 as $t \to +\infty$ for each fixed $s \ge 0$.

Consider the following DRDNNs with multiple impulses

$$\begin{cases} \frac{\partial \hat{z}_{i}(t,x)}{\partial t} = d_{i} \sum_{j=1}^{w} \frac{\partial^{2} \hat{z}_{i}(t,x)}{\partial x_{j}^{2}} - a_{i} \hat{z}_{i}(t,x) + \sum_{j=1}^{n} b_{ij} \hat{f}_{j}(\hat{z}_{j}(t,x)) + \sum_{j=1}^{n} p_{ij} \hat{f}_{j}(\hat{z}_{j}(t-\tau,x)) \\ + \sum_{j=1}^{n} q_{ij} \int_{0}^{+\infty} k(r) \hat{f}_{j}(\hat{z}_{j}(t-r,x)) dr + \hat{u}_{i}(t,x), \ t \ge 0, \ t \ne t_{k}, \end{cases}$$

$$(2.1)$$

$$\hat{z}_{i}(t_{k},x) = (1+c_{k}) \hat{z}_{i}(t_{k}^{-},x) + \hat{u}_{i}(t_{k}^{-},x), \ k \in \mathbb{N},$$

where $x \in O$, $i \in \bar{n}$, $\hat{z}_i(t, x)$ is the state variable of the *i*th neuron at time *t* and space *x*, d_i represents the positive transmission diffusion coefficient of the *i*th neuron, $\sum_{j=1}^{w} \frac{\partial^2 \hat{z}_i(t,x)}{\partial x_j^2}$ represents the reactiondiffusion term, $a_i > 0$ stands for the recovery rate, $b_{ij} > 0$, $p_{ij} > 0$, and $q_{ij} > 0$ are the connection weight strengths of the *j*th neuron on the *i*th neuron, \hat{f}_j stands for the activation function, \hat{u}_i is the external input, $i, j \in \bar{n}$. The delay kernel $k : [0, +\infty) \to \mathbb{R}_+$ is a nonnegative continuous function satisfying that there exists a positive constant λ^* such that $k(s) \le e^{-\lambda^* s}$ for $s \ge 0$. Then, the neuralnetwork model (2.1) can be rewritten in terms of the following vector form

$$\begin{cases} \frac{\partial \hat{z}(t,x)}{\partial t} = D\Delta \hat{z}(t,x) - A\hat{z}(t,x) + B\hat{f}(\hat{z}(t,x)) + P\hat{f}(\hat{z}(t-\tau,x)) \\ + Q \int_{0}^{+\infty} k(r)\hat{f}(\hat{z}(t-r,x))dr + \hat{u}(t,x), \ t \in [t_{k-1},t_k), \\ \hat{z}(t_k,x) = (1+c_k)\hat{z}(t_k^-,x) + \hat{u}(t_k^-,x), \ k \in \mathbb{N}, \end{cases}$$
(2.2)

where $\hat{z} = (\hat{z}_1, \hat{z}_2, \dots, \hat{z}_n)^T$, $\Delta = \sum_{j=1}^w \frac{\partial^2}{\partial x_j^2}$, $D = \text{diag}(d_1, d_2, \dots, d_n)$, $A = \text{diag}(a_1, a_2, \dots, a_n)$, $B = (b_{ij})_{n \times n}$, $P = (p_{ij})_{n \times n}$, $Q = (q_{ij})_{n \times n}$, $\hat{f}(z) = (\hat{f}_1(z_1), \hat{f}_2(z_2), \dots, \hat{f}_n(z_n))^T$, and $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_n)^T$. The Dirichlet boundary condition and initial condition, associated with (2.1) or (2.2), are given by

$$\hat{z}(t,x)|_{x\in\partial O} = 0, \ t \in \mathbb{R},$$
(2.3)

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$$\hat{z}(t,x) = \hat{\phi}(t,x) \in \mathcal{P}C^b, \ t \le 0, x \in O.$$
(2.4)

As standard hypotheses, we assume that

(H1) there exist positive constants l_i such that, for $\forall \hat{z}_1, \hat{z}_2 \in \mathbb{R}, i \in \bar{n}$,

$$|\hat{f}_i(\hat{z}_1) - \hat{f}_i(\hat{z}_2)| \le l_i |\hat{z}_1 - \hat{z}_2|;$$

(H2) there exist constants $N \in \mathbb{N}$ and $\sigma_k > 0$, $k \in \mathbb{N}$ such that $\sigma_{k+N} = \sigma_k$ and $|1 + c_k| \le \sigma_k$.

In this paper, we always assume that (H1) and (H2) are satisfied. The sets of stabilising strengths and destabilising strengths are denoted by $\{\hat{\sigma}_i\}_{i=1}^p$ and $\{\hat{\sigma}_i\}_{i=1}^{N-p}$, respectively. Define a linear operator \mathscr{D} from \mathcal{H} to \mathcal{L} by $\mathscr{D}\hat{z} = D\Delta\hat{z} - A\hat{z}$, then \mathscr{D} is an infinitesimal generator of a strongly continuous C_0 -semigroup S(t) [40]. Furthermore, the neural networks (2.2)–(2.4) can be reformulated in terms of the following abstract impulsive functional differential equation

$$\begin{cases} \frac{dz(t)}{dt} = \mathscr{D}z(t) + Bf(z(t)) + Pf(z(t-\tau)) \\ +Q \int_0^{+\infty} k(r)f(z(t-r))dr + u(t), \ t \in [t_{k-1}, t_k), \\ z(t_k) = (1+c_k)z(t_k^-) + u(t_k^-), \ k \in \mathbb{N}, \\ z_0 = \phi \in \mathcal{P}C^b, \end{cases}$$
(2.5)

where $z(t) = \hat{z}(t, x) \in \mathcal{L}$, $f : \mathcal{L} \to \mathcal{L}$, $u(t) = \hat{u}(t, x) \in \mathcal{L}$, and $z_0(\theta) = \phi(\theta) = \hat{\phi}(\theta, x) \in \mathcal{PC}^b$, $\theta \in (-\infty, 0]$.

Definition 1. An \mathcal{L} -valued functional $z(t) = z(t)(x, \phi, u)$ is said to be a mild solution of (2.5), if z(t) satisfies the following equation

$$z(t) = S(t)\phi(0) + \int_0^t S(t-s)Bf(z(s))ds + \int_0^t S(t-s)Pf(z(s-\tau))ds + \int_0^t S(t-s)Q \int_0^{+\infty} k(r)f(z(s-r))drds + \int_0^t S(t-s)u(s)ds + \sum_{t_k \le t} S(t-t_k)(c_k z(t_k^-) + u(t_k^-)).$$
(2.6)

Remark 1. From Lemma 2.2 and Theorem 5.3 of [41], we can obtain the local existence and uniqueness of mild solution under (H1) and (H2), and the mild solution is continuous between the impulse intervals. If the system (2.5) is input-to-state stable, the mild solution can not explode in finite time, which implies the global existence and uniqueness.

Definition 2 ([9]). For a given sequence $\{t_k\}_{k \in \mathbb{N}}$ of impulse times, the DRDNNs with multiple impulses (2.5) are called input-to-state stable, if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_{\infty}$ such that $\forall \phi \in \mathcal{PC}^{b}$, $\forall u \in \mathcal{U}$ it holds that

$$||z(t)|| \le \beta(||\phi||_{\mathcal{PC}^b}, t) + \sup_{s \le t} \gamma(||u(s)||).$$

The DRDNNs with multiple impulses (2.5) are called uniformly input-to-state stable (UISS) over a given set \mathcal{F} of admissible sequences of impulse times if it is input-to-state stable for every sequence in \mathcal{F} with β and γ independent of the choice of the sequence from the class \mathcal{F} .

Lemma 1 ([23]). Let O be a cube $|x_j| \le \rho_j$ ($j \in \bar{w}$) and let h(x) be a real-valued function belonging to $C^1(O)$, which vanishes on the boundary ∂O , that is, $h(x)|_{\partial O} = 0$. Then

$$\int_{O} h^{2}(x) dx \leq \rho_{j}^{2} \int_{O} \left(\frac{\partial h(x)}{\partial x_{j}}\right)^{2} dx, \ j \in \bar{w}.$$
(2.7)

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Lemma 2. Assume that (H1) holds. Then, the mild solution of (2.5) can be represented by

$$z(t) = T(t,0)\phi(0) + \int_0^t T(t,s)Bf(z(s))ds + \int_0^t T(t,s)Pf(z(s-\tau))ds + \int_0^t T(t,s)Q \int_0^{+\infty} k(r)f(z(s-r))drds + \int_0^t T(t,s)u(s)ds + \sum_{t_k \le t} T(t,t_k)u(t_k^-), \ t \ge 0,$$
(2.8)

where

$$T(t,s) = \begin{cases} S(t-s), & t, s \in [t_{k-1}, t_k), \\ (1+c_k)S(t-s), & t_{k-1} \le s < t_k \le t < t_{k+1}, \\ \prod_{j=i}^k (1+c_j)S(t-s), & t_{i-1} \le s < t_i < t_k \le t < t_{k+1}, \end{cases}$$

Proof. The proof is analogous to the proof of Lemma 2.2 in [43] so as to be omitted.

Lemma 3. Consider the following abstract Cauchy problem

$$\begin{cases} \frac{dz(t)}{dt} = \mathfrak{D}z(t), \ t \ge 0, \\ z(0) = \psi, \end{cases}$$
(2.9)

where $\psi \in \mathcal{H}$. Then, the strongly continuous semigroup S(t) generated by \mathfrak{D} is contractive and satisfies $||S(t)||^2 \leq e^{-2\vartheta t}$ for $t \geq 0$ where $\vartheta = \min_i \{d_i\} \sum_{i=1}^w (1/\rho_i^2) + \min_i \{a_i\}$.

Proof. Recalling that the solution of (2.9) is $z(t) = S(t)\psi$. Combining the Gaussian theorem, the homogeneous Dirichlet boundary condition, and Lemma 1, we obtain

$$\langle z, \mathfrak{D} z \rangle = \sum_{i=1}^{n} d_{i} \sum_{j=1}^{w} \int_{O} z_{i}(t,x) \frac{\partial^{2} z_{i}(t,x)}{\partial x_{j}^{2}} dx - \sum_{i=1}^{n} a_{i} \int_{O} (z_{i}(t,x))^{2} dx$$

$$= -\sum_{i=1}^{n} d_{i} \sum_{j=1}^{w} \int_{O} \left(\frac{\partial z_{i}(t,x)}{\partial x_{j}} \right)^{2} dx - \sum_{i=1}^{n} a_{i} \int_{O} (z_{i}(t,x))^{2} dx$$

$$\leq -\sum_{i=1}^{n} \sum_{j=1}^{w} \frac{d_{i}}{\rho_{k}^{2}} \int_{O} (z_{i}(t,x))^{2} dx - \sum_{i=1}^{n} a_{i} \int_{O} (z_{i}(t,x))^{2} dx \leq -\vartheta ||z||^{2},$$

$$(2.10)$$

which implies that

$$\frac{d||z(t)||^2}{dt} = 2\langle z(t), \mathfrak{D}z(t) \rangle \le -2\vartheta ||z(t)||^2.$$
(2.11)

Therefore, $||z(t)||^2 \le e^{-2\vartheta t} ||\psi||^2$ for all $\psi \in \mathcal{H}$. By the density of \mathcal{H} in \mathcal{L} [Theorem 1.2, 42], the result holds for all $\psi \in \mathcal{L}$ so as to complete the proof.

Lemma 4 ([43]). If $\dot{\eta} \leq t_k - t_{k-1} < \infty$ for $\forall k \in \mathbb{N}$, it holds that $\sum_{t_k \leq t} e^{-c(t-t_k)} < \frac{1}{1-e^{-c\dot{\eta}}}$, where c > 0 and $t \geq t_1$.

3. Input-to-state stability of DRDNNs with multiple impulses

In this section, the ISS of the DRDNNs with multiple impulses will be investigated by the direct estimate of the mild solution. First, let us consider the following integral inequality with infinite

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distributed delay:

$$\begin{cases} v(t) \leq \rho_1 e^{-ct} + \rho_2 \int_0^t e^{-c(t-s)} v(s) ds + \rho_3 \int_0^t e^{-c(t-s)} v(s-\tau) ds \\ + \rho_4 \int_0^t e^{-c(t-s)} \int_0^{+\infty} k(r) v(s-r) dr ds + \rho_5 \int_0^t e^{-c(t-s)} w(s) ds \\ + \rho_6 \sum_{t_k \leq t} e^{-c(t-t_k)} w(t_k^-), \ t \geq 0, \end{cases}$$

$$(3.1)$$

$$v(t) \leq M, \ t \leq 0,$$

where $v \in PC(\mathbb{R}, \mathbb{R}_+)$, $w \in PC(\mathbb{R}_+, \mathbb{R}_+)$, c > 0, $\rho_1 \ge M > 0$, $\rho_i > 0$, $i \in \overline{6}$, and $\hat{\eta} \le t_k - t_{k-1} < \infty$ for $\forall k \in \mathbb{N}$.

Lemma 5. If $\mu = c - \rho_2 - \rho_3 - \rho_4/\lambda^* > 0$, then there exists constant $0 < \lambda < c \land \lambda^*$ such that $\Theta(\lambda) < 1$ and

$$v(t) < Ne^{-\lambda t} + \kappa \sup_{0 \le s \le t} w(s), \tag{3.2}$$

where $\kappa = \frac{c}{\mu} \left(\frac{\rho_5}{c} + \frac{\rho_6}{1 - e^{-c\eta}} \right)$, $N = \frac{2\rho_1}{1 - \Theta(\lambda)} + M$, and

$$\Theta(\lambda) = \frac{\rho_2}{c - \lambda} + \frac{\rho_3}{c - \lambda} e^{\lambda \tau} + \frac{\rho_4}{(c - \lambda)(\lambda^* - \lambda)}.$$
(3.3)

Proof. Let us consider the function $\Theta(a)$ where $a \in [0, c \land \lambda^*)$. Since $\mu > 0$, $\Theta(0) < 1$ and $\Theta(a)$ converges to positive infinity or a constant as $a \to c \land \lambda^*$. Additionally, $\Theta(a)$ is monotonous and continuous with respect to a. Thus, there exists $\lambda \in (0, c \land \lambda^*)$ such that $\Theta(\lambda) < 1$, which further indicates that

$$\frac{\rho_1}{N} + \frac{\rho_2}{c - \lambda} + \frac{\rho_3}{c - \lambda} e^{\lambda \tau} + \frac{\rho_4}{(c - \lambda)(\lambda^* - \lambda)} = \frac{\rho_1}{N} + \Theta(\lambda)$$
$$= \frac{\rho_1}{\frac{2\rho_1}{1 - \Theta(\lambda)} + M} + \Theta(\lambda) < \frac{\rho_1}{\frac{2\rho_1}{1 - \Theta(\lambda)}} + \Theta(\lambda) = \frac{1 - \Theta(\lambda)}{2} + \Theta(\lambda) = \frac{1 + \Theta(\lambda)}{2} < 1.$$
(3.4)

If (3.2) is not true, there exists $t^* > 0$ such that

$$v(t^*) \ge Ne^{-\lambda t^*} + \kappa \sup_{0 \le s \le t^*} w(s),$$
 (3.5)

and

$$v(t) < Ne^{-\lambda t} + \kappa \sup_{0 \le s \le t} w(s), \ t < t^*.$$
 (3.6)

However, it follows from integral inequality (3.1) and Lemma 4 that

$$v(t^{*}) \leq \rho_{1}e^{-ct^{*}} + \rho_{2} \int_{0}^{t^{*}} e^{-c(t^{*}-s)}v(s)ds + \rho_{3} \int_{0}^{t^{*}} e^{-c(t^{*}-s)}v(s-\tau)ds + \rho_{4} \int_{0}^{t^{*}} e^{-c(t^{*}-s)} \int_{0}^{+\infty} k(r)v(s-r)drds + \rho_{5} \int_{0}^{t^{*}} e^{-c(t^{*}-s)}w(s)ds + \rho_{6} \sum_{t_{k} \leq t^{*}} e^{-c(t^{*}-t_{k})}w(t_{k}^{-}) \triangleq \sum_{i=1}^{6} I_{i}(t^{*}).$$
(3.7)

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From (3.6), we get that

$$I_{2}(t^{*}) \leq \rho_{2} \int_{0}^{t^{*}} e^{-c(t^{*}-s)} \Big[Ne^{-\lambda s} + \kappa \sup_{0 \leq p \leq s} w(p) \Big] ds$$

$$\leq \rho_{2} N \int_{0}^{t^{*}} e^{-c(t^{*}-s)} e^{-\lambda s} ds + \rho_{2} \kappa \sup_{0 \leq p \leq t^{*}} w(p) \int_{0}^{t^{*}} e^{-c(t^{*}-s)} ds$$

$$\leq \frac{\rho_{2}}{c-\lambda} N e^{-\lambda t^{*}} + \frac{\rho_{2} \kappa}{c} \sup_{0 \leq s \leq t^{*}} w(s), \qquad (3.8)$$

$$I_{3}(t^{*}) \leq \rho_{3} \int_{0}^{t^{*}} e^{-c(t^{*}-s)} \Big[N e^{-\lambda(s-\tau)} + \kappa \sup_{0 \leq p \leq s-\tau} w(p) \Big] ds$$

$$\leq \rho_{3} e^{\lambda \tau} N \int_{0}^{t^{*}} e^{-c(t^{*}-s)} e^{-\lambda s} ds + \rho_{3} \kappa \sup_{0 \leq p \leq t^{*}} w(p) \int_{0}^{t^{*}} e^{-c(t^{*}-s)} ds$$

$$\leq \frac{\rho_{3}}{c-\lambda} e^{\lambda \tau} N e^{-\lambda t^{*}} + \frac{\rho_{3} \kappa}{c} \sup_{0 \leq s \leq t^{*}} w(s), \qquad (3.9)$$

$$I_5(t^*) \le \rho_5 \int_0^{t^*} e^{-c(t^*-s)} ds \sup_{0 \le s \le t^*} w(s) \le \frac{\rho_5}{c} \sup_{0 \le s \le t^*} w(s).$$
(3.10)

From Cauchy-Schwarz inequality, we obtain

$$I_{4}(t^{*}) \leq \rho_{4} \int_{0}^{t^{*}} e^{-c(t^{*}-s)} \int_{0}^{+\infty} k(r) \Big[Ne^{-\lambda(s-r)} + \kappa \sup_{0 \leq p \leq s-r} w(p) \Big] dr ds$$

$$\leq \rho_{4} N \int_{0}^{+\infty} k(r) e^{\lambda r} dr \int_{0}^{t^{*}} e^{-c(t^{*}-s)} e^{-\lambda s} ds + \rho_{4} \kappa \sup_{0 \leq p \leq t^{*}} w(p) \int_{0}^{t^{*}} e^{-c(t^{*}-s)} ds$$

$$\leq \frac{\rho_{4}}{c-\lambda} N e^{-\lambda t^{*}} \int_{0}^{+\infty} k(r) e^{\lambda r} dr + \frac{\rho_{4} \kappa}{c} \sup_{0 \leq s \leq t^{*}} w(s)$$

$$\leq \frac{\rho_{4}}{(c-\lambda)(\lambda^{*}-\lambda)} N e^{-\lambda t^{*}} + \frac{\rho_{4} \kappa}{c} \sup_{0 \leq s \leq t^{*}} w(s).$$
(3.11)

From Lemma 4, we have

$$I_6(t^*) \le \rho_6 \sum_{t_k \le t^*} e^{-c(t^* - t_k)} \sup_{0 \le s \le t^*} w(s) \le \frac{\rho_6}{1 - e^{-c\hat{\eta}}} \sup_{0 \le s \le t^*} w(s).$$
(3.12)

Combining (3.4) and (3.7)–(3.12), we obtain that

$$v(t^{*}) \leq \left(\frac{\rho_{1}}{N} + \Theta(\lambda)\right) N e^{-\lambda t^{*}} + \left(\frac{(\rho_{2} + \rho_{3} + \rho_{4})\kappa}{c} + \frac{\rho_{5}}{c} + \frac{\rho_{6}}{1 - e^{-c\hat{\eta}}}\right) \sup_{0 \leq s \leq t^{*}} w(s) < N e^{-\lambda t^{*}} + \kappa \sup_{0 \leq s \leq t^{*}} w(s),$$
(3.13)

which contradicts (3.5) so as to complete the proof.

Theorem 1. Assume that $\mu = \delta^2 - 6n\chi \sum_{i=1}^n \sum_{j=1}^n (|b_{ij}|^2 + |p_{ij}|^2 + |q_{ij}|^2/\lambda^*)l_j^2 > 0$, where $\delta = \vartheta - \frac{1}{N} \sum_{i=1}^p \frac{\ln \check{\sigma}_i}{\check{\eta}} - \frac{1}{N} \sum_{i=1}^{N-p} \frac{\ln \check{\sigma}_i}{\check{\eta}} > 0$ and $\chi = \prod_{i=1}^{N-p} \check{\sigma}_i / \prod_{i=1}^p \check{\sigma}_i$. Then, the DRDNNs with multiple impulses (2.5) are UISS over the class $\mathcal{F}(\check{\eta},\check{\eta})$.

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Proof. From inequality $(\sum_{i=1}^{n} a_i)^2 \le n \sum_{i=1}^{n} a_i^2$ and Lemma 2, the moment of mild solution is estimated by

$$\begin{aligned} \|z(t)\|^{2} &\leq 6 \Big(\|T(t,0)\phi(0)\|^{2} + \|\int_{0}^{t} T(t,s)Bf(z(s))ds\|^{2} + \|\int_{0}^{t} T(t,s)Pf(z(s-\tau))ds\|^{2} \\ &+ \|\int_{0}^{t} T(t,s)Q\int_{0}^{+\infty} k(r)f(z(s-r))drds\|^{2} + \|\int_{0}^{t} T(t,s)u(s)ds\|^{2} + \|\sum_{t_{k} \leq t} T(t,t_{k})u(t_{k}^{-})\|^{2} \Big) \\ &\triangleq 6\sum_{i=1}^{6} \Gamma_{i}(t). \end{aligned}$$

$$(3.14)$$

It follows from (H2), Lemma 3, and the class $\mathcal{F}(\eta, \dot{\eta})$ that

$$\|T(t,s)\|^{2} \leq \prod_{s < t_{k} \leq t} (1+c_{k})^{2} \|S(t-s)\|^{2} \leq \prod_{s < t_{k} \leq t} \sigma_{k}^{2} e^{-2\vartheta(t-s)} \leq \chi \prod_{i=1}^{p} \check{\sigma}_{i}^{\frac{t-s}{\eta} \cdot \frac{2}{N}} \prod_{i=1}^{N-p} \check{\sigma}_{i}^{\frac{t-s}{\eta} \cdot \frac{2}{N}} e^{-2\vartheta(t-s)} \leq \chi e^{-2\delta(t-s)}.$$
(3.15)

Combining the Cauchy-Schwarz inequality, it yields

$$\Gamma_1(t) \le \chi e^{-2\delta t} \|\phi\|_{\mathcal{P}C^b}^2 \le \chi e^{-\delta t} \|\phi\|_{\mathcal{P}C^b}^2, \qquad (3.16)$$

$$\Gamma_{2}(t) = \sum_{i=1}^{n} ||\sum_{j=1}^{n} \int_{0}^{t} T(t,s)b_{ij}\hat{f}_{j}(\hat{z}_{j}(s,x))ds||^{2}
\leq n\sum_{i=1}^{n} \sum_{j=1}^{n} \left(\int_{0}^{t} ||T(t,s)b_{ij}\hat{f}_{j}(\hat{z}_{j}(s,x))||ds \right)^{2}
\leq n\chi \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\int_{0}^{t} e^{-\delta(t-s)} |b_{ij}|l_{j}||\hat{z}_{j}(s,x))||ds \right)^{2}
= n\chi \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\int_{0}^{t} e^{-\frac{\delta}{2}(t-s)} e^{-\frac{\delta}{2}(t-s)} |b_{ij}|l_{j}||\hat{z}_{j}(s,x))||ds \right)^{2}
\leq n\chi \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^{2} l_{j}^{2} \int_{0}^{t} (e^{-\frac{\delta}{2}(t-s)})^{2} ds \int_{t_{0}}^{t} (e^{-\frac{\delta}{2}(t-s)})^{2} ||\hat{z}_{j}(s,x))||^{2} ds
\leq \frac{n\chi}{\delta} \sum_{i=1}^{n} \sum_{j=1}^{n} |b_{ij}|^{2} l_{j}^{2} \int_{0}^{t} e^{-\delta(t-s)} ||z(s))||^{2} ds,$$
(3.17)

$$\begin{split} \Gamma_{3}(t) &= \sum_{i=1}^{n} \|\sum_{j=1}^{n} \int_{0}^{t} T(t,s) p_{ij} \hat{f}_{j}(\hat{z}_{j}(s-\tau,x)) ds \|^{2} \\ &\leq n \sum_{i=1}^{n} \sum_{j=1}^{n} \Big(\int_{0}^{t} \|T(t,s) p_{ij} \hat{f}_{j}(\hat{z}_{j}(s-\tau,x)) \| ds \Big)^{2} \\ &\leq n \chi \sum_{i=1}^{n} \sum_{j=1}^{n} \Big(\int_{0}^{t} e^{-\delta(t-s)} |p_{ij}| l_{j} \| \hat{z}_{j}(s-\tau,x)) \| ds \Big)^{2} \end{split}$$

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$$\leq n\chi \sum_{i=1}^{n} \sum_{j=1}^{n} |p_{ij}|^2 l_j^2 \int_0^t e^{-\delta(t-s)} ds \int_{t_0}^t e^{-\delta(t-s)} ||\hat{z}_j(s-\tau,x)||^2 ds$$

$$\leq \frac{n\chi}{\delta} \sum_{i=1}^{n} \sum_{j=1}^{n} |p_{ij}|^2 l_j^2 \int_0^t e^{-\delta(t-s)} ||z(s-\tau)||^2 ds, \qquad (3.18)$$

$$\Gamma_{4}(t) \leq \frac{n\chi}{\delta} \sum_{i=1}^{n} \sum_{j=1}^{n} |q_{ij}|^{2} l_{j}^{2} \int_{0}^{t} e^{-\delta(t-s)} \Big(\int_{0}^{+\infty} k(r) ||\hat{z}_{j}(s-r,x)|| dr \Big)^{2} ds
= \frac{n\chi}{\delta} \sum_{i=1}^{n} \sum_{j=1}^{n} |q_{ij}|^{2} l_{j}^{2} \int_{0}^{t} e^{-\delta(t-s)} \Big(\int_{0}^{+\infty} (k(r))^{\frac{1}{2}} ||\hat{z}(s-r,x)|| dr \Big)^{2} ds
\leq \frac{n\chi}{\delta} \sum_{i=1}^{n} \sum_{j=1}^{n} |q_{ij}|^{2} l_{j}^{2} \int_{0}^{t} e^{-\delta(t-s)} \int_{0}^{+\infty} k(r) dr \int_{0}^{+\infty} k(r) ||\hat{z}(s-r,x)||^{2} dr ds
\leq \frac{n\chi}{\delta} \sum_{i=1}^{n} \sum_{j=1}^{n} |q_{ij}|^{2} l_{j}^{2} \int_{0}^{t} e^{-\delta(t-s)} \int_{0}^{+\infty} k(r) ||z(s-r)||^{2} dr ds,$$
(3.19)

$$\Gamma_{5}(t) = \sum_{i=1}^{n} \|\int_{0}^{t} T(t,s)\hat{u}_{i}(s,x)ds\|^{2} \leq \sum_{i=1}^{n} \left(\int_{0}^{t} \|T(t,s)\hat{u}_{i}(s,x)\|ds\right)^{2} \leq \chi \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\int_{0}^{t} e^{-\delta(t-s)} \|\hat{u}_{i}(s,x)\|ds\right)^{2} \leq \chi \sum_{i=1}^{n} \int_{0}^{t} e^{-\delta(t-s)} \|\hat{u}_{i}(s,x)\|^{2} ds = \frac{\chi}{\delta} \int_{0}^{t} e^{-\delta(t-s)} \|u(s)\|^{2} ds.$$

$$(3.20)$$

Similarly, it follows from Lemma 4 that

$$\Gamma_{6}(t) \leq \sum_{i=1}^{n} \|\sum_{t_{k}\leq t} T(t,t_{k})\hat{u}_{i}(t_{k}^{-})\|^{2} \leq \sum_{i=1}^{n} \left(\sum_{t_{k}\leq t} \|T(t,t_{k})\hat{u}_{i}(t_{k}^{-})\|\right)^{2} \leq \chi \sum_{i=1}^{n} \left(\sum_{t_{k}\leq t} e^{-\delta(t-t_{k})} \|\hat{u}_{i}(t_{k}^{-})\|\right)^{2} \\
= \chi \sum_{i=1}^{n} \left(\sum_{t_{k}\leq t} e^{-\frac{\delta}{2}(t-t_{k})} e^{-\frac{\delta}{2}(t-t_{k})} \|\hat{u}_{i}(t_{k}^{-})\|\right)^{2} \leq \chi \sum_{i=1}^{n} \left(\sum_{t_{k}\leq t} \left(e^{-\frac{\delta}{2}(t-t_{k})}\right)^{2}\right) \left(\sum_{t_{k}\leq t} \left(e^{-\frac{\delta}{2}(t-t_{k})}\right)^{2} \|\hat{u}_{i}(t_{k}^{-})\|^{2}\right) \\
\leq \chi \left(\sum_{t_{k}\leq t} e^{-\delta(t-t_{k})}\right) \left(\sum_{t_{k}\leq t} e^{-\delta(t-t_{k})} \sum_{i=1}^{n} \|\hat{u}_{i}(t_{k}^{-})\|^{2}\right) \leq \frac{\chi}{1-e^{-\delta\eta}} \sum_{t_{k}\leq t} e^{-\delta(t-t_{k})} \|u(t_{k}^{-})\|^{2}.$$
(3.21)

Combining (3.14)–(3.21), we have

$$\begin{aligned} \|z(t)\|^{2} &\leq \rho_{1}e^{-\delta t} + \rho_{2}\int_{0}^{t}e^{-\delta(t-s)}\|z(s)\|^{2}ds + \rho_{3}\int_{0}^{t}e^{-\delta(t-s)}\|z(s-\tau)\|^{2}ds \\ &+ \rho_{4}\int_{0}^{t}e^{-\delta(t-s)}\int_{0}^{+\infty}k(r)\|z(s-r)\|^{2}drds + \rho_{5}\int_{0}^{t}e^{-\delta(t-s)}\|u(s)\|^{2}ds \\ &+ \rho_{6}\sum_{t_{k}\leq t}e^{-\delta(t-t_{k})}\|u(t_{k}^{-})\|^{2}, \end{aligned}$$
(3.22)

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where $\rho_1 = 6\chi ||\phi||_{\mathcal{P}C^b}^2$, $\rho_2 = \frac{6n\chi}{\delta} \sum_{i=1}^n \sum_{j=1}^n |b_{ij}|^2 l_j^2$, $\rho_3 = \frac{6n\chi}{\delta} \sum_{i=1}^n \sum_{j=1}^n |p_{ij}|^2 l_j^2$, $\rho_4 = \frac{6n\chi}{\delta} \sum_{i=1}^n \sum_{j=1}^n |q_{ij}|^2 l_j^2$, $\rho_5 = \frac{6\chi}{\delta}$, $\rho_6 = \frac{6\chi}{1-e^{-\delta \eta}}$. Combining $||z(t)||^2 \le ||\phi||_{\mathcal{P}C^b}^2$ for $t \le 0$, it follows from Lemma 5 that there exist constants $0 < \lambda < \delta \land \lambda^*$ such that $\Theta(\lambda) < 1$

$$||z(t)||^{2} \le Ne^{-\lambda t} + \kappa \sup_{0 \le s \le t} ||u(s)||^{2},$$
(3.23)

where $N = (\frac{2\chi}{1-\Theta(\lambda)} + 1) \|\phi\|_{\mathcal{P}C^b}^2$ and $\kappa = \frac{\chi}{\mu} (1 + \frac{\delta^2}{(1-e^{\delta \hat{\eta}})^2})$. Therefore, the DRDNNs with multiple impulses are UISS over the class $\mathcal{F}(\hat{\eta}, \hat{\eta})$.

Remark 2. In [28], the ISS property of stochastic delayed neural networks are investigated to show that the ISS of continuous dynamics can be retained under certain destabilising impulses. Then, the ISS criteria of DRDNNs with impulses were established by an impulsive delay inequality in [27], where two scenarios are considered: stabilising continuous dynamics with destabilising impulses and destabilising continuous dynamics with stabilising impulses. One can notice that these results in [27,28] focused on the single impulse effect (stabilising or destabilising impulses), and ignored the hybrid effect of multiple impulses. In comparison, the results established here indicate that the ISS property of continuous dynamics can be retained under certain multiple impulsive disturbance and the unstable continuous dynamics can be stabilised by multiple impulsive control, if the intervals between the multiple impulses are bounded.

Remark 3. In most of the existing works on ISS of neural networks [20, 27, 28], the ISS criteria are usually established by the Lyapunov method and extended Halanay-type inequalities. In comparison, the ISS criteria in this paper are established by direct estimate of mild solution and an integral inequality to handle the multiple impulses and infinite distributed delays.

If the multiple impulses degenerate into single impulses, that is, stabilising impulses or destabilising impulses, we have the following corollaries from Theorem 1.

Corollary 1. Assume that $\dot{\mu} = \dot{\delta}^2 - \frac{6n}{\dot{\sigma}} \sum_{i=1}^n \sum_{j=1}^n (|b_{ij}|^2 + |p_{ij}|^2 + |q_{ij}|^2/\lambda^*) l_j^2 > 0$, where $\dot{\delta} = \vartheta - \frac{\ln \dot{\sigma}}{\dot{\eta}} > 0$, $\dot{\sigma} = \dot{\sigma}_k < 1, k \in \mathbb{N}$, and N = 1. Then, the DRDNNs with stabilising impulses (2.5) are UISS over the class $\mathcal{F}^+(\dot{\eta}) \cap \mathcal{F}_-(\dot{\eta})$ for arbitrary $\dot{\eta} > 0$.

Corollary 2. Assume that $\dot{\mu} = \dot{\delta}^2 - 6n\dot{\sigma} \sum_{i=1}^n \sum_{j=1}^n (|b_{ij}|^2 + |p_{ij}|^2 + |q_{ij}|^2/\lambda^*)l_j^2 > 0$, where $\dot{\delta} = \vartheta - \frac{\ln \dot{\sigma}}{\dot{\eta}} > 0$, $\dot{\sigma} = \dot{\sigma}_k > 1$, $k \in \mathbb{N}$, and N = 1. Then, the DRDNNs with destabilising impulses (2.5) are UISS over the class $\mathcal{F}_-(\dot{\eta})$.

Remark 4. The corollaries with stabilising or destabilising impulses accord with the results in [27,43]. But the infinite distributed delays are additionally included in neural-network model, so our results are more general.

4. Numerical examples

In this section, the effectiveness of theoretical results are demonstrated by two numerical examples. **Example 1.** Consider the DRDNNs with multiple impulses which consist of two neurons on $O = \{x | -1 \le x \le 1\}$, where the parameters are given by B = 0, and

$$D = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.3 \end{pmatrix}, A = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}, P = \begin{pmatrix} 1 & -0.5 \\ 0.5 & 1 \end{pmatrix}, Q = \begin{pmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \end{pmatrix},$$

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and $\tau = 0.1$, $k(s) = e^{-s}$, $f(s) = \tanh(s)/3$. The initial condition is given by

$$z_1(t,x) = \begin{cases} \frac{1}{10}\cos(\frac{x\pi}{2}), & t \in [-5,0], \\ 0, & t \in (-\infty,-5), \end{cases} \quad z_2(t,x) = \begin{cases} \frac{1}{10}\sin(x\pi), & t \in [-5,0], \\ 0, & t \in (-\infty,-5), \end{cases}$$

where $x \in O$, and the boundary condition is the homogeneous Dirichlet boundary condition. Then we have the following result from Theorem 1 and Corollary 1.

Corollary 3. The DRDNNs (2.5) with the above parameters are input-to-state stable via the following multiple impulsive control (I) or stabilising impulsive control (II): (I): $\frac{\ln 4}{\hat{\eta}} - \frac{\ln 4/3}{\hat{\eta}} > 7.9244$, $\sigma_{2k-1} = \frac{4}{3}$, $\sigma_{2k} = \frac{1}{4}$, $k \in \mathbb{N}$; (II): $\hat{\eta} < 0.3499$, $\hat{\eta} > 0$, $\sigma_k = \frac{1}{4}$, $k \in \mathbb{N}$.

Figure 1 illustrates the state norms of the DRDNNs via the impulsive control (I) and (II) under the external input by $u_1(t, x) = \sin(x\pi)$ and $u_2(t, x) = \frac{1}{t+1}\cos(t\pi/2)$. We can see that the DRDNNs are bounded under the bounded spatiotemporal external input, which corresponds to the ISS property.



Figure 1. The state norms of the DRDNNs via the multiple impulsive control (I) and stabilising impulsive control (II) in Example 1.

Example 2. Consider the DRDNNs with multiple impulses which consist of two neurons on $O = \{x | -1 \le x \le 1\}$, where the parameters are given by

$$D = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.5 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 1.9 \end{pmatrix}, B = \begin{pmatrix} -0.6 & 0.8 \\ 0.4 & -0.4 \end{pmatrix}, P = \begin{pmatrix} 0.1 & 0.4 \\ 0.4 & -0.6 \end{pmatrix}, Q = \begin{pmatrix} 0.4 & 0.3 \\ 0.5 & 0.4 \end{pmatrix},$$

and $\tau = 1.8$, $k(s) = e^{-s}$, $f(s) = 0.1 \tanh(s)$. The initial condition is given by

$$z_1(t,x) = \begin{cases} \cos(\frac{x\pi}{2}), & t \in [-5,0], \\ 0, & t \in (-\infty, -5), \end{cases} \quad z_2(t,x) = \begin{cases} \sin(x\pi), & t \in [-5,0], \\ 0, & t \in (-\infty, -5), \end{cases}$$

where $x \in O$, and the boundary condition is the homogeneous Dirichlet boundary condition. Then we have the following result from Theorem 1 and Corollary 2.

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Corollary 4. The DRDNNs (2.5) with the above parameters are input-to-state stable with the following multiple impulsive disturbance (III) or destabilising impulsive disturbance (IV): (III): $\frac{\ln 3}{\hat{\eta}} - \frac{\ln 2}{\hat{\eta}} < 1.2423$, $\sigma_{3k-2} = \frac{3}{2}$, $\sigma_{3k-1} = 2$, $\sigma_{3k} = \frac{1}{2}$, $k \in \mathbb{N}$; (IV): $\hat{\eta} > 0.9791$, $\sigma_k = \frac{3}{2}$, $k \in \mathbb{N}$.

Figure 2 illustrates the state norms of the DRDNNs with the impulsive disturbance (III) and (IV) under the external input by $u_1(t, x) = \cos(t\pi/10) \sin(x\pi)$ and $u_2(t, x) = (\sin(t\pi/10) + 1) \cos(t\pi/2)$ in the continuous dynamics, where the ISS property is also observed.



Figure 2. The state norms of the DRDNNs with the multiple impulsive disturbance (III) and destabilising impulsive disturbance (IV) in Example 2.

Remark 5. Because of the multiple impulses and the infinite distributed delays, the existing results in [27, 28, 43] are invalid for these two numerical examples.

5. Conclusions

This work addresses the ISS issues of DRDNNs with multiple impulses after reformulating the neural-network model in term of an abstract impulsive functional differential equation. The ISS property is studied by the direct estimate of mild solution and an integral inequality with infinite distributed delay. The obtained results show that the ISS property can be ensured if the intervals between the multiple impulses are bounded. Note that the impulsive sequences considered here have a fixed dwell-time. A more general class of impulsive sequences satisfying the average dwell-time condition is considered in ISS literature [8,9]. Thus, the further work will focus on ISS analysis of impulsive DRDNNs under the average dwell-time condition.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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