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Research article

Existence and multiplicity of solutions for Schrödinger equations with sublinear nonlinearities

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Abstract: In the paper, we investigate a class of Schrödinger equations with sign-changing potentials $V(x)$ and sublinear nonlinearities. We remove the coercive condition on $V(x)$ usually required in the existing literature and also weaken the conditions on nonlinearities. By proving a Hardy-type inequality, extending the results in [1], and using it together with variational methods, we get at least one or infinitely many small energy solutions for the problem.

Keywords: Schrödinger equation; sublinear nonlinearity; Hardy-type inequality; variant fountain theorem; infinitely many solutions
Mathematics Subject Classification: 35A15, 35J60, 58E05

1. Introduction and main results

In this paper, we are devoted to studying the existence and multiplicity of solutions for the following Schrödinger equation with a sublinear nonlinearity,

$$-\Delta u + V(x)u = K(x)f(u),\ \forall x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 3$, V is sign-changing, K is positive and $f \in C(\mathbb{R})$. We remove the coercive condition usually imposed on $V(x)$ and obtain the existence of at least one or infinitely many small energy solutions to (1.1) for sublinear nonlinearities $K(x)f(u)$.

As mentioned in [1, 10, 12], this type of equations is essentially related to seeking for the standing waves $\psi(t,x) = e^{-i\omega t}u(x)$ for the time-dependent Schrödinger equation,

$$i\hbar\partial_t\Psi = -\Delta\Psi + U(x)\Psi - g(x,\Psi),\ x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{1.2}$$

where the potential V is given by $V(x) = U(x) - \omega$. Hence V may be indefinite in sign for large ω(see [1, 23]).

Much attention has been paid on the following equation,

$$-\Delta u + V(x)u = f(x,u),\ x \in \mathbb{R}^N,\ N \geq 3, \tag{1.3}$$

involving a continuous term $V(x)$. We refer, for instance, to [2–8, 11, 14, 15, 17–20, 22, 23] and the references therein. It is known to all that the main difficulty in dealing with problem (1.3) arises from the lack of the compactness of Sobolev embeddings, which prevents from checking directly that the energy functional associated with (1.3) satisfies the PS-condition.

To obtain the compactness in $\mathbb{R}^N$, some feasible methods are provided in the existing papers. For example, Bartsch, Pankov and Wang [6] have studied a class of Schrödinger equations, where $V(x)$ is continuous function verifying the following conditions,
(v_1) $\operatorname{ess\,inf} V(x) > 0$;
(v_2) for any $M > 0$, there exists x_0 such that $\lim_{|y|\to\infty} \operatorname{meas}(\{x \in \mathbb{R}^N : |x-y| \leq x_0,\ V(x) \leq M\}) = 0$, where meas devotes the Lebesgue measure on $\mathbb{R}^N$. Under conditions (v_1) and (v_2), the compactness of Sobolev embedding can be recovered. With the assumptions (v_1) and (v_2), equation (1.3) has been investigated by the variational methods by [6] and some other authors.

In [22], the authors studied a class of sublinear Schrödinger equations, where $f(x,u) = \xi(x)|u|^{\mu-2}u$ with $1 < \mu < 2$ and $\xi(x) : \mathbb{R}^N \to \mathbb{R}$ being a positive continuous function. Under conditions (v_1) and (v_2), they established a theorem on the existence of infinitely many small energy solutions.

The results of [22] were improved in the recent paper [7], where they improved the results of [22] by removing assumption (v_2) and relaxing the assumptions on $f(x,t)$. By using the genus properties in critical point theory, they established some existence criteria to guarantee that the problem has at least one or infinitely many nontrivial solutions.

In [8], for problem (1.3), Cheng and Wu studied a sublinear problem and used conditions on $V(x)$ below:
$(V1)$ $V \in C(\mathbb{R}^N)$ is bounded below;
$(V2)$ for every $M > 0$, $\operatorname{meas}\{x : V(x) \leq M\} < \infty$.
Under some additional conditions of f, two theorems are obtained in [8]. One theorem states that equation (1.3) possesses at least one nontrivial solution. By using a variant fountain theorem, they obtained the existence of infinitely many small energy solutions in another theorem.

Bao and Han [4] also considered a nonlinear sublinear Schrödinger equation,

$$-\Delta u + V(x)u = a(x)|u|^{\mu-2}u,\ \forall x \in \mathbb{R}^N, \tag{1.4}$$

where $V(x) \in L^\infty(\mathbb{R}^N)$ is sign-changing and $a(x) \in L^\infty(\mathbb{R}^N)$ with $a(x) > 0$ a.e. in $\mathbb{R}^N$. Under some conditions on $V(x)$ and by using bounded domain approximation technique, infinitely many small energy solutions are obtained.

In those above papers, (v_2), $(V2)$ or the coercive condition on V plays an important role in obtaining the compact embedding. In this paper, we remove the coercive condition of $V(x)$ and also weaken the conditions on f.

We remark that there have been many interesting results for the similar sublinear problems (1.1) but on bounded domains $\Omega \subset \mathbb{R}^N$. We refer to [13] for some results for $p-$Laplacian equation problems and the references therein.

Before stating our main results, we make some assumptions, where $V^+(x) = \max\{V(x), 0\}$, $V^-(x) = \max\{-V(x), 0\}$.

(K_1) $K(x) > 0,\ \forall x \in \mathbb{R}^N$ and $K(x) \in L^\infty(\mathbb{R}^N)$.
(V_1) $V = V^+ - V^-$, where $V^+ \in L^1(\mathbb{R}^N, \mathbb{R})$, $V^- \in L^{\frac{N}{2}}(\mathbb{R}^N, \mathbb{R})$.

$$\Omega = \{x \in \mathbb{R}^N |\ V(x) < 0\} \neq \emptyset,$$

meas$\Omega > 0$ and there exists a large constant R_0 such that $V(x) > 0$ for a.e. $|x| \geq R_0$.
(V_2) There exists a constant $\eta_0 > 1$ such that

$$\eta_1 := \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V^+ u^2 dx}{\int_{\mathbb{R}^N} V^- u^2 dx} \geq \eta_0.$$

(KV) $\frac{K}{|V|} \in L^\infty(\mathbb{R}^N)$.
(f_1) $f \in C(\mathbb{R})$ and there exist constants $\tau_1, \tau_2 \in (1, 2)$ with $\tau_1 < \tau_2$ such that

$$0 \leq f(u)u \leq |u|^{\tau_1} + |u|^{\tau_2} \ \text{ for all } u \in \mathbb{R}.$$

(f_2) $F(u) \geq C|u|^{\tau_1}$, $\forall u \in \mathbb{R}$, where C is some positive constant, $F(u) = \int_0^u f(\tau) d\tau$.

Remark 1.1. *Conditions similar to (V_2) can be found in [9] and [16]. By condition (V_1) and the Hölder and Sobolev inequalities,*

$$\begin{aligned}
\frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V^+(x)|u|^2 dx}{\int_{\mathbb{R}^N} V^-(x)|u|^2 dx} \geq & \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{|V^-|_{\frac{N}{2}} |u|_{2^*}^2} \\
\geq & \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{S^{-1} |V^-|_{\frac{N}{2}} \int_{\mathbb{R}^N} |\nabla u|^2 dx} \\
= & \frac{S}{|V^-|_{\frac{N}{2}}},
\end{aligned} \tag{1.5}$$

where S is the best constant for the Sobolev embedding of $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $2^* = \frac{2N}{N-2}$. It implies that if $|V^-|_{\frac{N}{2}} < S$, then $\mu_1 \geq \frac{S}{|V^-|_{\frac{N}{2}}} > 1$. Hence, (V_2) is satisfied for $V(x)$ with sufficiently small $|V^-|_{\frac{N}{2}}$.

By (V_2) and a simple calculation,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V^+|u|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 dx \geq \frac{\eta_0 - 1}{\eta_0} \Big(\int_{\mathbb{R}^N} |\nabla u|^2 + V^+|u|^2 dx \Big). \tag{1.6}$$

More details on condition (V_2) can be found in [9].

Theorem 1.2. *Assume that conditions (K_1), (V_1), (V_2), (KV), (f_1) and (f_2) hold. Then Eq (1.1) possesses at least one nontrivial solution.*

Theorem 1.3. *Assume that conditions (K_1), (V_1), (V_2), (KV), (f_1) and (f_2) hold. Moreover, $f(u) = -f(-u)$, $\forall u \in \mathbb{R}$. Then equation (1.1) possesses infinitely many small energy solutions.*

We emphasize that the conditions on $V(x)$ in this paper are essentially different from those in [8] and [22]. In fact, we are dealing with the vanishing potentials $V(x)$. As far as we know, for problem (1.1) with sublinearity, few works in this case seem to have appeared in the literature. Since $V(x)$ is sign-changing and vanishing, it seems not to be obvious from the literature to obtain the compactness suitable to deal with the problem. By proving a Hardy-type inequality, which extends the results in [1], we can obtain the needed compactness. Our theorems also extend the results in [8, 22] and our hypotheses on nonlinearities are more general.

The paper is organized as follows. In Section 2, we introduce the variational setting and state some preliminary results which will be needed later. In Section 3, the proofs of our main results are given.

2. Variational setting

In this paper, we define

$$E = \Big\{u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V^+(x)|u|^2 dx < \infty\Big\}.$$

We know E is a separable Hilbert space with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} \Big(\nabla u \cdot \nabla v + V^+(x)u(x)v(x)\Big)dx$$

and the norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. Let $L^q_K(\mathbb{R}^N)$ be the weighted space of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$|u|_{K,q} = \Big[\int_{\mathbb{R}^N} K(x)|u|^q dx\Big]^{\frac{1}{q}} < +\infty.$$

Denote $L^q(\mathbb{R}^N)$ with

$$|u|_q = \Big[\int_{\mathbb{R}^N} |u|^q dx\Big]^{\frac{1}{q}} < +\infty,$$

where $1 \le q < +\infty$. And set

$$\|u\|_\infty = \operatorname{ess}\sup_{x\in\mathbb{R}^N} |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

It is well known that the embedding $E \subset L^s(\mathbb{R}^N)(2 \le s \le 2^*)$ is continuous.

Now we give a Hardy-type inequality which extends the one in [1] and is suitable for dealing with our sublinear problems. Before stating the result, we recall condition (A),
(A) if $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \le R$ for some $R > 0$ and all n, then

$$\lim_{r\to\infty} \int_{A_n\cap B_r^c(0)} K(x)dx = 0, \text{ uniformly in } n \in \mathbb{N}. \tag{2.1}$$

As stated in [1], if $K \in L^1(\mathbb{R}^N\backslash B_\rho(0))$ for some $\rho > 0$, we know that K satisfies condition (A).

Lemma 2.1. *Suppose that* (K_1), (V_1) *and* (KV) *hold. Then* E *is compactly embedded in* $L^r_K(\mathbb{R}^N)$ *for* $r \in (1, 2]$.

Proof. By (V_1) and noticing that $V^+ \in L^1$, for any $\varepsilon > 0$, we can choose $R_\varepsilon > R_0$ such that

$$\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} V(x)dx = \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} V^+(x)dx < \varepsilon^2. \tag{2.2}$$

Fixed $1 < r \leq 2$. For given ε, there are $0 < T_\varepsilon^0 < T_\varepsilon$ and $C_\varepsilon > 0$ such that, for a.e. $|x| \geq R_\varepsilon$,

$$K(x)|s|^r \leq C\varepsilon(V(x)|s| + |s|^{2^*}) + C_\varepsilon K(x)\chi_{(T_\varepsilon^0, T_\varepsilon)}|s|^{2^*}, \ \forall s \in \mathbb{R}^N. \tag{2.3}$$

Hence,

$$\begin{aligned} &\int_{B^c_{R_\varepsilon}(0)} K(x)|u|^r dx \\ \leq & C\varepsilon\Big(\int_{B^c_{R_\varepsilon}(0)} V(x)|u|dx + \int_{B^c_{R_\varepsilon}(0)} |u|^{2^*}dx\Big) + C_\varepsilon T_\varepsilon^{2^*} \int_{A\cap B^c_{R_\varepsilon}(0)} K(x)dx, \ \forall u \in E, \end{aligned} \tag{2.4}$$

where

$$A = \{x \in \mathbb{R}^N : T_\varepsilon^0 \leq |u(x)| \leq T_\varepsilon\}.$$

If $\{v_n\}$ is a sequence such that $v_n \rightharpoonup v$ in E, then there is $M > 0$ such that

$$\int_{\mathbb{R}^N} (|\nabla v_n|^2 + V^+(x)|v_n|^2)dx \leq M^2 \tag{2.5}$$

and

$$\int_{\mathbb{R}^N} |v_n|^{2^*}dx \leq M^2, \ \forall n \in \mathbb{N}. \tag{2.6}$$

It follows that

$$\int_{B^c_{R_\varepsilon}} |v_n|^{2^*}dx < M^2 \tag{2.7}$$

and

$$\int_{B^c_{R_\varepsilon}(0)} V^+(x)|v_n|dx \leq \Big[\int_{\mathbb{R}^N} V^+(x)|v_n|^2dx\Big]^{\frac{1}{2}}\Big[\int_{B^c_{R_\varepsilon}(0)} V^+(x)\Big]^{\frac{1}{2}} \leq M\varepsilon. \tag{2.8}$$

Thus, by (V_1), (2.7) and (2.8), we obtain that

$$\begin{aligned} &\int_{B^c_{R_\varepsilon}(0)} V(x)|v_n|dx + \int_{B^c_{R_\varepsilon}(0)} |v_n|^{2^*}dx \\ =&\int_{B^c_{R_\varepsilon}(0)} V^+(x)|v_n|dx + \int_{B^c_{R_\varepsilon}(0)} |v_n|^{2^*}dx \leq M\varepsilon + M^2. \end{aligned} \tag{2.9}$$

By (2.2) and (KV), we have

$$\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} K(x)dx = \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} \frac{K(x)}{V(x)}V(x)dx \leq C\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} V(x)dx < C\varepsilon. \tag{2.10}$$

Furthermore, set

$$A_n = \{x \in \mathbb{R}^N : T_\varepsilon^0 \leq |v_n(x)| \leq T_\varepsilon\}. \tag{2.11}$$

By (2.6),

$$(T_\varepsilon^0)^{2^*}|A_n| \leq \int_{An} |v_n|^{2^*} dx \leq M^2, \ \forall n \in \mathbb{N}, \tag{2.12}$$

that is,

$$\sup_{n\in\mathbb{N}} |A_n| < +\infty. \tag{2.13}$$

Therefore, by (2.1), (2.10) and (2.13), there is a constant $\bar{R}_\varepsilon > 0$ such that

$$\int_{A_n \cap B^c_{\bar{R}_\varepsilon}(0)} K(x)dx < \frac{\varepsilon}{C_\varepsilon T_\varepsilon^{2^*}}, \ \text{ for all } n \in \mathbb{N}. \tag{2.14}$$

Hence, for $\hat{R}_\varepsilon = \max\{\bar{R}_\varepsilon, R_\varepsilon\}$, (2.4), (2.9) and (2.14) lead to

$$\begin{aligned}
&\int_{B^c_{\hat{R}_\varepsilon}(0)} K(x)|v_n|^r dx \\
\leq& C\varepsilon\Big(\int_{B^c_{\hat{R}_\varepsilon}(0)} V(x)|v_n|dx + \int_{B^c_{\hat{R}_\varepsilon}(0)} |v_n|^{2^*}\Big) + C_\varepsilon T_\varepsilon^{2^*} \int_{A_n\cap B^c_{\hat{R}_\varepsilon}(0)} K(x)dx \\
\leq& C\varepsilon\Big(\int_{B^c_{R_\varepsilon}(0)} V(x)|v_n|dx + \int_{B^c_{R_\varepsilon}(0)} |v_n|^{2^*}\Big) + C_\varepsilon T_\varepsilon^{2^*} \int_{A_n\cap B^c_{\bar{R}_\varepsilon}(0)} K(x)dx \\
\leq& C\varepsilon(M\varepsilon + M^2) + \varepsilon \leq \hat{C}\varepsilon, \ \forall n \in \mathbb{N}.
\end{aligned} \tag{2.15}$$

Furthermore, for that $\varepsilon > 0$ and large n, it is easy to obtain that

$$\int_{B_{\hat{R}_\varepsilon}(0)} K(x)(|v_n|^r - |v|^r)dx < \varepsilon. \tag{2.16}$$

Therefore, from (2.15) and (2.16), we obtain that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} K(x)(|v_n|^r - |v|^r)dx\Big| \\
=&\Big|\int_{B_{\hat{R}_\varepsilon}(0)} K(x)(|v_n|^r - |v|^r)dx\Big| + \Big|\int_{B^c_{\hat{R}_\varepsilon}(0)} K(x)(|v_n|^r - |v|^r)dx\Big| \\
\leq& \int_{B_{\hat{R}_\varepsilon}(0)} K(x)||v_n|^r - |v|^r|dx + \int_{B^c_{\hat{R}_\varepsilon}(0)} K(x)|v_n|^r dx \leq \bar{C}\varepsilon,
\end{aligned} \tag{2.17}$$

which completes the proof. □

Lemma 2.2. *(Lemma 2.13 [21]) Let $V(x) \in L^{\frac{N}{2}}(\Omega)$ and suppose that $u_n \rightharpoonup u$ in E. Then*

$$\int_\Omega V^-(x)|u_n|^2 dx \to \int_\Omega V^-(x)|u|^2 dx. \tag{2.18}$$

Lemma 2.3. *Suppose that (K_1), (V_1) and (KV) hold. Then the functional $\mathcal{J} : E \to \mathbb{R}$ defined by*

$$\mathcal{J}(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)|u|^2 dx - \int_{\mathbb{R}^N} K(x)F(u)dx \tag{2.19}$$

is well defined and belongs to $C^1(E, \mathbb{R})$. Moreover,

$$\langle \mathcal{J}'(u), v\rangle = \langle u, v\rangle - \int_{\mathbb{R}^N} V^-(x)uvdx - \int_{\mathbb{R}^N} K(x)f(u)vdx. \tag{2.20}$$

Proof. By (V_1) and Lemma 2.13 in [21], we know $\int_{\mathbb{R}^N} V^-(x)|u|^2 dx$ is well defined for $u \in E$. By virtue of (f_1),

$$|F(u)| \leq C|u|^{\tau_1} + C|u|^{\tau_2}. \tag{2.21}$$

Hence, by Lemma 2.1, we get

$$\int_{\mathbb{R}^N} K(x)|F(u)|dx \leq C\int_{\mathbb{R}^N} (K(x)|u|^{\tau_1} + K(x)|u|^{\tau_2})dx \leq C\|u\|^{\tau_1} + C\|u\|^{\tau_2}, \tag{2.22}$$

which means that $\mathcal{J}$ is well defined for $u \in E$.

By a direct computation, it is not difficult to prove that (2.20) holds. Furthermore, by a standard argument, we obtain that the critical points of $\mathcal{J}$ in E are solutions of problem (1.1).

Finally, we will show that $\mathcal{J}'(u)$ is weakly continuous, that is, if $u_n \rightharpoonup u$ in E, then

$$\langle \mathcal{J}'(u_n) - \mathcal{J}'(u), v\rangle \to 0, \ \forall v \in E. \tag{2.23}$$

Arguing directly, by $u_n \rightharpoonup u$ in E, choose a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}(x) \to u(x)$ a.e. in $\mathbb{R}^N$ and $Q_1(x) \in L^2_K(\mathbb{R}^N)$, where

$$Q_1(x) = \left(\Sigma_{k=1}^{\infty}|u_{n_k}(x) - u(x)|^2\right)^{\frac{1}{2}}.$$

It is clear that

$$\begin{aligned} K(x)|f(u_{n_k}) - f(u)|^2 \leq & 2K(x)\Big(|f(u_{n_k})|^2 + |f(u)|^2\Big) \\ \leq & 4\sum_{i=1}^{2}\left(K(x)|u_{n_k}|^{2(\tau_i-1)} + K(x)|u|^{2(\tau_i-1)}\right) \\ = & 4\sum_{i=1}^{2}\left(K(x)|u_{n_k} - u + u|^{2(\tau_i-1)} + K(x)|u|^{2(\tau_i-1)}\right) \\ \leq & \sum_{i=1}^{2} C(\tau_i)\Big(K(x)|u_{n_k} - u|^{2(\tau_i-1)} + K(x)|u|^{2(\tau_i-1)}\Big) \\ \leq & \sum_{i=1}^{2} C(\tau_i)\Big[K(x)Q_1^{2(\tau_i-1)}(x) + K(x)|u|^{2(\tau_i-1)}\Big]. \end{aligned} \tag{2.24}$$

Write $Q_2(x) = \sum_{i=1}^{2} C(\tau_i)\Big[K(x)Q_1^{2(\tau_i-1)}(x)+K(x)|u|^{2(\tau_i-1)}\Big]$. By (V_1) and (KV), we obtain that $K \in L^1(\mathbb{R}^N)$. By $\frac{1}{\tau_i-1} > 1$ and $\frac{1}{2-\tau_i} > 1$, one has

$$\begin{aligned}
&\int_{\mathbb{R}^N} Q_2(x)dx \\
=&\sum_{i=1}^{2} C(\tau_i)\int_{\mathbb{R}^N}\Big[K(x)Q_1^{2(\tau_i-1)}(x) + K(x)|u|^{2(\tau_i-1)}\Big]dx \\
=&\sum_{i=1}^{2} C(\tau_i)\int_{\mathbb{R}^N} K^{(\tau_i-1)+(2-\tau_i)}(x)Q_1^{2(\tau_i-1)}dx + \sum_{i=1}^{2} C(\tau_i)\int_{\mathbb{R}^N} K^{(\tau_i-1)+(2-\tau_i)}(x)|u|^{2(\tau_i-1)}dx \\
\leq&\sum_{i=1}^{2} C(\tau_i)\Big(\int_{\mathbb{R}^N} K(x)Q_1^2dx\Big)^{\tau_i-1}\Big(\int_{\mathbb{R}^N} K(x)dx\Big)^{2-\tau_i} + \sum_{i=1}^{2} C(\tau_i)\Big(\int_{\mathbb{R}^N} K(x)u^2dx\Big)^{\tau_i-1}\Big(\int_{\mathbb{R}^N} K(x)dx\Big)^{2-\tau_i} \\
\leq&\sum_{i=1}^{2} C(\tau_i)\Big[|Q_1|_{K,2}^{2(\tau_i-1)}\Big(\int_{\mathbb{R}^N} K(x)dx\Big)^{2-\tau_i} + |u|_{K,2}^{2(\tau_i-1)}\Big(\int_{\mathbb{R}^N} K(x)dx\Big)^{2-\tau_i}\Big] < \infty.
\end{aligned} \tag{2.25}$$

This together with Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)|f(u_n) - f(u)|^2dx = 0. \tag{2.26}$$

Therefore, for any $v \in E$,

$$\begin{aligned}
&\langle \mathcal{J}'(u_n) - \mathcal{J}'(u), v\rangle \\
=&\Big|\langle u_n - u, v\rangle - \int_{\mathbb{R}^N} V^-(x)(u_n - u)vdx - \int_{\mathbb{R}^N} K(x)|f(u_{n_k}) - f(u)|vdx\Big| \\
\leq&|\langle u_n - u, v\rangle| + \int_{\mathbb{R}^N} V^-(x)|(u_n - u)v|dx + C\Big(\int_{\mathbb{R}^N} K(x)|f(u_{n_k}) - f(u)|^2dx\Big)^{\frac{1}{2}}\|v\| \\
\to&0, \text{ as } n \to \infty.
\end{aligned}$$

Hence, $\mathcal{J}'(u)$ is weakly continuous in E. The proof is complete. □

Definition 2.4. *(PS condition) Let E be a Banach space, $c \in \mathbb{R}$ and $\mathcal{J} \in C^1(E,\mathbb{R})$. The function $\mathcal{J}$ is said to satisfy the $(PS)_c$-condition on E if any $(PS)_c$-sequence $\{u_n\}$ such that*

$$\mathcal{J}(u_n) \to c \text{ and } \mathcal{J}'(u_n) \to 0, \text{ as } n \to \infty$$

has a strongly convergent subsequence in E.

Let $\{e_j\}$ be an orthonormal basis of the Hilbert space E and define $X_j = Re_j,\ Y_k = \overset{k}{\underset{j=1}{\oplus}} X_j,\ Z_k = \overline{\overset{\infty}{\underset{j=k}{\oplus}} X_j}$. For the statement of Dual Fountain Theorem, we need the following condition. More details can be found in [21].

(A$_1$) A compact group G acts isometrically on the Hilbert space $E = \overline{\bigoplus_{j\in N} X_j}$, the spaces X_j are invariant and there exists a finite dimensional space V such that, for every $j \in N, X_j \simeq V$ and the action of G on V is admissible.

Lemma 2.5. *(Theorem 3.18 in [21] Dual Fountain Theorem, Bartsch-Willem, 1995) Assume that condition* ($\boldsymbol{A}_1$) *holds and let* $\mathcal{J} \in C^1(E, \mathbb{R})$ *be an invariant functional. If, there exist two sequences* $0 < r_k < \rho_k \to 0$ *as* $k \to \infty$ *and the following conditions* (D_1)*–*(D_4) *hold, then* $\mathcal{J}$ *has a sequence of negative critical values converging to* 0*, where*
(D_1) $a_k := \inf_{u \in Z_k, \|u\| = \rho_k} \mathcal{J}(u) \geq 0$;
(D_2) $b_k := \max_{u \in Y_k, \|u\| = r_k} \mathcal{J}(u) < 0$;
(D_3) $d_k := \inf_{u \in Z_k, \|u\| \leq \rho_k} \mathcal{J}(u) \to 0$ *as* $k \to \infty$;
(D_4) *for every* $c \in [d_k, 0)$, $\mathcal{J}$ *satisfies the* $(PS)_c^*$ *condition, that is, every sequence* $u_{n_j} \in E$ *satisfying*

$$u_{n_j} \in Y_{n_j}, \mathcal{J}(u_{n_j}) \to c, \ \mathcal{J}|'_{Y_{n_j}}(u_{n_j}) \to 0, n_j \to \infty$$

contains a subsequence converging to a critical point of $\mathcal{J}$.

3. Proof of theorems

Lemma 3.1. *Assume that conditions* (K_1)*,* (V_1)*,* (V_2)*,* (KV) *and* (f_1) *hold. Then the functional* $\mathcal{J}$ *is coercive and bounded below on* E.

Proof. It is obvious to obtain that

$$\begin{aligned}\mathcal{J}(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)|u|^2 dx - \int_{\mathbb{R}^N} K(x)F(u)dx \\ \geq& \frac{\eta_0 - 1}{2\eta_0}\|u\|^2 - c\|u\|^{\tau_1} - c\|u\|^{\tau_2}.\end{aligned} \tag{3.1}$$

Since $\tau_1, \tau_2 \in (1, 2)$, the above inequality implies that $\mathcal{J}$ is coercive and bounded below on E. □

By Lemma 3.1 and Ekeland's variational method, there exists a minimizing sequence $\{u_n\}$ such that

$$\mathcal{J}(u_n) \to \inf_E \mathcal{J} \ \text{ and } \mathcal{J}'(u_n) \to 0, \ \text{ as } n \to \infty.$$

By Lemma 3.1, it is clear that the minimizing sequence $\{u_n\}$ is bounded in E.

Lemma 3.2. *Assume that conditions* (K_1)*,* (V_1)*,* (V_2)*,* (KV) *and* (f_1) *hold. Then there exists a strong convergent subsequence of the minimizing sequence* $\{u_n\}$.

Proof. By Lemma 3.1, the minimizing sequence $\{u_n\}$ is bounded. Passing to a subsequence, one has

$$\begin{cases} u_n \rightharpoonup u \text{ in } E, \\ u_n \to u \text{ in } \ L_K^2(\mathbb{R}^N), \\ u_n(x) \to u(x) \ \text{ a.e. in } \ \mathbb{R}^N. \end{cases} \tag{3.2}$$

By a direct computation, we derive that

$$\begin{aligned}\|u_n - u\|^2 =& \langle \mathcal{J}'(u_n) - \mathcal{J}'(u), u_n - u \rangle + \int_{\mathbb{R}^N} V^-(x)|u_n - u|^2 dx \\ &+ \int_{\mathbb{R}^N} K(x)(f(u_n) - f(u))(u_n - u)dx.\end{aligned} \tag{3.3}$$

By Lemma 2.2, $\int_{\mathbb{R}^N} V^-(x)|u_n-u|^2dx \to 0$. It is obvious that $\langle \mathcal{J}'(u_n)-\mathcal{J}'(u), u_n-u\rangle \to 0$. Thus, it is enough to show

$$\int_{\mathbb{R}^N} K(x)(f(u_n)-f(u))(u_n-u)dx \to 0, \text{ as } n\to\infty.$$

We can see that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} K(x)(f(u_n)-f(u))(u_n-u)dx\Big| \\
\le&\Big|\int_{\mathbb{R}^N} K(x)f(u_n)(u_n-u)+K(x)f(u)(u_n-u)dx\Big| \\
\le&\int_{\mathbb{R}^N}\Big(K(x)|f(u_n)(u_n-u)|+K(x)|f(u)(u_n-u)|\Big)dx.
\end{aligned}\tag{3.4}$$

Since $\frac{2-\tau_i}{2}+\frac{1}{2}+\frac{\tau_i-1}{2}=1,\ i=1,2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} K(x)|f(u_n)(u_n-u)|dx \\
=&\int_{\mathbb{R}^N} K(x)|(|u_n|^{\tau_1-1}+|u_n|^{\tau_2-1})(u_n-u)|dx \le \sum_{i=1}^{2}\Big(\int_{\mathbb{R}^N}K(x)dx\Big)^{\frac{2-\tau_i}{2}}|u_n|_{K,2}^{\tau_i-1}|u_n-u|_{K,2}.
\end{aligned}\tag{3.5}$$

This together with (3.2), for large n, we obtain that $\int_{\mathbb{R}^N} K(x)|f(u_n)(u_n-u)|<C\varepsilon$. Similarly, for that ε and large n, $\int_{\mathbb{R}^N} K(x)|f(u)(u_n-u)|dx\le\varepsilon$. The proof is complete. □

Proof of Theorem 1.2. Assume that conditions (K_1), (V_1), (V_2), (KV), (f_1) and (f_2) hold. Then the limit u_0 of the minimizing sequence $\{u_n\}$ is nontrivial.

Proof. We will argue directly. First, we take a subspace $\hat{E}$ of E with $\dim\hat{E}<\infty$. By Lemma 2.1 and a similar discussion to the proof of Lemma 2.4 in Zhang and Wang [22](see (5) of Lemma 3.1 in [20]), there exists a constant $\kappa>0$ such that

$$\text{meas}\{x: K(x)|u(x)|^{\tau_1}\ge\kappa\|u\|^{\tau_1},\ \forall\, u\in\hat{E}\}\ge\kappa.\tag{3.6}$$

We consider the sets $\Lambda=\{x: K(x)F(u)\ge\kappa\|u\|^{\tau_1},\ u\in\hat{E}\}$ and $\Omega=\{x: K(x)|u(x)|^{\tau_1}\ge\kappa\|u\|^{\tau_1},\ u\in\hat{E}\}$. By (f$_2$), we obtain that $\Omega\subset\Lambda$. Hence, meas$(\Lambda)\ge$ meas$(\Omega)\ge\kappa$. Then for any fixed $u\in\hat{E}\backslash\{0\}$ and $s>0$, it follows from (f_2) that

$$\begin{aligned}
\mathcal{J}(su)\le&\frac{1}{2}\|su\|^2-\int_{\mathbb{R}^N}K(x)F(su)dx \\
\le&\frac{1}{2}\|su\|^2-\int_{\Lambda}\kappa\|su\|^{\tau_1}dx \\
\le&\frac{1}{2}\|su\|^2-\kappa\|su\|^{\tau_1}\text{meas}(\Lambda) \\
\le&\frac{1}{2}s^2\|u\|^2-\kappa^2s^{\tau_1}\|u\|^{\tau_1}.
\end{aligned}\tag{3.7}$$

Since $1<\tau_1<2$, $\mathcal{J}(su)<0$ for s sufficient small and $u\in E\backslash\{0\}$. Since $\mathcal{J}$ is coercive and by Lemma 3.2, we obtain that

$$\mathcal{J}(u_0)=\inf_E\mathcal{J}(u)<0,$$

which implies that $u_0\ne0$. □

Now, we show that the energy functional $\mathcal{J}$ has the geometric properties in Lemma 2.5 under the conditions of Theorem 1.3.

Lemma 3.3. *Assume that conditions* (K_1), *(V_1), (V_2)*, (KV), (f_1) *and* (f_2) *hold. Moreover,* $f(u) = -f(-u), \forall u \in \mathbb{R}$. *Then there exists a sequence* $0 < \rho_k$ ($\rho_k \to 0$ *as* $k \to \infty$) *such that*

$$a_k := \inf_{u \in Z_k, \|u\| = \rho_k} \mathcal{J}(u) \geq 0.$$

Proof. By (V_2) and (f_1), we obtain that

$$\begin{aligned} \mathcal{J}(u) =& \frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)|u|^2 dx - \int_{\mathbb{R}^N} K(x)F(u)dx \\ \geq& \frac{\eta_0 - 1}{2\eta_0}\|u\|^2 - c|u|_{K,\tau_1}^{\tau_1} - c|u|_{K,\tau_2}^{\tau_2}. \end{aligned} \tag{3.8}$$

Let

$$\beta_{k,\tau_i} := \sup_{u \in Z_k, \|u\| = 1} |u|_{K,\tau_i}, \ i = 1, 2, \ \forall k \in \mathbb{N}.$$

Based on Lemma 3.8 in [21] and Lemmas 2.1, $\beta_{k,\tau_i} \to 0, i = 1, 2$, as $k \to \infty$. We have that

$$\mathcal{J}(u) \geq \frac{\eta_0 - 1}{2\eta_0}\|u\|^2 - c\beta_{k,\tau_1}^{\tau_1}\|u\|^{\tau_1} - c\beta_{k,\tau_2}^{\tau_2}\|u\|^{\tau_2} = \|u\|^2\Big(\frac{\eta_0 - 1}{2\eta_0} - c\beta_{k,\tau_1}^{\tau_1}\|u\|^{\tau_1 - 2} - c\beta_{k,\tau_2}^{\tau_2}\|u\|^{\tau_2 - 2}\Big). \tag{3.9}$$

Choose $\|u\| = \rho_k := \left(\frac{\eta_0}{\eta_0 - 1}\right)^{\frac{1}{2-\tau_1}}\left[8c\beta_{k,\tau_1}^{\tau_1} + \left(\frac{\eta_0}{\eta_0-1}\right)^{\frac{2-\tau_1}{2-\tau_2}-1}\left(8c\beta_{k,\tau_2}^{\tau_2}\right)^{\frac{2-\tau_1}{2-\tau_2}}\right]^{\frac{1}{2-\tau_1}}$. By the definition of ρ_k, a direct computation implies

$$c\beta_{k,\tau_1}^{\tau_1}\rho_k^{\tau_1 - 2} = \frac{c\beta_{k,\tau_1}^{\tau_1}}{\frac{\eta_0}{\eta_0-1}\left[8c\beta_{k,\tau_1}^{\tau_1} + \left(\frac{\eta_0}{\eta_0-1}\right)^{\frac{2-\tau_1}{2-\tau_2}-1}\left(8c\beta_{k,\tau_2}^{\tau_2}\right)^{\frac{2-\tau_1}{2-\tau_2}}\right]} \leq \frac{\eta_0 - 1}{8\eta_0} \tag{3.10}$$

and

$$c\beta_{k,\tau_2}^{\tau_2}\rho_k^{\tau_2 - 2} = \frac{c\beta_{k,\tau_2}^{\tau_2}}{\left(\frac{\eta_0}{\eta_0-1}\right)^{\frac{2-\tau_2}{2-\tau_1}}\left[8c\beta_{k,\tau_1}^{\tau_1} + \left(\frac{\eta_0}{\eta_0-1}\right)^{\frac{2-\tau_1}{2-\tau_2}-1}\left(8c\beta_{k,\tau_2}^{\tau_2}\right)^{\frac{2-\tau_1}{2-\tau_2}}\right]^{\frac{2-\tau_2}{2-\tau_1}}} \leq \frac{\eta_0 - 1}{8\eta_0}. \tag{3.11}$$

Then, we get

$$\mathcal{J}(u) \geq \rho_k^2\Big(\frac{\eta_0 - 1}{2\eta_0} - c\beta_{k,\tau_1}^{\tau_1}\rho_k^{\tau_1 - 2} - c\beta_{k,\tau_2}^{\tau_2}\rho_k^{\tau_2 - 2}\Big) \geq \rho_k^2\frac{\eta_0 - 1}{\eta_0}\Big(\frac{1}{2} - \frac{1}{4}\Big) = \frac{\eta_0 - 1}{4\eta_0}\rho_k^2 > 0. \tag{3.12}$$

Thus, for every k, $u \in Z_k$ and $\|u\| = \rho_k$, we have $a_k \geq 0$. Since $\beta_{k,\tau_1}, \beta_{k,\tau_2} \to 0$ as $k \to \infty$, it follows that $\rho_k \to 0$ as $k \to \infty$. □

Lemma 3.4. *Assume that conditions* (K_1), (V_1), (V_2), (KV), (f_1) *and* (f_2) *hold. Moreover,* $f(u) = -f(-u), \forall u \in \mathbb{R}$. *Then there exists a sequence* r_k , $0 < r_k < \rho_k$, $r_k \to 0$ *as* $k \to \infty$ *such that*

$$b_k := \max_{u \in Y_k, \|u\| = r_k} \mathcal{J}(u) < 0.$$

Proof. Noticing that Y_k is of finite dimension for each $k \in \mathbb{N}$. By a similar discussion to the proof of Theorem 1.2, it follows from (f_1) and (f_2) that

$$\begin{aligned}\mathcal{J}(u) \leq &\frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F(u)dx\\ \leq &\frac{1}{2}\|u\|^2 - \int_{\Lambda} \kappa\|u\|^{\tau_1}dx\\ \leq &\frac{1}{2}\|u\|^2 - \kappa^2\|u\|^{\tau_1}\\ = &\|u\|^{\tau_1}\Big(\frac{1}{2}\|u\|^{2-\tau_1} - \kappa^2\Big).\end{aligned} \tag{3.13}$$

Choosing $\|u\| := r_k = \min\{(\kappa^2)^{\frac{1}{2-\tau_1}}, \frac{1}{2}\rho_k\}$, we obtain $0 < r_k < \rho_k$ and $\frac{1}{2}\|u\|^{2-\tau_1} - \kappa^2 = -\frac{1}{2}\kappa^2 < 0$ for $\|u_n\| = r_k$. Hence, for each k, we have $b_k < 0$. This completes the proof. □

Lemma 3.5. *Assume that conditions (K_1), (V_1), (V_2), (KV), (f_1) and (f_2) hold. Moreover, $f(u) = -f(-u), \forall u \in \mathbb{R}$. Then it holds that*

$$d_k := \inf_{u\in Z_k, \|u\|\leq\rho_k} \mathcal{J}(u) \to 0 \text{ as } k \to \infty.$$

Proof. For $u \in Z_k$, $\|u\| \leq \rho_k$, we derive that

$$\mathcal{J}(u) \leq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} K(x)F(u)dx \leq \frac{1}{2}\|u\|^2 \leq \frac{1}{2}\rho_k^2. \tag{3.14}$$

On the other hand, by (3.9), we obtain that

$$\begin{aligned}\mathcal{J}(u) = &\frac{1}{2}\|u\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^- u^2dx - \int_{\mathbb{R}^N} K(x)F(u)dx\\ \geq &\frac{\eta_0 - 1}{2\eta_0}\|u\|^2 - c\beta_{k,\tau_1}^{\tau_1}\|u\|^{\tau_1} - c\beta_{k,\tau_2}^{\tau_2}\|u\|^{\tau_2}\\ \geq &- c\beta_{k,\tau_1}^{\tau_1}\rho_k^{\tau_1} - c\beta_{k,\tau_2}^{\tau_2}\rho_k^{\tau_2}.\end{aligned} \tag{3.15}$$

Since $\beta_{k,\tau_1} \to 0, \beta_{k,\tau_2} \to 0$ and $\rho_k \to 0$ as $k \to \infty$, it follows from (3.14) and (3.15) that

$$d_k := \inf_{u\in Z_k, \|u\|\leq\rho_k} \mathcal{J}(u) \to 0 \text{ as } k \to \infty.$$

This proof is complete. □

Proof of Theorem 1.3. We just need to prove the $(PS)_c^*$ condition. Consider a sequence $\{u_{n_j}\}$ such that

$$n_j \to \infty,\ u_{n_j} \in Y_{n_j},$$
$$\mathcal{J}(u_{n_j}) \to c,\ \mathcal{J}'|_{Y_{n_j}}(u_{n_j}) \to 0.$$

For the proof of boundedness of $\{u_{n_j}\}$, arguing indirectly, $\|u_{n_j}\| \to +\infty$, as $n_j \to +\infty$. It follows that $\mathcal{J}'|_{Y_{n_j}}(u_{n_j}) \to 0$, that is,

$$\frac{\eta_0 - 1}{\eta_0}\left\|u_{n_j}\right\|^2 \leq \int_{\mathbb{R}^N} |\nabla u_{n_j}|^2 + V(x)|u_{n_j}|^2dx = \int_{\mathbb{R}^N} K(x)f(u_{n_j})u_{n_j}dx \tag{3.16}$$

and for $1 < \tau_1 < \tau_2 < 2$, we derive that

$$\begin{aligned}\frac{\eta_0 - 1}{\eta_0} &\leq \frac{\int_{\mathbb{R}^N} K(x) f(u_{n_j}) u_{n_j} dx}{\|u_{n_j}\|^2} \leq \frac{|u_{n_j}|_{K,\tau_1}^{\tau_1} + |u_{n_j}|_{K,\tau_2}^{\tau_2}}{\|u_{n_j}\|^2} \\ &\leq \frac{C\|u\|^{\tau_1} + C\|u\|^{\tau_2}}{\|u_{n_j}\|^2} = \frac{C}{\|u_{n_j}\|^{2-\tau_1}} + \frac{C}{\|u_{n_j}\|^{2-\tau_2}} \to 0,\end{aligned} \tag{3.17}$$

as $j \to \infty$, which is contradiction. Therefore, we derive that $\{u_{n_j}\}$ is bounded in E.

Since $\{u_{n_j}\}$ is bounded in E, by Lemma 2.1, we get that the sequence $\{u_{n_j}\}$ has strong convergent subsequence in E. Passing to a sequence, we suppose that $u_{n_j} \to u_k$ in E. Thus, by Lemma 2.5, for each k, $\{u_k\}$ is a critical point of $\mathcal{J}$ and $\mathcal{J}(u_k) \to 0$, as $k \to \infty$. Hence, (1.1) possesses infinitely many small energy solutions. The proof of Theorem 1.3 is complete. □

Conflict of interest

The authors declare no conflict of interest.

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