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Invariance of separation in covering approximation spaces

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Abstract: In this paper, the invariance of separation in covering approximation spaces are discussed. This paper proves that some separations in covering approximation spaces are invariant to reducts of coverings, invariant to covering approximation subspaces and invariant under *CAP*-transformations of covering approximation spaces. These results deepen and enrich theory of separations in covering approximation spaces, which is helpful to give further researches and applications of Pawlak rough set theory in information sciences.

1. Introduction

In information sciences, in order to extract useful information hidden in voluminous data, many methods were proposed, including classical logics and classical mathematics. Rough set theory, introduced by Z. Pawlak in [1], plays an important role in applications of these methods (see [2–9], for example). In the classical rough set theory, Pawlak approximation spaces are based on partitions of the universe of discourse, but this requirement is not satisfied in some situations [10]. In the past years, Pawlak approximation spaces were extended to covering approximation spaces, and used in information sciences (see [10–22], for example).

It is easy to see that topological spaces on finite universes are special covering approximation spaces. Thus, some topological properties are considered in covering approximation spaces. In addition, Separations play important roles in topological spaces. Recently, S_i-separations

$(i = 0, 1, 2, 3, d, r)$ in covering approximation spaces draw our attention. And some interesting results were obtained ([12, 13]).

However, just as separations in topological spaces [23–25], the theoretical research framework about separations in covering approximation spaces can be constructed by relations, characterizations and invariance. More precisely, this framework can be described as follows.

Framework 1. *The theoretical research framework about separations in covering approximation spaces consists of the following three parts.*

Part 1: Establish some relations among separations in covering approximation spaces. These relations mainly include implications, not implications, and equivalences.

Part 2: Give some equivalent characterizations of separations in covering approximation spaces. These equivalent characterizations are mainly shown by elements of covering, the covering upper approximation of subsets and the covering lower approximation of subsets, respectively.

Part 3: Study invariance of separation in covering approximation spaces. The invariance of separation mainly includes invariance of separation about reducts of covering, subspaces and transformations.

Part 1 and Part 2 of Framework 1 were researched in [12, 13]. However, there are not results on Part 3 of Framework 1. Obviously, this research can be performed by means of investigations of the following questions.

Question 1. *Let $i = 0, 1, 2, 3, d, r$.*

(1) Is S_i-separation in covering approximation spaces invariant to reducts of coverings?

(2) Is S_i-separation in covering approximation spaces invariant to covering approximation subspaces?

(3) Is S_i-separation in covering approximation spaces invariant under some transformations of covering approximation spaces?

This paper investigates covering upper approximations and covering lower approximations of subsets in covering approximation spaces and their reduction spaces (resp. their subspaces, their transformations), and give some relations among covering approximations of these subsets. Based on these results, we answer Question 1 positively.

The remainder of this paper is organized as follows. Section 2 reviews some definitions and remarks. The invariance of separations in covering approximation spaces are considered in Sections 3–5. In section 6, some concluding remarks are provided.

2. Definitions and remarks

To begin with, the definition of covering approximation space is reviewed.

Definition 1. *(cf. [1, 22]) Let U, the universe of discourse, be a set and C be a family of subsets of U.*

(1) C is called a covering of U if $\bigcup \{K : K \in C\} = U$. Furthermore, C is called a partition on U if also $K \bigcap K' = \emptyset$ for all $K, K' \in C$, where $K \neq K'$.

(2) The pair $(U; C)$ is called a covering approximation space (resp. a Pawlak approximation space) if C is a covering (resp. a partition) of U.

All results this paper proposed are based on the assumption that the universe U is a finite set. Next we review the following upper approximation operator $\overline{C}$ and lower approximation operator $\underline{C}$ on $(U; C)$ [17].

Definition 2. *(cf. [17]) Let $(U; C)$ be a covering approximation space. For each $X \subseteq U$, put*

$\underline{C}(X) = \bigcup\{K : K \in C \text{ and } K \subseteq X\}$,

$\overline{C}(X) = U - \underline{C}(U - X)$.

(1) $\overline{C} : 2^U \longrightarrow 2^U$ is called a covering upper approximation operator, simply denoted by X^*.

(2) $\underline{C} : 2^U \longrightarrow 2^U$ is called a covering lower approximation operator, simply denoted by X_*.

Before introducing concepts of separations, we give some notations as follows.

Notation 1. *Let $(U; C)$ be a covering approximation space. Throughout this paper, we use the following notations, where $\mathcal{F}$ is a family of subsets of U and $x \in U$.*

(1) $\bigcap \mathcal{F} = \bigcap_{F \in \mathcal{F}} F$.

(2) $\bigcup \mathcal{F} = \bigcup_{F \in \mathcal{F}} F$.

(3) $C_x = \{K : x \in K \in C\}$.

(4) $N(x) = \bigcap C_x$.

(5) $\overline{N}(x) = \bigcap\{K^* : K \in C_x\}$.

(6) $D(x) = U - \bigcup(C - C_x)$.

Now we define separations in covering approximation spaces.

Definition 3. *(cf. [12]) Let $(U; C)$ be a covering approximation space. Some separations are defined as follows.*

(1) S_0-separation: if $x, y \in U$ and $x \neq y$, then there exists $K \in C$ such that $K \bigcap\{x, y\} = \{x\}$ or $K \bigcap\{x, y\} = \{y\}$.

(2) S_1-separation: if $x, y \in U$ and $x \neq y$, then there exist $K_x, K_y \in C$ such that $K_x \bigcap\{x, y\} = \{x\}$ and $K_y \bigcap\{x, y\} = \{y\}$.

(3) S_2-separation: if $x, y \in U$ and $x \neq y$, then there exist $K_x, K_y \in C$ such that $x \in K_x$, $y \in K_y$ and $K_x \bigcap K_y = \emptyset$.

(4) S_3-separation: if $x \in U$ and $x \notin X \subseteq U$, then there exists $K \in C$ such that $x \in K$ and $K \bigcap X = \emptyset$.

(5) S_d-separation: if $x \in U$, then there exists $K \in C$ such that $\{x\} = K \bigcap D(x)$.

(6) S_r-separation: if $x \in K \in C$, then $D(x) \subseteq K$.

Some explanations of separations are provided in the following.

Remark 1. *Let $(U; C)$ be a covering approximation space.*

(1) If $(U; C)$ satisfies S_0-separation, then for every pair of distinct $x, y \in U$ there is $K \in C$ containing exactly one of x and y.

(2) If $(U; C)$ satisfies S_1-separation, then for every pair of distinct $x, y \in U$ there is $K \in C$ containing x and $K' \in C$ containing y such that K does not contain y and K' does not contain x.

(3) If $(U; C)$ satisfies S_2-separation, then every pair of distinct $x, y \in U$ can be separated from elements of C.

(4) If $(U; C)$ satisfies S_3-separation, then for each $x \in U$ and each subset X of U not containing x there is $K \in C$ containing x such that K and X are separated.

(5) If $(U; C)$ satisfies S_d-separation, then for each $x \in U$ there is $K \in C$ containing x such that whenever $y \in U$ and $y \neq x$, either K does not contain y or K' contains y for $K' \in C$ satisfying $x \notin K'$.

(6) $(U; C)$ satisfies S_r-separation, i.e., for each $x \in U$ and each $K \in C$ containing x, if $y \in U$ such that $K' \in C$ does not contain x implying K' does not contain y, then K contains y.

Let $i = 0, 1, 2, 3, d, r$. Throughout this paper, we call a covering approximation space $(U; C)$ with S_i-separation an S_i-covering approximation space. For short, S_i-covering approximation spaces are called S_i-spaces (Note: S_i-separations and S_i-spaces in [13] are called G_i-separations and G_i-spaces, respectively).

The following example illustrates the concepts of separation.

Example 1. *(1) Let $U = \{a, b, c\}$ and $C = \{\{a, b\}, \{c\}\}$. Then (U, C) is a Pawlak approximation space. Each Pawlak approximation space is an S_r-space ([12, Remark 3.4]). Hence, (U, C) is an S_r-space. On the other hand, $N(a) = \{a, b\} \neq \{a\}$, so (U, C) is not an S_1-space ([12, Theorem 3.6]).*

(2) Let $U = \{a, b, c\}$ and $C = \{\{a, c\}, \{b, c\}\}$. It is easy to check that (U, C) is an S_0-space. Since $D(c) = U$ and $K \cap D(x) = K \neq \{c\}$ for each $K \in C$, $(U; C)$ is not an S_d-space.

(3) Let $U = \{a, b\}$ and $C = \{\{a, b\}, \{a\}\}$. Since $D(a) = \{a, b\}$ and $D(b) = \{b\}$, we have $\{a\} = \{a\} \cap D(a)$ and $\{b\} = \{a, b\} \cap D(b)$, Note that $\{a\}, \{a, b\} \in C$. So (U, C) is an S_d-space. On the other hand, $N(b) = \{a, b\} \neq \{b\}$, so (U, C) is not an S_1-space ([12, Theorem 3.6]).

(4) Let $U = \{a, b, c\}$ and $C = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Since $N(a) = \{a\}$, $N(b) = \{b\}$ and $N(c) = \{c\}$, (U, C) is an S_1-space. Whenever $K, K' \in C$, $K \cap K' \neq \emptyset$, So (U, C) is not an S_2-space.

(5) Let $U = \{a, b, c, d\}$ and $C = \{\{a, b\}, \{a, c\}, \{c, d\}, \{b, d\}\}$. It is not difficult to check that (U, C) is an S_2-space. Since $\{a\} \notin C$, (U, C) is not an S_3-space ([12, Theorem 3.13]).

The following lemma shows the relation among all separations in covering approximation spaces.

Lemma 1. *[13] The following implications are known and none of these implications can be reversed by Example 1.*
(1) S_3-space $\Longrightarrow S_2$-space $\Longrightarrow S_1$-space $\Longrightarrow S_d$-space $\Longrightarrow S_0$-space.
(2) S_1-space $\Longrightarrow S_r$-space.

However, we have the following relations among separations in Pawlak approximation spaces, which further illustrates the concepts of separation.

Corollary 1. *Let $(U; C)$ be a Pawlak approximation space. Then the following are equivalent:*
(1) $(U; C)$ is an S_3-space,
(2) $(U; C)$ is an S_2-space,
(3) $(U; C)$ is an S_1-space,
(4) $(U; C)$ is an S_d-space,
(5) $(U; C)$ is an S_0-space.

Proof. By Lemma 1, we have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). So we only need to prove that (5) $\Longrightarrow$ (1).

Suppose $(U; C)$ is an S_0-space. Since $(U; C)$ is a Pawlak approximation space, C is a partition of U. Let $x \in U$ and $x \notin X \subset U$. Take $K \in C$ satisfying $x \in K$. It suffices to prove that $K \cap X = \emptyset$. In fact, if $K \cap X \neq \emptyset$, then there is $y \in K \cap X$. So we obtain $y \neq x$. Since $(U; C)$ is an S_0-space, there is $K' \in C$

such that $K' \bigcap \{x, y\} = \{x\}$ or $K' \bigcap \{x, y\} = \{y\}$. Without loss of generality, we assume $K' \bigcap \{x, y\} = \{x\}$. Then $y \notin K'$. It is not hard to see that $K' \neq K$ and $x \in K' \bigcap K$. This contradicts that C is a partition of U. So $K \bigcap X = \emptyset$. $\qquad\square$

Remark 2. *Separations in covering approximation spaces play an important role in not only applications but also theoretical research of rough set theory. For example, in the fields of rough set data analysis, it is an interesting question how to characterize the conditions under which $\{N(x) : x \in U\}$ forms a partition of U for a covering approximation space $(U; C)$ ([17, 26]). Reference [12] proves that a covering approximation space $(U; C)$ is an S_r-space if and only if $\{N(x) : x \in U\}$ forms a partition of U. Thus, we can investigate the above question by discussing S_r-spaces. Furthermore, one concludes that $\{N(x) : x \in U\}$ forms a partition of U if C is a partition of U.*

The following results are known.

Lemma 2. *(cf. [13, 17, 22]) Let $(U; C)$ be a covering approximation space. Then the following results hold:*
  (1) $U^* = U_* = U$ and $\emptyset^* = \emptyset_* = \emptyset$,
  (2) if $X \subseteq U$, then $X_* \subseteq X \subseteq X^*$,
  (3) if $X \subseteq Y \subseteq U$, then $X_* \subseteq Y_*$ and $X^* \subseteq Y^*$,
  (4) if $X \subseteq U$, then $(X_*)_* = X_*$ and $(X^*)^* = X^*$,
  (5) if $K \in C$, then $K_* = K$,
  (6) if $X \subseteq U$, then $(U - X)_* = U - X^*$ and $(U - X)^* = U - X_*$,
  (7) if $x \in U$, then $D(x) = \{x\}^*$.

The characteristics about all separations in covering approximation spaces are provided.

Theorem 1. *(cf. [13]) Let $(U; C)$ be a covering approximation space. Then the following* results *hold:*
  (1) $(U; C)$ is an S_0-space if and only if $\{x\}^* \neq \{y\}^*$ for each pair $x, y \in U$ with $x \neq y$,
  (2) $(U; C)$ is an S_1-space if and only if $\{x\} = N(x)$ for each $x \in U$,
  (3) $(U; C)$ is an S_2-space if and only if $\{x\} = \overline{N}(x)$ for each $x \in U$,
  (4) $(U; C)$ is an S_3-space if and only if $\{x\} \in C$ for each $x \in U$,
  (5) $(U; C)$ is an S_d-space if and only if whenever $x \in U$, $\{x\} = X_* \bigcap Y^*$ for some $X, Y \subseteq U$,
  (6) $(U; C)$ is an S_r-space if and only if $x \in \{y\}^* \implies y \in \{x\}^*$ for each pair $x, y \in U$.

3. On reducts of coverings

To consider the invariance of separation in covering approximation spaces, the reduct of covering is introduced firstly.

Definition 4. *(cf. [20, 21]) Let U be the universe of discourse and C be a covering of U.*
  (1) $K \in C$ is called reducible in C if $K = \bigcup \mathcal{F}$ for $\mathcal{F} \subseteq C - \{K\}$. Otherwise, K is called irreducible in C.
  (2) C is called irreducible if K is irreducible in C for each $K \in C$; otherwise C is called reducible.
  (3) C' is called the reduct of C if C' is obtained by deleting all reducible elements in C. It is clear that C' is an irreducible subcovering of C.

Proposition 1. *Let C be a covering of the set U. Each set in C is a union of some irreducible elements of C.*

However, Proposition 1 does not hold for infinite set U. A counter example is provided to show it.

Example 2. *Let U be the set of all positive integers. Assume $C = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}, \ldots\} \cup \{\{2\} \cup \{3, 4, 5, 6, \ldots\}, \{2\} \cup \{4, 5, 6, \ldots\}, \{2\} \cup \{5, 6, \ldots\}, \ldots\}$ is a covering of U. It is easy to see that $\{1,2\}, \{3\}, \{4\}, \{5\}, \{6\}, \ldots$ are irreducible elements of C. But $\{2\} \cup \{n, n + 1, n + 2, \ldots\}$ is a reducible element of C because $\{2\} \cup \{n, n + 1, n + 2, \ldots\} = \{2\} \cup \{n + 1, n + 2, \ldots\} \cup \{n\}$ holds, where $n \in U$ and $n > 2$. Thus, $\{2\} \cup \{n, n + 1, n + 2, \ldots\}$ is not a union of some irreducible elements of C.*

Let $(U; C)$ be a covering approximation space, C' be the reduct of C and $X \subseteq U$. According to Definition 4, C' is the unique reduct of C. In this section, the covering lower approximation of X and the covering upper approximation of X in covering approximation space $(U; C')$ are denoted by $X_\#$ and $X^\#$, respectively. For each $x \in U$, $D'(x) = U - \bigcup(C' - C'_x)$, $N'(x) = \bigcap\{K : K \in C'_x\} = \bigcap C'_x$ and $\overline{N'}(x) = \bigcap\{K^\# : K \in C'_x\}$. The following propositions can be provided.

Proposition 2. *Let $(U; C)$ be a covering approximation space and C' be the reduct of C. For each $x \in U$, if $K \in C_x$, then $K' \subseteq K$ for some $K' \in C'_x$.*

Proof. (1) Let $x \in U$ and $K \in C_x$. If $K \in C'_x$, then $K' \subseteq K$ for $K' = K \in C'_x$. If $K \notin C'_x$, then K is a reducible elements in C. It is not difficult to see that there exists $\mathcal{F} \subseteq C'$ such that $K = \bigcup \mathcal{F}$. Since $x \in K$, we choose $K' \in \mathcal{F}$ satisfying $x \in K'$. Thus, one derives $K' \in C'_x$ and $K' \subseteq K$.

$\square$

Proposition 3. *[21] Let $(U; C)$ be a covering approximation space and C' be the reduct of C. For $X \subseteq U$, $X_* = X_\#$ and $X^* = X^\#$.*

Proposition 4. *Let $(U; C)$ be a covering approximation space, C' be the reduct of C and $x \in U$. Then the following conclusions hold:*
(1) $D(x) = D'(x)$,
(2) $N(x) = N'(x)$,
(3) $\overline{N}(x) = \overline{N'}(x)$.

Proof. (1) It suffices to prove that $\bigcup(C - C_x) = \bigcup(C' - C'_x)$. Assume $y \in \bigcup(C - C_x)$. Then $y \in K$ for some $K \in C - C_x$. By Proposition 2 (1), there is $K' \in C'$ such that $y \in K' \in C'$ and $K' \subseteq K$. $K' \in C' - C'_x$ because $x \notin K$ and $x \notin K'$, i.e., $K' \notin C'_x$. It follows that $y \in \bigcup(C' - C'_x)$. Conversely, we assume $y \in \bigcup(C' - C'_x)$. Then $y \in K$ for some $K \in C' - C'_x$. So $K \in C' \subseteq C$ and $x \notin K$. Thus $K \in C - C_x$. It follows that $y \in \bigcup(C - C_x)$. It proves that $\bigcup(C - C_x) = \bigcup(C' - C'_x)$.

(2) Since $C' \subseteq C$, $C'_x \subseteq C_x$, one sees $N(x) = \bigcap C_x \subseteq \bigcap C'_x = N'(x)$. On the other hand, we assume $y \in N'(x)$. Whenever $K \in C_x$, by Proposition 2 (1), there is $K' \in C'_x$ such that $K' \subseteq K$, hence $y \in N'(x) = \bigcap C'_x \subseteq K' \subseteq K$. It proves that $y \in N(x)$. So $N'(x) \subseteq N(x)$. Consequently, we have $N(x) = N'(x)$.

(3) Since $C' \subseteq C$, $C'_x \subseteq C_x$, we have $\overline{N}(x) = \bigcap\{K^* : K \in C_x\} \subseteq \bigcap\{K^\# : K \in C'_x\} = \overline{N'}(x)$. On the other hand, if $y \in \overline{N'}(x)$, then $x \in K'^\#$ for each $K' \in C'_x$. Whenever $K \in C_x$, by Proposition 2 (1), there is $K' \in C'_x$ such that $K' \subseteq K$. Based on Lemma 2 (3) and Proposition 2 (2), it is not hard to see that

$y \in \overline{N'}(x) \subseteq K'^{\#} = K'^{*} \subseteq K^{*}$. This proves that $y \in \overline{N}(x)$. So $\overline{N'}(x) \subseteq \overline{N}(x)$. Consequently, one gets $\overline{N}(x) = \overline{N'}(x)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Now we give the main theorem of this section.

Theorem 2. *Let $(U; C)$ be a covering approximation space and C' be the reduct of C. Then the following conclusions hold:*

(1) $(U; C)$ is an S_0-space if and only if $(U; C')$ is an S_0-space,

(2) $(U; C)$ is an S_1-space if and only if $(U; C')$ is an S_1-space,

(3) $(U; C)$ is an S_2-space if and only if $(U; C')$ is an S_2-space,

(4) $(U; C)$ is an S_3-space if and only if $(U; C')$ is an S_3-space,

(5) $(U; C)$ is an S_d-space if and only if $(U; C')$ is an S_d-space,

(6) $(U; C)$ is an S_r-space if and only if $(U; C')$ is an S_r-space.

Proof. (1) By Proposition 2 (2), $\{x\}^{*} \neq \{y\}^{*}$ if and only if $\{x\}^{\#} \neq \{y\}^{\#}$ for each pair $x, y \in U$ with $x \neq y$. By Theorem 1 (1), $(U; C)$ is an S_0-space if and only if $(U'; C')$ is an S_0-space.

(2) By Proposition 4 (2), $N(x) = N'(x)$ for each $x \in U$. It follows that $\{x\} = N(x)$ if and only if $\{x\} = N'(x)$. By Theorem 1 (2), $(U; C)$ is an S_1-space if and only if $(U'; C')$ is an S_1-space.

(3) By Proposition 4 (3), $\overline{N}(x) = \overline{N'}(x)$ for each $x \in U$. It follows that $\{x\} = \overline{N}(x)$ if and only if $\{x\} = \overline{N'}(x)$. By Theorem 1 (3), $(U; C)$ is an S_2-space if and only if $(U'; C')$ is an S_2-space.

(4) By Definition 4, $\{x\} \in C$ if and only if $\{x\} \in C'$ for each $x \in U$. By Theorem 1 (4), $(U; C)$ is an S_3-space if and only if $(U'; C')$ is an S_3-space.

(5) By Proposition 2 (2), whenever $x \in U$, $\{x\} = X_{*} \bigcap Y^{*}$ if and only if $\{x\} = X_{\#} \bigcap Y^{\#}$ for some $X, Y \subseteq U$. By Theorem 1 (5), $(U; C)$ is an S_d-space if and only if $(U'; C')$ is an S_d-space.

(6) By Proposition 2 (2), for each pair $x, y \in U$, $x \in \{y\}^{*} \implies y \in \{x\}^{*}$ if and only if $x \in \{y\}^{\#} \implies y \in \{x\}^{\#}$. By Theorem 1 (6), $(U; C)$ is an S_r-space if and only if $(U'; C')$ is an S_r-space. $\qquad$ $\square$

Remark 3. *For each $i = 0, 1, 2, 3, d, r$, Theorem 2 shows that S_i-separation in covering approximation spaces is invariant with respect to the reduct of coverings. It indicates that we can reduce some elements of C without influencing separations. In network applications, some existing approaches deal with the covering directly. In contrast, we deal with C', which is the reduct of C. Therefore, both time and space requirements are reduced, and network securities are preserved.*

4. On covering approximation subspaces

The following definition presents covering approximation subspaces.

Definition 5. *(cf. [11]) Let $(U; C)$ be a covering approximation space and $U' \subseteq U$. Take $C' = \{K \bigcap U' : K \in C\}$, then $(U'; C')$ is a covering approximation space. $(U'; C')$ is called a subspace of $(U; C)$.*

Let $(U; C)$ be a covering approximation space, $(U'; C')$ be a subspace of $(U; C)$ and $X \subseteq U'$. In this section, the covering lower approximation of X and the covering upper approximation of X in covering approximation space $(U'; C')$ are denoted by $X_{\#}$ and $X^{\#}$, respectively. For each $x \in U'$, $D'(x) = U' - \bigcup(C' - C'_x)$, $N'(x) = \bigcap\{K : K \in C'_x\} = \bigcap C'_x$ and $\overline{N'}(x) = \bigcap\{K^{\#} : K \in C'_x\}$. The following two results are proved.

Proposition 5. *Let $(U;C)$ be a covering approximation space and $(U';C')$ be a subspace of $(U;C)$. Then the following conclusions hold:*

(1) If $X \subseteq U$, then $(X \cap U')^\# \subseteq X^* \cap U'$.

(2) If $X \subseteq U$, then $X_* \cap U' \subseteq (X \cap U')_\#$.

Proof. (1) Because $(X \cap U')^\# = U' - (U' - X \cap U')_\# = U' - (U' - X)_\#$ and $X^* \cap U' = (U - (U - X)_*) \cap U' = U' - U' \cap (U - X)_*$, it suffices to prove that $U' \cap (U - X)_* \subseteq (U' - X)_\#$. Assume $x \in U' \cap (U - X)_*$. Then $x \in U'$ and there is $L \in C$ such that $x \in L \subseteq U - X$. It follows that $x \in L \cap U' \subseteq U' - X$. Note that $L \cap U' \in C'$. So $x \in (U' - X)_\#$. It proves that $U' \cap (U - X)_* \subseteq (U' - X)_\#$.

(2) Assume $x \in X_* \cap U'$. Then $x \in U'$ and there is $K \in C$ such that $x \in K \subseteq X$. Thus $x \in K \cap U' \subseteq X \cap U'$. Note that $K \cap U' \in C'$. So $x \in (X \cap U')_\#$. It proves that $X_* \cap U' \subseteq (X \cap U')_\#$. $\square$

Remark 4. *In Proposition 5 (1) and (2), "$\subseteq$" can not be replaced by "$=$" (see Examples 3 and 4). However, we have the following result.*

Proposition 6. *Let $(U;C)$ be a covering approximation space and $(U';C')$ be a subspace of $(U;C)$. If $X \subseteq U'$, then $X^\# = X^* \cap U'$. Specially, $\{x\}^\# = \{x\}^* \cap U'$ if $x \in U'$.*

Proof. Assume $X \subseteq U'$. By Proposition 5 (1), $X^\# \subseteq X^* \cap U'$, we only need to prove $X^* \cap U' \subseteq X^\#$. Note that $X^* \cap U' = (U - (U - X)_*) \cap U' = U' - (U - X)_* \cap U'$ and $X^\# = U' - (U' - X)_\#$. It suffices to prove that $(U' - X)_\# \subseteq (U - X)_* \cap U'$. Assume $y \in (U' - X)_\#$. Then there is $K' \in C'$ such that $y \in K' \subseteq U' - X$, i.e., there is $K \in C$ such that $y \in K \cap U' \subseteq U' - X$. Thus $(K \cap U') \cap X = \emptyset$, i.e., $K \cap X = \emptyset$. Hence, we derive $y \in K \subseteq U - X$. It follows that $y \in (U - X)_* \cap U'$. It proves that $(U' - X)_\# \subseteq (U - X)_* \cap U'$. $\square$

The following examples show that "$\subseteq$" is not replaced by "$=$" in Proposition 5 (1) and (2).

Example 3. *Let $U = \{a, b, c\}$, $C = \{\{a, b\}, \{c\}\}$, $U' = \{b, c\}$, $C' = \{\{b\}, \{c\}\}$ and $X = \{a, c\}$. Then $X^* \cap U' \nsubseteq (X \cap U')^\#$. In fact, $(X \cap U')^\# = \{c\}^\# = U' - (U' - \{c\})_\# = U' - \{b\}_\# = U' - \{b\} = \{c\}$. On the other hand, $X^* = U - (U - X)_* = U - \{b\}_* = U - \emptyset = U$. It follows that $X^* \cap U' = U \cap U' = U' = \{b, c\}$. So $X^* \cap U' \nsubseteq (X \cap U')^\#$.*

Example 4. *Let $U = \{a, b\}$, $C = \{\{a, b\}\}$, $U' = \{a\}$, $C' = \{\{a\}\}$ and $X = \{a\}$. Then $(X \cap U')_\# \nsubseteq X_* \cap U'$. In fact, $X_* \cap U' = \emptyset$ and $(X \cap U')_\# = \{a\}$.*

To show the invariance of separation, the following results are proved.

Proposition 7. *Let $(U;C)$ be a covering approximation space, $(U';C')$ be a subspace of $(U;C)$ and $x \in U$. Then the following conclusions hold:*

(1) $D(x) \cap U' = D'(x)$,

(2) $N(x) \cap U' = N'(x)$,

(3) $\overline{N}(x) \cap U' \supseteq \overline{N'}(x)$.

Proof. By computing directly, we have the following results.

(1) $D(x) \cap U' = (U - \bigcup(C - C_x)) \cap U' = U' - U' \cap (\bigcup(C - C_x)) = U' - U' \cap (\bigcup\{K : x \notin K \in C\}) = U' - \bigcup\{K \cap U' : x \notin K \in C\} = U' - \bigcup\{K' : x \notin K \cap U' \in C'\} = U' - \bigcup(C' - C'_x) = D'(x)$.

(2) $N(x) \cap U' = (\bigcap\{K : x \in K \in C\}) \cap U' = \bigcap\{K \cap U' : x \in K \in C\} = \bigcap\{K' : x \in K' \in C'\} = N'(x)$.

(3) $\overline{N}(x) \cap U' = (\bigcap\{K^* : x \in K \in C\}) \cap U' = \bigcap\{K^* \cap U' : x \in K \in C\} \supseteq \bigcap\{(K \cap U')^\# : x \in K' \in C'\} = \bigcap\{K'^\# : x \in K' \in C'\} = \overline{N'}(x)$. $\square$

Now we give the main theorem of this section.

Theorem 3. *Let $(U; C)$ be a covering approximation space and $(U'; C')$ be a subspace of $(U; C)$. Then the following conclusions hold:*

(1) if $(U; C)$ is an S_0-space, then $(U'; C')$ is an S_0-space,

(2) if $(U; C)$ is an S_1-space, then $(U'; C')$ is an S_1-space,

(3) if $(U; C)$ is an S_2-space, then $(U'; C')$ is an S_2-space,

(4) if $(U; C)$ is an S_3-space, then $(U'; C')$ is an S_3-space,

(5) if $(U; C)$ is an S_d-space, then $(U'; C')$ is an S_d-space,

(6) if $(U; C)$ is an S_r-space, then $(U'; C')$ is an S_r-space.

Proof. (1) Assume $(U; C)$ is an S_0-space. By Theorem 1 (1), $\{x\}^* \neq \{y\}^*$ for any $x, y \in U'$ satisfying $x \neq y$. We claim that $\{x\}^\# \neq \{y\}^\#$. In fact, if $\{x\}^\# = \{y\}^\#$, then $\{x\}^* \cap U' = \{y\}^* \cap U'$ from Proposition 6. Thus $x \in \{x\}^* \cap U' = \{y\}^* \cap U' \subseteq \{y\}^*$. That is $\{x\}^* \subseteq \{y\}^*$. Similarly, we obtain $\{y\}^* \subseteq \{x\}^*$. It follows that $\{x\}^* = \{y\}^*$. This is a contradiction.

(2) Assume $(U; C)$ is an S_1-space. By Theorem 1 (2), $N(x) = \{x\}$ for each $x \in U'$. Thus, we have $N(x) \cap U' = \{x\}$. From Proposition 7 (2), one sees $N'(x) = N(x) \cap U' = \{x\}$. According to Theorem 1 (2), $(U'; C')$ is an S_1-space.

(3) Assume $(U; C)$ is an S_2-space. By Theorem 1 (3), $\overline{N}(x) = \{x\}$ for each $x \in U'$. Thus, we have $\overline{N}(x) \cap U' = \{x\}$. By Proposition 7 (3), $\{x\} \subseteq \overline{N'}(x) \subseteq \overline{N}(x) \cap U' = \{x\}$, i.e., $\overline{N'}(x) = \{x\}$. By Theorem 1 (3), $(U'; C')$ is an S_2-space.

(4) Assume $(U; C)$ is an S_3-space. By Theorem 1 (4), $\{x\} \in C$ for each $x \in U'$. Thus, we have $\{x\} = \{x\} \cap U' \in C'$. By Theorem 1 (4), $(U'; C')$ is an S_3-space.

(5) Assume $(U; C)$ is an S_d-space. Take $x \in U'$. Then there is $K \in C$ such that $\{x\} = K \cap D(x)$. Thus, we obtain $\{x\} = (K \cap U') \cap (D(x) \cap U')$. By Proposition 7 (1), $D(x) \cap U' = D'(x)$. It follows that $\{x\} = (K \cap U') \cap D'(x)$. Note that $K \cap U' \in C'$. So $(U'; C')$ is an S_d-space.

(6) Assume $(U; C)$ is an S_r-space. Let $x, y \in U'$ and $x \in \{y\}^\#$. By Theorem 1 (6), we only need to prove that $y \in \{x\}^\#$. Since $\{y\}^\# = \{y\}^* \cap U'$ from Proposition 6, $x \in \{y\}^*$. By Theorem 1 (6), $y \in \{x\}^*$ because $(U; C)$ is an S_r-space. It follows that $y \in \{x\}^* \cap U' = \{x\}^\#$. $\qquad\square$

Remark 5. *For each $i = 0, 1, 2, 3, d, r$, Theorem 3 shows that S_i-separation in covering approximation spaces is invariant to covering approximation subspaces. It indicates that some dynamic methods [27] are applicable to covering based rough sets theory. We can employ different subsets of U to produce different coverings, and obtain more stable rules.*

5. On transformations of covering approximation spaces

In this section, we use the following notations for covering approximation space $(U'; C')$. For $X \subseteq U'$, the covering lower approximation of X and the covering upper approximation of X in covering approximation space $(U'; C')$ are denoted by $X_\#$ and $X^\#$, respectively. For $x \in U'$, $D'(x) = U' - \bigcup(C' - C'_x)$, $N'(x) = \bigcap\{K : K \in C'_x\} = \bigcap C'_x$ and $\overline{N'}(x) = \bigcap\{K^\# : K \in C'_x\}$. The definition of transformation is proposed.

Definition 6. *Let $(U; C)$ and $(U'; C')$ be two covering approximation spaces. If $f : U \longrightarrow U'$ is a bijective mapping, then f is called a transformation from $(U; C)$ to $(U'; C')$.*

Remark 6. *It is clear that if f is a transformation from $(U;C)$ to $(U';C')$, then f^{-1} is a transformation from $(U';C')$ to $(U;C)$.*

Remark 7. *Let $(U;C)$ and $(U';C')$ be two covering approximation spaces, f be a transformation from $(U;C)$ to $(U';C')$. Whenever $A, B \subseteq U$ and $X, Y \subseteq U'$. Then the following conclusions hold:*

(1) $f(f^{-1}(X)) = X$,
(2) $f^{-1}(f(A)) = A$,
(3) $f(A \bigcup B) = f(A) \bigcup f(B)$,
(4) $f^{-1}(X \bigcup Y) = f^{-1}(X) \bigcup f^{-1}(Y)$,
(5) $f(A \bigcap B) = f(A) \bigcap f(B)$,
(6) $f^{-1}(X \bigcap Y) = f^{-1}(X) \bigcap f^{-1}(Y)$,
(7) $f(A - B) = f(A) - f(B)$,
(8) $f^{-1}(X - Y) = f^{-1}(X) - f^{-1}(Y)$,
(9) $A \neq B \Longrightarrow f(A) \neq f(B)$,
(10) $X \neq Y \Longrightarrow f^{-1}(X) \neq f^{-1}(Y)$.

In general, S_i-separation in covering approximation spaces is not invariant under transformations of covering approximation spaces for $i = 0, 1, 2, 3, d, r$.

Example 5. *Let $(U;C)$ and $(U';C')$ be two covering approximation spaces, where $U = \{a, b, c, d\}$, $U' = \{x, y, z, w\}$, $C = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}\}$ and $C' = \{\{x, y\}, \{x, y, z\}, \{z, w\}\}$. Take $f : U \longrightarrow U'$, where $f(a) = x$, $f(b) = y$, $f(c) = z$ and $f(d) = w$. Then f is a transformation from $(U;C)$ to $(U';C')$.*

(1) Since $\{u\} \in C$ for each $u \in U$, $(U;C)$ is an S_3-space from Theorem 1 (4). By Remark 1, $(U;C)$ is an S_i-space for each $i = 0, 1, 2, 3, d, r$.

(2) Note that $\{x\}^{\#} = U' - (U' - \{x\})_{\#} = U' - \{y, z, w\}_{\#} = U' - \{z, w\} = \{x, y\}$ and $\{y\}^{\#} = U' - (U' - \{y\})_{\#} = U' - \{x, z, w\}_{\#} = U' - \{z, w\} = \{x, y\}$. So we derive $\{x\}^{\#} = \{y\}^{\#}$. Hence, $(U';C')$ is not an S_0-space from Theorem 1 (1). By Remark 1 (1), $(U';C')$ is not an S_i-space for each $i = 0, 1, 2, 3, d$.

(3) It is clear that $N'(z) = \{z\}$ and $N'(w) = \{z, w\}$. So $z \in N'(w)$ and $w \notin N'(z)$. By Theorem 1 (6), $(U';C')$ is not an S_r-space.

However, which classes of transformations preserve separations in covering approximation spaces? It is one of important topics in research framework about separations in covering approximation spaces. In order to answer this question, we introduce the following definitions.

Definition 7. *Let $(U;C)$ and $(U';C')$ be two covering approximation spaces, and f be a transformation from $(U;C)$ to $(U';C')$.*

(1) f is called a covering lower approximation-preserving transformation (abbr. CLAP-transformation) if $f(A_*) = (f(A))_{\#}$ for all $A \subseteq U$.

(2) f is called a covering upper approximation-preserving transformation (abbr. CUAP-transformation) if $f(A^*) = (f(A))^{\#}$ for all $A \subseteq U$.

(3) f is called a covering approximation-preserving transformation (abbr. CAP-transformation) if f is a covering both lower and upper approximation-preserving transformation.

The following proposition is obvious.

Proposition 8. *Let* $(U; C)$ *and* $(U'; C')$ *be two covering approximation spaces, and* f *be a transformation from* $(U; C)$ *to* $(U'; C')$. *Then the following conditions are equivalent:*

(1) f is a CLAP-transformation,

(2) f is a CUAP-transformation,

(3) f is a CAP-transformation.

According to Proposition 8, one obtains Lemma 3.

Lemma 3. *Let* $(U; C)$ *and* $(U'; C')$ *be two covering approximation spaces. Then the following conditions are equivalent:*

(1) f is a CAP-transformation from $(U; C)$ to $(U'; C')$,

(2) f^{-1} is a CAP-transformation from $(U'; C')$ to $(U; C)$.

Proof. We only need to prove (1) $\Longrightarrow$ (2) because $f = (f^{-1})^{-1}$.

Assume f be a *CAP*-transformation from $(U; C)$ to $(U'; C')$. Then f^{-1} is a transformation from $(U'; C')$ to $(U; C)$ by Remark 6. Whenever $X \subseteq U'$, we get $(f(f^{-1}(X)))_\# = f((f^{-1}(X))_*)$ because f is a *CLAP*-transformation. Combined with Lemma 7 (1) and (2), $f^{-1}(X_\#) = f^{-1}((f(f^{-1}(X)))_\#) = f^{-1}(f((f^{-1}(X))_*)) = (f^{-1}(X))_*$. It shows that f^{-1} is a *CLAP*-transformation. By Proposition 8, f^{-1} is a *CAP*-transformation. $\qquad\square$

Lemma 4. *Let* $(U; C)$ *and* $(U'; C')$ *be two covering approximation spaces, and* f *be a CAP-transformation from* $(U; C)$ *to* $(U'; C')$. *If* $a \in U$ *and* $x = f(a) \in U'$, *then the following conclusions hold:*

(1) $f(N(a)) = N'(x)$,

(2) $f(\overline{N}(a)) = \overline{N'}(x)$.

Proof. For each $K \in C_a$, $K = K_*$ from Lemma 2 (5). Then one sees f is a *CAP*-transformation. So we get $x = f(a) \in f(K) = f(K_*) = (f(K))_\#$. Hence, there is $H_K \in C'_x$ such that $H_K \subseteq f(K)$.

(1) $f(N(a)) = f(\bigcap\{K : K \in C_a\}) = \bigcap\{f(K) : K \in C_a\} \supseteq \bigcap\{H_K : K \in C_a\} \supseteq N'(x)$. Similarly, $f^{-1}(N'(x)) \supseteq N(a)$ because f^{-1} is a *CAP*-transformation from Proposition 3. By Lemma 7 (1), $N'(x) = ff^{-1}(N'(x)) \supseteq f(N(a))$. Consequently, we obtain $f(N(a)) = N'(x)$.

(2) $f(\overline{N}(a)) = f(\bigcap\{K^* : K \in C_a\}) = \bigcap\{f(K^*) : K \in C_a\} \supseteq \bigcap\{H_K^\# : K \in C_a\} \supseteq \overline{N'}(x)$. By the same method, $f^{-1}(\overline{N'}(x)) \supseteq \overline{N}(a)$ because f^{-1} is a *CAP*-transformation from Proposition 4.7. By Lemma 4.2(1), $\overline{N'}(x) = ff^{-1}(\overline{N'}(x)) \supseteq f(\overline{N}(a))$. Consequently, $f(\overline{N}(a)) = \overline{N'}(x)$. $\qquad\square$

Based on the analysis above, the following results are proved.

Theorem 4. *Let* $(U; C)$ *and* $(U'; C')$ *be two covering approximation spaces, and* f *be a CAP-transformation from* $(U; C)$ *to* $(U'; C')$. *Then the following conclusions hold:*

(1) $(U; C)$ is an S_0-space if and only if $(U'; C')$ is an S_0-space,

(2) $(U; C)$ is an S_1-space if and only if $(U'; C')$ is an S_1-space,

(3) $(U; C)$ is an S_2-space if and only if $(U'; C')$ is an S_2-space,

(4) $(U; C)$ is an S_3-space if and only if $(U'; C')$ is an S_3-space,

(5) $(U; C)$ is an S_d-space if and only if $(U'; C')$ is an S_d-space,

(6) $(U; C)$ is an S_r-space if and only if $(U'; C')$ is an S_r-space.

Proof. By Proposition 3, we only need to prove "only if" parts.

(1) Assume $(U; C)$ is an S_0-space. For any $x, y \in U'$ satisfying $x \neq y$, there are $a, b \in U$ with $a \neq b$ such that $x = f(a)$ and $y = f(b)$. Because $(U; C)$ is an S_0-space, $\{a\}^* \neq \{b\}^*$ from Theorem 1 (1). It follows that $f(\{a\}^*) \neq f(\{b\}^*)$ from Lemma 3 (9). Since f is a *CAP*-transformation, one sees $\{f(a)\}^{\#} = f(\{a\}^*)$ and $\{f(b)\}^{\#} = f(\{b\}^*)$. Consequently, $\{x\}^{\#} = \{f(a)\}^{\#} = f(\{a\}^*) \neq f(\{b\}^*) = \{f(b)\}^{\#} = \{y\}^{\#}$. By Theorem 1 (1), $(U'; C')$ is an S_0-space.

(2) Assume $(U; C)$ is an S_1-space. For each $x \in U'$, there is $a \in U$ such that $x = f(a)$. Because $(U; C)$ is an S_1-space, we have $\{a\} = N(a)$ from Theorem 1 (2). By Lemma 4 (1), one gets $\{x\} = \{f(a)\} = f(\{a\}) = f(N(a)) = N'(x)$. It follows that $(U'; C')$ is an S_1-space from Theorem 1 (2).

(3) Assume $(U; C)$ is an S_2-space. For each $x \in U'$, there is $a \in U$ such that $x = f(a)$. Because $(U; C)$ is an S_2-space, $\{a\} = \overline{N}(a)$ from Theorem 1 (3). By Lemma 4 (2), we have $\{x\} = \{f(a)\} = f(\{a\}) = f(\overline{N}(a)) = \overline{N'}(x)$. It follows that $(U'; C')$ is an S_2-space from Theorem 1 (3).

(4) Assume $(U; C)$ is an S_3-space. For each $x \in U'$, there is $a \in U$ such that $x = f(a)$. Because $(U; C)$ is an S_3-space, we obtain $\{a\} \in C$ from Theorem 1 (4). Hence, one gets $\{a\} = \{a\}_*$. Since f is a *CAP*-transformation, it is easy to see that $f(\{a\}_*) = (f(\{a\}))_{\#}$. Consequently, $\{x\} = \{f(a)\} = f(\{a\}) = f(\{a\}_*) = (f(\{a\}))_{\#} = \{f(a)\}_{\#} = \{x\}_{\#}$. It follows that $\{x\} \in C'$. By Theorem 1 (4), $(U'; C')$ is an S_3-space.

(5) Assume $(U; C)$ is an S_d-space. For each $x \in U'$, there is $a \in U$ such that $x = f(a)$. Because $(U; C)$ is an S_d-space, there are $A, B \subseteq U'$ such that $\{a\} = A^* \cap B_*$ from Theorem 1 (5). Since f is a *CAP*-transformation, one sees $f(A_*) = (f(A))_{\#}$ and $f(B^*) = (f(B))^{\#}$. Combined with Lemma 7 (5), we have $\{x\} = \{f(a)\} = f(A^* \cap B_*) = f(A^*) \cap f(B_*) = (f(A))_{\#} \cap (f(B))^{\#}$. Note that $f(A), f(B) \subseteq U'$. So $(U'; C')$ is an S_d-space from Theorem 1 (5).

(6) Assume $(U; C)$ is an S_r-space. For any $x, y \in U'$ satisfying $x \in \{y\}^{\#}$, there are $a, b \in U$ such that $x = f(a)$ and $y = f(b)$. Thus, we find $a = f^{-1}(x)$ and $b = f^{-1}(y)$. Since f is a *CAP*-transformation from $(U; C)$ to $(U'; C')$, f^{-1} is a *CAP*-transformation from $(U'; C')$ to $(U; C)$ by Proposition 3. So $f^{-1}(\{y\}^{\#}) = (f^{-1}(\{y\}))^*$. Consequently, $a = f^{-1}(x) \in f^{-1}(\{y\}^{\#}) = (f^{-1}(\{y\}))^* = \{b\}^*$. Because $(U; C)$ is an S_r-space, we obtain $b \in \{a\}^*$ from Theorem 1 (6). It follows that $y = f(b) \in f(\{a\}^*) = (f(\{a\}))^{\#} = \{x\}^{\#}$ because f is a *CAP*-transformation. Hence, $(U'; C')$ is an S_r-space from Theorem 1 (6). $\square$

Remark 8. *In Theorem 4, "CAP" can be replaced by "CLAP" or "CUAP" from Proposition 8 and "f be a CAP-transformation from $(U; C)$ to $(U'; C')$" can be replaced by "f^{-1} be a CAP-transformation from $(U'; C')$ to $(U; C)$" from Proposition 3.*

Remark 9. *For each $i = 0, 1, 2, 3, d, r$, Theorem 4 shows that S_i-separation in covering approximation spaces is invariant under CAP-transformations of covering approximation spaces. It indicates that some transformations methods are applicable to covering-based rough sets theory. We can employ different the universe of discourse with different coverings to produce the same separations.*

6. Conclusions

This paper studies some invariance of separations in covering approximation spaces. Three main theorems, Theorems 2–4, are obtained. These are important results in the theoretical research framework about separations in covering approximation spaces. These results give answers to questions posed in the background section.

It is a strong assumption that the universe is finite, because usually coverings are defined as a collection of nonempty subsets of an arbitrary universe. In this work we have restricted to finite universes, but we will prove whether the results this paper presented hold for covering approximation spaces with an infinite universe in the future. In addition, since there are more than 20 covering covering approximation operators, the invariance of separation can also be considered in other types of covering approximation spaces.

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Conflict of interest

The authors declare that they have no conflict of interest.

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