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Research article

Optimality necessary conditions for an optimal control problem on time scales

Qiu-Yan Ren[1] **and Jian-Ping Sun**[1,2,*]

[1] College of Electrical and Information Engineering, Lanzhou University of Technology, Lanzhou 730050, China
[2] Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou 730050, China

* **Correspondence:** Email: jpsun@lut.cn; Tel: +8613619318935.

Abstract: An optimal control problem with quadratic cost functional on time scales is studied and some optimality necessary conditions are derived. The main tool used is the integration by parts on time scales.

Keywords: time scale; optimal control problem; optimality; necessary condition
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1. Introduction

In recent years, the calculus of variations and optimal control problems on time scales have attracted much attention. For example, the calculus of variations on time scales was discussed in [1–4], some maximum principles on time scales were studied in [5–9], while the existence of optimal solutions or the necessary conditions of optimality for some optimal control problems on time scales were investigated in [10–15]. In particular, Peng et al. [11] presented the necessary conditions of optimality for the Lagrange problem of systems governed by linear dynamic equations on time scales with quadratic cost functional. It is necessary to point out that the controlled state variable in [11] satisfies the initial value condition.

Throughout this paper, we always assume that $\mathbb{T}$ is a time scale, that is, $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers [16], $T > 0$ is fixed, $0,\ T \in \mathbb{T}$ and $\sigma(T) = T$. For each interval $\mathbf{I}$ of $\mathbb{R}$, we denote by $\mathbf{I}_{\mathbb{T}} = \mathbf{I} \cap \mathbb{T}$. The notation σ, which is standard in the study of time scales will be recalled in section 2 as well as the related tools required to follow the paper.

Let U_{ad} be the admissible control set. For any given control policy $u \in U_{ad}$, it is assumed that the change in the controlled state variable $x(t)$ can be described by the following dynamic equation

$$x^{\Delta}(t) + p(t)x(\sigma(t)) = f(t) + q(t)u(t),\ t \in [0, T]_{\mathbb{T}}. \tag{1.1}$$

At the same time, we assume that $x(t)$ satisfies the following loop condition

$$x(0) = x(T). \tag{1.2}$$

Suppose that x_u is the solution of the controlled system (1.1)–(1.2) corresponding to the control policy u and x_d is the desired value. In this paper, we will study optimality necessary conditions for the optimal control problem (P): Find a $u_0 \in U_{ad}$ such that

$$J(u) \geq J(u_0) \text{ for all } u \in U_{ad},$$

where

$$J(u) = \int_0^T [x_u(\sigma(t)) - x_d(t)]^2 \Delta t + \int_0^T u^2(t)\Delta t, \ u \in U_{ad}$$

is the quadratic cost functional. By using the integration by parts on time scales, we obtain some optimality necessary conditions for the problem (P).

2. Preliminaries

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his PhD thesis [17] in 1988 in order to unify continuous and discrete analysis. For more details, one can see [16, 18, 19]. In this section, we will recall some foundational definitions and results from the calculus on time scales which will be used in the paper.

Definition 2.1. *The forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \text{ for all } t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\} \text{ for all } t \in \mathbb{T}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ *and* $\sup \emptyset = \inf \mathbb{T}$, *where* $\emptyset$ *denotes the empty set. If* $\sigma(t) > t$, *then* t *is called right-scattered, while if* $\rho(t) < t$, *then* t *is called left-scattered. Also, if* $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$, *then* t *is called right-dense, and if* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$, *then* t *is called left-dense. If* $\mathbb{T}$ *has a left-scattered maximum* m, *then we define* $\mathbb{T}^k = \mathbb{T} - \{m\}$, *otherwise* $\mathbb{T}^k = \mathbb{T}$. *Finally, the graininess function* $\mu : \mathbb{T} \to [0, +\infty)$ *is defined by*

$$\mu(t) := \sigma(t) - t \text{ for all } t \in \mathbb{T}.$$

Definition 2.2. *Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^k$. *Then* $f^{\Delta}(t)$ *is defined to be the number (provided it exists) with the property that given any* $\epsilon > 0$, *there is a neighborhood* U *of* t *(i.e.,* $U = (t - \delta, t + \delta)_{\mathbb{T}}$ *for some* $\delta > 0$*) such that*

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s| \text{ for all } s \in U.$$

In this case, $f^{\Delta}(t)$ *is called the delta derivative of* f *at* t.

Moreover, f is called delta differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta derivative of f on $\mathbb{T}^k$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ holds for all } t \in \mathbb{T}^k.$$

If $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$, then the Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \text{ for all } a, b \in \mathbb{T}.$$

Definition 2.3. *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*

In the following we will provide some important properties of the exponential function which is specific to time scales. Their proofs can be found in [16].

Definition 2.4. *A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided*

$$1 + \mu(t)p(t) \neq 0 \text{ for all } t \in \mathbb{T}^k$$

holds. The set of all regressive and rd-continuous functions will be denoted by $\mathcal{R}$. The set of positively regressive functions $\mathcal{R}^+$ is defined as the set consisting of those $p \in \mathcal{R}$ satisfying

$$1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}.$$

Lemma 2.1. *[16] Let $p \in \mathcal{R}$, $t_0, s \in \mathbb{T}$ and $e_p(\cdot, t_0)$ be the exponential function on $\mathbb{T}$. Then*

(i) $e_p(t,t) \equiv 1 \text{ for all } t \in \mathbb{T}$;
(ii) $e_p^{\Delta}(t,t_0) = p(t)e_p(t,t_0) \text{ for all } t \in \mathbb{T}^k$;
(iii) $e_p(t,t_0) = \frac{1}{e_p(t_0,t)} \text{ for all } t \in \mathbb{T}$;
(iv) $e_p(t,s)e_p(s,t_0) = e_p(t,t_0) \text{ for all } t \in \mathbb{T}$;
(v) $\left(\frac{1}{e_p(t,t_0)}\right)^{\Delta} = -\frac{p(t)}{e_p(\sigma(t),t_0)} \text{ for all } t \in \mathbb{T}^k$.

Moreover, if $p \in \mathcal{R}^+$, then

$$e_p(t, t_0) > 0 \text{ for all } t \in \mathbb{T}.$$

Lemma 2.2. *[16] Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)),\ t \in \mathbb{T}^k.$$

Lemma 2.3. *[16] If $a, b \in \mathbb{T}$ and $f, g : \mathbb{T} \to \mathbb{R}$ are rd-continuous functions, then*

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t.$$

In the remainder of this paper, we always assume that Banach space

$$C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R}) := \{x \mid x : [0,T]_{\mathbb{T}} \to \mathbb{R} \text{ is rd-continuous}\}$$

is equipped with the norm $\|x\| = \max_{t \in [0,T]_{\mathbb{T}}} |x(t)|$, $p : [0,T]_{\mathbb{T}} \to (0, +\infty)$ is rd-continuous and denote

$$L = \frac{e_p(T,0)}{1 - e_p(T,0)} \text{ and } M = \frac{1}{e_p(T,0) - 1}.$$

Lemma 2.4. *For any $g \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$, the following first-order linear periodic boundary value problem (PBVP for short)*

$$\begin{cases} y^{\Delta}(t) = p(t)y(t) + g(t), \ t \in [0,T]_{\mathbb{T}}, \\ y(0) = y(T) \end{cases} \tag{2.1}$$

has a unique solution

$$y(t) = e_p(t,0)\left[\int_0^t e_p(0,\sigma(s))g(s)\,\Delta s + L\int_0^T e_p(0,\sigma(s))g(s)\,\Delta s\right], \ t \in [0,T]_{\mathbb{T}}. \tag{2.2}$$

Proof. Since g and e_p are rd-continuous, we know that the right side of (2.2) is well defined. By the equation in (2.1), Lemma 2.1 and Lemma 2.2, we get

$$[y(t)e_p(0,t)]^{\Delta} = e_p(0,\sigma(t))g(t), \ t \in [0,T]_{\mathbb{T}}.$$

So,

$$y(t) = e_p(t,0)\left[y(0) + \int_0^t e_p(0,\sigma(s))g(s)\Delta s\right], \ t \in [0,T]_{\mathbb{T}}. \tag{2.3}$$

It follows from (2.3) and the boundary condition in (2.1) that

$$y(0) = L\int_0^T e_p(0,\sigma(s))g(s)\Delta s.$$

And so,

$$y(t) = e_p(t,0)\left[\int_0^t e_p(0,\sigma(s))g(s)\,\Delta s + L\int_0^T e_p(0,\sigma(s))g(s)\,\Delta s\right], \ t \in [0,T]_{\mathbb{T}}.$$

□

Lemma 2.5. *[20] For any $h \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$, the following first-order linear PBVP*

$$\begin{cases} x^{\Delta}(t) + p(t)x(\sigma(t)) = h(t), \ t \in [0,T]_{\mathbb{T}}, \\ x(0) = x(T) \end{cases}$$

has a unique solution

$$x(t) = \frac{1}{e_p(t,0)}\left[\int_0^t e_p(s,0)h(s)\,\Delta s + M\int_0^T e_p(s,0)h(s)\,\Delta s\right], \ t \in [0,T]_{\mathbb{T}}.$$

3. Main results

From now on, we always suppose that the control space is $C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$ and the admissible control set U_{ad} is a nonempty convex subset of $C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$.

Theorem 3.1. *Assume that $f, q \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$. Let $(x_{u_0}, u_0) \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R}) \times U_{ad}$ be an optimal pair of the problem (P). Then*

$$\begin{cases} x_{u_0}^{\Delta}(t) + p(t)x_{u_0}(\sigma(t)) = f(t) + q(t)u_0(t), \ t \in [0,T]_{\mathbb{T}}, \\ x_{u_0}(0) = x_{u_0}(T) \end{cases} \tag{3.1}$$

and there exists a function $\varphi \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$ such that

$$\int_0^T [u(t) - u_0(t)][\varphi(t)q(t) + u_0(t)]\Delta t \geq 0 \text{ for any } u \in U_{ad}.$$

Proof. Since $(x_{u_0}, u_0) \in C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R}) \times U_{ad}$ is an optimal pair of the problem (P), it must satisfy (3.1).

According to Lemma 2.4, we know that the following PBVP

$$\begin{cases} \varphi^{\Delta}(t) = p(t)\varphi(t) + x_d(t) - x_{u_0}(\sigma(t)), \ t \in [0,T]_{\mathbb{T}}, \\ \varphi(0) = \varphi(T) \end{cases} \tag{3.2}$$

has a unique solution φ.

In what follows, we shall show that

$$\int_0^T [u(t) - u_0(t)][\varphi(t)q(t) + u_0(t)]\Delta t \geq 0 \text{ for any } u \in U_{ad}. \tag{3.3}$$

For any fixed $u \in U_{ad}$, we first consider the following PBVP

$$\begin{cases} z^{\Delta}(t) + p(t)z(\sigma(t)) = q(t)[u(t) - u_0(t)], \ t \in [0,T]_{\mathbb{T}}, \\ z(0) = z(T). \end{cases} \tag{3.4}$$

By Lemma 2.5, we know that the PBVP (3.4) has a unique solution z.

Next, for $\epsilon \in [0,1]$, we denote

$$u_\epsilon = u_0 + \epsilon(u - u_0). \tag{3.5}$$

Then, the hypothesis that U_{ad} is a nonempty convex set yields $u_\epsilon \in U_{ad}$ for $\epsilon \in [0,1]$ and moreover from Lemma 2.5 we obtain

$$x_{u_\epsilon} - x_{u_0} = \epsilon z, \ \epsilon \in [0,1]. \tag{3.6}$$

In view of (3.2), (3.4) and Lemma 2.3, we have

$$\begin{aligned} \int_0^T z(\sigma(t))[x_{u_0}(\sigma(t)) - x_d(t)]\Delta t &= \int_0^T z(\sigma(t))[-\varphi^{\Delta}(t) + p(t)\varphi(t)]\Delta t \\ &= \int_0^T \varphi(t)[z^{\Delta}(t) + p(t)z(\sigma(t))]\Delta t \\ &= \int_0^T \varphi(t)q(t)[u(t) - u_0(t)]\Delta t, \end{aligned}$$

which together with (3.5) and (3.6) indicates that for any $\epsilon \in [0,1]$,

$$J(u_\epsilon) - J(u_0)$$

$$
\begin{aligned}
&= \int_0^T \{[x_{u_\epsilon}(\sigma(t)) - x_d(t)]^2 - [x_{u_0}(\sigma(t)) - x_d(t)]^2\}\Delta t + \int_0^T [u_\epsilon^2(t) - u_0^2(t)]\Delta t \\
&= \int_0^T [x_{u_\epsilon}(\sigma(t)) - x_{u_0}(\sigma(t))][x_{u_\epsilon}(\sigma(t)) + x_{u_0}(\sigma(t)) - 2x_d(t)]\Delta t \\
&\quad + \int_0^T [u_\epsilon(t) + u_0(t)][u_\epsilon(t) - u_0(t)]\Delta t \\
&= \epsilon \int_0^T z(\sigma(t))[x_{u_\epsilon}(\sigma(t)) + x_{u_0}(\sigma(t)) - 2x_d(t)]\Delta t \\
&\quad + \epsilon \int_0^T [u(t) - u_0(t)]\{2u_0(t) + \epsilon[u(t) - u_0(t)]\}\Delta t \\
&= \epsilon \int_0^T z(\sigma(t))\{[x_{u_\epsilon}(\sigma(t)) - x_{u_0}(\sigma(t))] + 2[x_{u_0}(\sigma(t)) - x_d(t)]\}\Delta t \\
&\quad + \epsilon^2 \int_0^T [u(t) - u_0(t)]^2\Delta t + 2\epsilon \int_0^T u_0(t)[u(t) - u_0(t)]\Delta t \\
&= \epsilon^2 \int_0^T z^2(\sigma(t))\Delta t + 2\epsilon \int_0^T \varphi(t)q(t)[u(t) - u_0(t)]\Delta t \\
&\quad + \epsilon^2 \int_0^T [u(t) - u_0(t)]^2\Delta t + 2\epsilon \int_0^T u_0(t)[u(t) - u_0(t)]\Delta t \\
&= \epsilon^2 \int_0^T \{z^2(\sigma(t)) + [u(t) - u_0(t)]^2\}\Delta t + 2\epsilon \int_0^T [u(t) - u_0(t)][\varphi(t)q(t) + u_0(t)]\Delta t.
\end{aligned}
$$

Since u_0 is an optimal solution of the problem (P), for any $\epsilon \in [0, 1]$, we get

$$
\epsilon\left(\epsilon \int_0^T \{z^2(\sigma(t)) + [u(t) - u_0(t)]^2\}\Delta t + 2\int_0^T [u(t) - u_0(t)][\varphi(t)q(t) + u_0(t)]\Delta t\right) \geq 0,
$$

which implies that for any $u \in U_{ad}$,

$$
\int_0^T [u(t) - u_0(t)][\varphi(t)q(t) + u_0(t)]\Delta t \geq 0.
$$

□

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Conflict of interest

The authors declare that there are no conflict of interest regarding the publication of this paper.

References

1. Z. Bartosiewicz, N. Martins, D. F. M. Torres, The second Euler-Lagrange equation of variational calculus on time scales, *Eur. J. Control*, **17** (2011), 9–18.
2. R. A. C. Ferreira, A. B. Malinowska, D. F. M. Torres, Optimality conditions for the calculus of variations with higher-order delta derivatives, *Appl. Math. Lett.*, **24** (2011), 87–92.
3. R. Hilscher, V. Zeidan, First order conditions for generalized variational problems over time scales, *Comput. Math. Appl.*, **62** (2011), 3490–3503.
4. A. B. Malinowska, N. Martins, D. F. M. Torres, Transversality conditions for infinite horizon variational problems on time scales, *Optim. Lett.*, **5** (2011), 41–53.
5. P. Stehlik, B. Thompson, Maximum principles for second order dynamic equations on time scales, *J. Math. Anal. Appl.*, **331** (2007), 913–926.
6. R. Hilscher, V. Zeidan, Weak maximum principle and accessory problem for control problems on time scales, *Nonlinear Anal.*, **70** (2009), 3209–3226.
7. L. Bourdin, E. Trélat, Pontryagin maximum principle for finite dimensional nonlinear optimal control problems on time scales, *SIAM J. Control Optim.*, **51** (2013), 3781–3813.
8. S. Zhou, H. Li, Maximum principles for dynamic equations on time scales and their applications, *J. Appl. Math.*, **2014** (2014), 434582.
9. M. Bohner, K. Kenzhebaev, O. Lavrova, O. Stanzhytskyi, Pontryagin's maximum principle for dynamic systems on time scales, *J. Difference Equ. Appl.*, **23** (2017), 1161–1189.
10. Y. Gong, X. Xiang, A class of optimal control problems of systems governed by the first order linear dynamic equations on time scales, *J. Ind. Manage. Optim.*, **5** (2009), 1–10.
11. Y. Peng, X. Xiang, Y. Gong, G. Liu, Necessary conditions of optimality for a class of optimal control problems on time scales, *Comput. Math. Appl.*, **58** (2009), 2035–2045.
12. G. Liu, X. Xiang, Y. Peng, Nonlinear integro-differential equations and optimal control problems on time scales, *Comput. Math. Appl.*, **61** (2011), 155–169.
13. D. A. Carlson, The existence of optimal controls for problems defined on time scales, *J. Optim. Theory Appl.*, **166** (2015), 351–376.
14. O. E. Lavrova, Conditions for the existence of optimal control for some classes of differential equations on time scales, *J. Math. Sci.*, **222** (2017), 276–295.
15. J. P. Sun, Q. Y. Ren, Y. H. Zhao, An optimal control problem governed by nonlinear first order dynamic equation on time scales, *Math. Probl. Eng.*, **2020** (2020), 3869089.
16. M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
17. S. Hilger, *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, PhD thesis, Universität Würzburg, 1988.
18. V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Boston, 1996.

19. M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

20. J. P. Sun, W. T. Li, Existence of solutions to nonlinear first-order PBVPs on time scales, *Nonlinear Anal.*, **67** (2007), 883–888.

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