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Research article

Fixed point theorems for generalized α-ψ-contractive mappings in extended b-metric spaces with applications

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Abstract: In this paper, we introduce a new concept of locally α-ψ-contractive mapping, generalized $\alpha - \psi$ rational contraction and establish fixed point theorems for such mappings in the setting of extended b-metric space. Our main results extend and improve some results given by some authors. We also provide a non trivial example to show the validity of our main results. As an application, we derive some new fixed point result for ψ-graphic contraction defined on an extended b-metric space endowed with a graph.

Keywords: fixed point; b-metric spaces; extended b-metric spaces; $\alpha - \psi$-contractive mapping
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1. Introduction

One of the most important results in fixed point theory which is well-known as Banach's fixed point theorem or Banach contraction principle [1]. This principle guarantees the existence and uniqueness of fixed points of certain self-mappings of metric spaces, and provides a constructive method to find those fixed points. The Banach contraction principle was also used to establish the existence of a unique solution for a nonlinear integral equation. For instance, it has been utilized to find the existence of solutions of nonlinear Volterra integral equations, nonlinear integro-differential equations in Banach spaces and to demonstrate the convergence of algorithms in computational mathematics. Due to its significance and usefulness for mathematical theory, it has turn into an exceptionally well known tool in solving existence problems in numerous directions. Many researchers have established different theorems to extend, unify and generalized Banach's theorem by defining a variety of contractive type conditions for self and non-self mappings on metric spaces.

On the other hand, in 1989, Bakhtin [2] introduced the concept of a b-metric space and proved some fixed point theorems for some contractive mappings in b-metric spaces. In 1993, Czerwik [3] extended the results of b-metric spaces. Recently, many research [4–6] was conducted on b-metric space under different contraction conditions. The notion of extended b-metric space has been introduced recently by Kamran et al [7]. For more details in this direction, we refer the readers to [8–14] On the other hand, Samet et al. [15] introduced a new concept of α-ψ-contractive mappings and establish various fixed point theorems in complete metric spaces. Recently, Shatanawi et al. [18] utilized the notion of α-ψ-contractive mappings in extended b-metric spaces and proved a fixed point theorem to generalize the main result of Kamran et al [7]. In this paper, we will define the notion of locally α-ψ-contractive mappings in the setting of extended b-metric space and obtain some new fixed point results.

2. Preliminaries

Czerwik [3] introduced the notion of b-metric space in this way.

Definition 1. *(see [3]) Let X be a non empty set and $s \geq 1$ be a given real number. A function d_b:$X\times X\rightarrow [0,\infty)$ is called b-metric if it satisfies the following properties for each $x, y, z \in X$.*
(b1) $d_b(x, y) = 0$ if and only if x=y;
(b2) $d_b(x, y) = d_b(y, x)$;
(b3) $d_b(x, y) \leq s[d_b(x, z) + d_b(z, y)]$.
Then (X, d_b) is called a b-metric space with coefficient s.

Kamran et al [7] defined extended b-metric space to generalize b-metric space as follows:

Definition 2. *(see [7])Let X be a non-empty set and θ :$X\times X\rightarrow [1,\infty)$. A function d_θ:$X\times X\rightarrow [0,\infty)$ is called an extended b-metric if for all $x, y, z \in X$, it satisfies:*
(d_θ1) $d_\theta(x, y) = 0$ if and only if $x = y$,
(d_θ2) $d_\theta(x, y) = d_\theta(y, x)$,
(d_θ3) $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.
The pair (X, d_θ) is called an extended b-metric space.
For $x \in X$ and $\epsilon > 0$, $\overline{B(x,\epsilon)} = \{y \in X{:}d_\theta(x, z) \leq \epsilon\}$ is a closed ball in extended b-metric space (X, d_b).

Example 3. *Consider the set X= $\{-1, 1, 2\}$, define the function θ on $X\times X$ to be the function $\theta(x, y) = |x| + |y|$. We define the function $d_\theta(x, y)$ as follows:*
$d_\theta(2, 2) = d_\theta(1, 1) = d_\theta(-1,-1) = 0, d_\theta(1, 2) = \frac{1}{2} = d_\theta(2, 1)$ and $d_\theta(1,-1) = d_\theta(-1, 1) = d_\theta(2,-1) = d_\theta(-1, 2) = \frac{1}{3}$. Then it is clear that $d_\theta(x, y)$ satisfies the first two conditions of Definition 2. We need to verify the last condition:

$$\begin{aligned}
d_\theta(1,2) = \frac{1}{2} &\leq 3\left[\frac{1}{3}+\frac{1}{3}\right] = \theta(1,2)[d_\theta(1,-1)+d_\theta(-1,2)]\\
d_\theta(1,-1) = \frac{1}{3} &\leq 2\left[\frac{1}{2}+\frac{1}{3}\right] = \theta(1,-1)[d_\theta(1,2)+d_\theta(2,-1)]\\
d_\theta(-1,2) = \frac{1}{3} &\leq 3\left[\frac{1}{3}+\frac{1}{2}\right] = \theta(-1,2)[d_\theta(-1,1)+d_\theta(1,2)]
\end{aligned}$$

Therefore, $d_\theta(x, y)$ satisfies the last condition of the definition and hence (X, d_θ) is an extended b-metric space.

The concepts of convergence, Cauchy sequence and completeness can easily be extended to the case of an extended b-metric space.

Definition 4. *(see [7]) Let (X, d_θ) be an extended b-metric space.*

(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$ and x is called the limit point of sequence x_n. In extended b-metric space, the limit of a convergent sequence is unique.

(ii) A sequence $\{x_n\}$ in X is said to be 6 Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$, for all $m, n \geq N$.

Definition 5. *(see [7]) An extended b-metric space (X, d_θ) is complete if every Cauchy sequence in X is convergent.*

Note that, in general a b-metric is not a continuous functional and thus so is an extended b-metric. In this paper,we have give some new notions of locally α-ψ-contractive mapping and generalized $\alpha - \psi$ rational contraction of Dass and Gupta [17] type in the context of complete extended b-metric spaces and investigate some existence and uniqueness fixed point theorems.

3. Main results

In 2012, Samet et al. [15] introduced the concept of α-admissible mapping and $\alpha - \psi$-contractive mapping in complete metric space as follows:

Definition 6. *(see [15]) Let $\mathcal{J}$ be a self-mapping on X and α :X×X→ $[0, +\infty)$ be a function. We say that $\mathcal{J}$ is an α-admissible mapping if*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(\mathcal{J}x, \mathcal{J}y) \geq 1 \quad \forall x, y \in X.$$

Let Ψ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ such that

(1) ψ is nondecreasing,

(2) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n-th composition of ψ.

Lemma 7. *If $\psi \in \Psi$, then $\psi(t) < t$, for all $t \in (0, +\infty)$.*

Definition 8. *(see [15]) A be a self-mapping $\mathcal{J}$ on metric space (X,d) is said to be $\alpha - \psi$-contractive mapping if there exist α :X×X→ $[0, +\infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(\mathcal{J}x, \mathcal{J}y) \leq \psi\,(d(x, y))$$

$\forall x, y \in X.$

Later on, Samet [16] extended this concept of $\alpha - \psi$-contractive mapping to the context of b-metric space and obtained some generalized fixed point results. Samet [16] also defined $\alpha - \psi$ rational contraction in the setting of b-metric space by using rational expression of Dass and Gupta [17] type in his contraction.

Very recently, Shatanawi et al. [18] utilized the notion of $\alpha - \psi$-contractive mapping in extended b-metric space in this way:

Definition 9. *(see [18]) Let X be a set and* $\theta : X\times X\to [1,+\infty)$ *be a mapping. A function* $\psi : [0,\infty) \to [0,\infty)$ *is said to be an extended comparison function if it satisfies the following conditions:*

(1) ψ *is nondecreasing,*

(2) $\sum_{n=1}^{\infty} \psi^n(t) \prod_{i=0}^{n} \theta(x_i, x_m) < +\infty$, *any sequence* $\{x_n\}_{n=1}^{\infty}$ *in X, for all* $t > 0$ *and* $m \in \mathbb{N}$ *where* ψ^n *is the n-th iterate of* ψ.
The set of all extended comparison functions is denoted by Ψ_θ. *Note that if* $\psi \in \Psi_\theta$, *then we have* $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$, *since*

$$\sum_{n=1}^{\infty} \psi^n(t) \prod_{i=0}^{n} \theta(x_i, x_m) \geq \sum_{n=1}^{\infty} \psi^n(t),$$

for all $t > 0$. *Hence, by Lemma 7, we have* $\psi(t) < t$. *Also, note that the family* Ψ_θ *is a non-empty set, as it can be shown in the following examples.*

Example 10. *Consider the extended b-metric space* (X,d_θ) *that was defined in Example 3. Define the mapping* $\psi(t) = \frac{\lambda t}{4}$, *where* $\lambda < 1$. *Note that* $\theta(x, y) \leq 4$. *Then we have*

$$\psi^n(t) \prod_{i=0}^{n} \theta(x_i, x) \leq \frac{\lambda^n t}{4^n}.4^n = \lambda^n t.$$

Therefore,

$$\sum_{n=1}^{\infty} \psi^n(t) \prod_{i=0}^{n} \theta(x_i, x_m) \leq \sum_{n=1}^{\infty} \lambda^n t < \infty.$$

Definition 11. *Let* (X,d_θ) *be an extended b-metric space and* $A \subseteq X$. *A mapping* $\mathcal{J} : X\to X$ *is said to be locally* α-ψ*-contractive mapping if there exists* $\alpha : X\times X\to [0,\infty)$ *such that*

$$\alpha(x, y)d_\theta(\mathcal{J}x, \mathcal{J}y) \leq \psi(d_\theta(x, y)) \tag{3.1}$$

$\forall x, y \in A$.

Theorem 12. *Let* (X,d_θ) *be a complete extended b-metric space such that* d_θ *is continuous functional,* $x_0 \in X$, $r > 0$ *and let* $\mathcal{J} : X\to X$ *be a locally* α-ψ*-contractive mapping. Suppose that the following conditions hold:*
(i) $\mathcal{J}$ *is* α*-admissible;*
(ii) for such $x_0 \in X$, $\alpha(x_0, \mathcal{J}x_0) \geq 1$*;*
(iii)

$$\sum_{i=0}^{j} \psi^i (d_\theta(x_0, \mathcal{J}x_0)) \prod_{k=0}^{i} \theta\left(x_k, x_{j+1}\right) \leq r, \tag{3.2}$$

(iv) $\mathcal{J}$ *is continuous or* $\alpha(x_n, x) \geq 1$ *for a sequence* $\{x_n\}$ *in* $\overline{B(x_0, r)}$ *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for* $n \in \mathbb{N}$ *and* $x_n \to x$ *as* $n \to +\infty$.
Then $\mathcal{J}$ *has a fixed point in* $\overline{B(x_0, r)}$.
Moreover, for all $x, y \in \overline{B(x_0, r)}$, *there exists* $z \in \overline{B(x_0, r)}$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$. *Then the fixed point is unique.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0, \mathcal{J}x_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = \mathcal{J}^n x_0 = \mathcal{J}x_{n-1} \quad \forall n \in N$. Assume that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, we get $x_{n_0} = \mathcal{J}x_{n_0}$ and the proof is completed. Now, we suppose that $x_n \neq x_{n+1} \quad \forall n \in \mathbb{N}$. Since $\mathcal{J}$ is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, \mathcal{J}x_0) \geq 1 \Longrightarrow \alpha(\mathcal{J}x_0, \mathcal{J}x_{01}) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1$$

$\forall n \in \mathbb{N}$. First, we show that $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$. Using inequality (3.2), we have

$$d_\theta(x_0, \mathcal{J}x_0) \leq d_\theta(x_0, \mathcal{J}x_0)\theta(x_0, x_1) \leq r.$$

It follows that $x_1 \in \overline{B(x_0, r)}$. Let $x_2, x_3, .., x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$. Since $\mathcal{J}$ is a locally α-ψ-contractive mapping, so

$$\begin{aligned} d_\theta(x_j, x_{j+1}) &\leq \alpha(x_{j-1}, x_j) d_\theta(x_j, x_{j+1}) \\ &= \alpha(x_{j-1}, x_j) d_\theta(\mathcal{J}x_{j-1}, \mathcal{J}x_j) \\ &\leq \psi(d_\theta(x_{j-1}, x_j)) \\ &\leq \psi(\alpha(x_{j-2}, x_{j-1}) d_\theta(x_{j-1}, x_j)) \\ &= \psi(\alpha(x_{j-2}, x_{j-1}) d_\theta(\mathcal{J}x_{j-2}, \mathcal{J}x_{j-1})) \\ &\leq \psi^2(d_\theta(x_{j-2}, x_{j-1})) \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \psi^j(d_\theta(x_0, x_1)). \end{aligned} \tag{3.3}$$

Now

$$\begin{aligned} d_\theta(x_0, x_{j+1}) &\leq \theta(x_0, x_{j+1}) d_\theta(x_0, x_1) + \theta(x_0, x_{j+1})\theta(x_1, x_{j+1}) d_\theta(x_1, x_2) \\ &\quad + ... + \theta(x_0, x_{j+1})\theta(x_1, x_{j+1}) \cdots \theta(x_j, x_{j+1}) d_\theta(x_j, x_{j+1}) \\ &\leq \theta(x_0, x_{j+1}) d_\theta(x_0, x_1) + \theta(x_0, x_{j+1})\theta(x_1, x_{j+1})\psi(d_\theta(x_0, x_1)) \\ &\quad + ... + \theta(x_0, x_{j+1})\theta(x_1, x_{j+1}) \cdots \theta(x_j, x_{j+1})\psi^j(d_\theta(x_0, x_1)) \\ &= \sum_{i=0}^{j} \psi^i \left(d_\theta(x_0, \mathcal{J}x_0)\right) \prod_{k=0}^{i} \theta\left(x_k, x_{j+1}\right) \\ &\leq r. \end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in \mathbb{N}$. Now for $x_n, x_{n+1} \in \overline{B(x_0, r)}$, we have

$$\begin{aligned} d_\theta(x_n, x_{n+1}) &\leq \alpha(x_{n-1}, x_n) d_\theta(x_n, x_{n+1}) \\ &= \alpha(x_{n-1}, x_n) d_\theta(\mathcal{J}x_{n-1}, \mathcal{J}x_n) \end{aligned}$$

$$\begin{aligned}
&\leq \psi(d_\theta(x_{n-1}, x_n)) \\
&\leq \psi(\alpha(x_{n-2}, x_{n-1})d_\theta(x_{n-1}, x_n)) \\
&= \psi(\alpha(x_{n-2}, x_{n-1})d_\theta(\mathcal{J}x_{n-2}, \mathcal{J}x_{n-1})) \\
&\leq \psi^2(d_\theta(x_{n-2}, x_{n-1})) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \psi^n(d_\theta(x_0, x_1)).
\end{aligned} \tag{3.4}$$

Now for any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
d_\theta(x_n, x_m) &\leq \theta(x_n, x_m)[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_m)] \\
&= \theta(x_n, x_m)d_\theta(x_n, x_{n+1}) + \theta(x_n, x_m)d_\theta(x_{n+1}, x_m) \\
&\leq \theta(x_n, x_m)d_\theta(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)[d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_m)] \\
&\leq \theta(x_n, x_m)\psi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\psi^{n+1}(d_\theta(x_0, x_1)) + ... \\
&+ \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)...\theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)\psi^{m-1}(d_\theta(x_0, x_1)) \\
&= \sum_{j=n}^{m-1} \psi^j (d_\theta(x_0, x_1)) \prod_{i=n}^{j} \theta(x_i, x_m) \\
&= \sum_{j=1}^{m-1} \psi^j (d_\theta(x_0, x_1)) \prod_{i=n}^{j} \theta(x_i, x_m) - \sum_{j=1}^{n-1} \psi^j (d_\theta(x_0, x_1)) \prod_{i=n}^{j} \theta(x_i, x_m) \\
&= S_{m-1} - S_{n-1}
\end{aligned}$$

where $S_{m-1} = \sum\limits_{j=1}^{m-1} \psi^j (d_\theta(x_0, x_1)) \prod_{i=n}^{j} \theta(x_i, x_m)$. Since, $\psi \in \Psi_\theta$, so

$$\lim_{n,m\to\infty} [S_{m-1} - S_{n-1}] = 0.$$

Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. The completeness of the space (X, d_θ) implies that the sequence is convergent to a point $x^* \in X$, that is, $\lim_{n\to\infty} x_n = x^*$. Now, we show that x^* is a fixed point for $\mathcal{J}$. Since $\mathcal{J}$ is continuous, we have $\lim_{n\to\infty} \mathcal{J}x_n = \mathcal{J}x^*$ and so $\mathcal{J}x^* = x^*$. Now as $\lim_{n\to\infty} x_n = x^*$, so by the second part of assumption (iv), we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}
d_\theta(x_{n+1}, \mathcal{J}x^*) &= d_\theta(Tx_n, \mathcal{J}x^*) \\
&\leq \alpha(x_n, x^*)d_\theta(Tx_n, \mathcal{J}x^*) \\
&\leq \psi(d_\theta(x_n, x^*))
\end{aligned}$$

Letting $n \to +\infty$, since ψ is continuous at $t = 0$, we obtain $d_\theta(x^*, \mathcal{J}x^*) = 0$, that is, $x^* = \mathcal{J}x^*$. The last assumption of the theorem implies that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$. Since $\mathcal{J}$ is α-admissible, we have $\alpha(\mathcal{J}^n u; \mathcal{J}^n z) = \alpha(u, \mathcal{J}^n z) \geq 1$ and $\alpha(\mathcal{J}^n v; \mathcal{J}^n z) = \alpha(v, \mathcal{J}^n z) \geq 1$, for all $n \in \mathbb{N}$. Using (3.1), we get

$$
\begin{aligned}
d_\theta(u,\mathcal{J}^n z) &= d_\theta(\mathcal{J}u,\mathcal{J}(\mathcal{J}^{n-1}z)) \\
&\le \alpha(u,\mathcal{J}^{n-1}z)d_\theta(\mathcal{J}u,\mathcal{J}(\mathcal{J}^{n-1}z)) \\
&\le \psi(d_\theta(u,\mathcal{J}^{n-1}z)) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\le \psi^n(d_\theta(u,z))
\end{aligned}
$$

Since $\psi \in \Psi s$, the sequence $\{\psi^n(d_\theta(u,z))\}$ converges to 0. Therefore, $\mathcal{J}^n z$ converges to u. Similarly, we can show that $\mathcal{J}^n z$ converges to v. The uniqueness of the limit implies that $u = v$. □

In Theorem 12, if we put $\alpha(x,y) = 1$, then we obtain the following result.

Corollary 13. *Let (X,d_θ) be a complete extended b-metric space such that d_θ is continuous functional, $x_0 \in X$, $r > 0$ and let $\mathcal{J}$:$X \to X$ be a mapping satisfying the following conditions:*
(i) there exists $\psi \in \Psi s$ such that

$$d_\theta(\mathcal{J}x,\mathcal{J}y) \le \psi(d_\theta(x,y)) \quad \forall\, x,y \in \overline{B(x_0,r)},$$

(ii) for such $x_0 \in X$ such that

$$\sum_{i=0}^{j} \psi^i\,(d_\theta(x_0,\mathcal{J}x_0)) \prod_{k=0}^{i} \theta\left(x_k,x_{j+1}\right) \le r,$$

(iii) $\mathcal{J}$ is continuous.

Then $\mathcal{J}$ has a unique fixed point in $\overline{B(x_0,r)}$.

If we put $\psi(t) = \lambda t$, where $0 \le \lambda < 1$, in Theorem 12, we get the following result.

Corollary 14. *Let (X,d_θ) be a complete extended b-metric space such that d_θ is continuous functional, $x_0 \in X$, $r > 0$ and let $\mathcal{J}$:$X \to X$ be a mapping satisfying the following conditions:*
(i) there exists $\lambda \in [0,1)$ such that

$$\alpha(x,y)d_\theta(\mathcal{J}x,\mathcal{J}y) \le \lambda d_\theta(x,y) \quad \forall\, x,y \in \overline{B(x_0,r)},$$

(ii) for any sequence $\{x_n\}_{n=1}^{\infty}$ in $\overline{B(x_0,r)}$ and $x \in \overline{B(x_0,r)}$, we have

$$\sum_{n=1}^{\infty} \lambda^n \prod_{i=0}^{n} \theta\,(x_i,x) < +\infty,$$

(ii)$\mathcal{J}$ is α- admissible;
(iv) for such $x_0 \in X$, $\alpha(x_0,\mathcal{J}x_0) \ge 1$;
(v)

$$\sum_{i=0}^{j} \lambda^n \prod_{k=0}^{i} \theta\left(x_k,x_{j+1}\right) \le r,$$

(vi) $\mathcal{J}$ is continuous or $\alpha(x_n, x) \geq 1$ for a sequence $\{x_n\}$ in $\overline{B(x_0, r)}$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$.
Then $\mathcal{J}$ has a fixed point in $\overline{B(x_0, r)}$.
Moreover, for all $x, y \in \overline{B(x_0, r)}$, there exists $z \in \overline{B(x_0, r)}$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. Then the fixed point is unique.

Example 15. *Let the set $X = [0, \infty)$. Define $d_\theta : X \times X \to [0, \infty)$ and $\theta : X \times X \to [1, \infty)$ as: $d_\theta(x, y) = (x-y)^2$, $\theta(x, y) = x+y+2$. Then d_θ is a complete extended b-metric on X. Define $\mathcal{J} : X \to X$ by $\mathcal{J}x = \frac{x}{2}$ $\forall x \in X$ and $\alpha : X \times X \to \mathbb{R}$ such that*

$$\alpha(x, y) = \begin{cases} 1 & \text{if} \quad x, y \in [0, 1], \\ 0 & \quad\ \text{otherwise}. \end{cases}$$

Considering, $x_0 = \frac{1}{3}$, $x_1 = \frac{1}{6}$, $r = \frac{1}{2}$, then $\overline{B(x_0, r)} = [0, 1]$ and $\psi(t) = \frac{t}{3}$. Clearly T is an α-ψ-Contractive mapping,

$$d_\theta(x_0, \mathcal{J}x_0) = \left| \frac{1}{3} - \frac{1}{6} \right| = \frac{1}{6}$$

$$\sum_{i=1}^{n} \psi^i \left(d_\theta(x_0, \mathcal{J}x_0) \right) \prod_{i=0}^{j} \theta\left(x_i, x_{j+1}\right) = \frac{1}{6} \sum_{i=1}^{n} \frac{1}{3^n} \prod_{j=0}^{i} \theta\left(x_i, x_{j+1}\right) = \frac{1}{6}\frac{1}{2}\frac{5}{2} = \frac{5}{24} \leq \frac{1}{2}.$$

We prove the condition of our theorem 12 are satisfied for $x, y \in \overline{B(x_0, r)}$. We suppose that $x = y$,

$$\alpha(x, x) d_\theta(\mathcal{J}x, \mathcal{J}x) = 0 \leq \psi(d_\theta(x, x)) = \psi(0) = 0 \qquad \forall x \in \overline{B(x_0, r)}.$$

Now, we suppose that $x < y$. Then,

$$\alpha(x, y) d_\theta(\mathcal{J}x, \mathcal{J}y) = \frac{1}{4}(x - y)^2 < \frac{1}{3}(x - y)^2 \leq \psi(d_\theta(x, y))$$

Thus, $\mathcal{J}$ satisfies the conditions in Theorem 12 and hence it has a unique fixed point $x = 0 \in \overline{B(x_0, r)}$.

Now we define a notion of *generalized α-ψ-rational contraction of Dass and Gupta [17] and establish some fixed point results in the setting of complete extended b-metric space.*

Definition 16. *Let (X, d_θ) be an extended b-metric space and $\mathcal{J} : X \to X$ is said to be generalized α-ψ-rational contraction if there exists $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha(x, y) d_\theta(\mathcal{J}x, \mathcal{J}y) \leq \psi\left(\max\left\{ \frac{[1 + d_\theta(x, \mathcal{J}x)]\, d_\theta(y, \mathcal{J}y)}{1 + d_\theta(x, y)}, d_\theta(x, y) \right\} \right) \tag{3.5}$$

$\forall x, y \in X$.

Theorem 17. *Let (X, d_θ) be a complete extended b-metric space such that d_θ is continuous functional and let $\mathcal{J} : X \to X$ be generalized α-ψ-rational contraction. Suppose that the following conditions hold:*
(i) $\mathcal{J}$ is α-admissible;
(ii) $\exists$ $x_0 \in X$ such that $\alpha(x_0, \mathcal{J}x_0) \geq 1$;
(iii) $\mathcal{J}$ is continuous or $\alpha(x_n, x) \geq 1$ for a sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for $n \in \mathbb{N}$ and

$x_n \to x$ as $n \to +\infty$.
Then $\mathcal{J}$ has a fixed point in X.
Moreover, for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$. Then the fixed point is unique.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, \mathcal{J}x_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_n = \mathcal{J}^n x_0 = \mathcal{J}x_{n-1} \quad \forall n \in N$. Assume that $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, we get $x_{n_0} = \mathcal{J}x_{n_0}$ and the proof is completed. Now, we suppose that $x_n \neq x_{n+1}$ $\forall n \in \mathbb{N}$. This implies that $d_\theta(x_n, x_{n+1}) > 0$. The second condition of the Theorem implies that $\alpha(\mathcal{J}x_0, \mathcal{J}x_1) = \alpha(x_1, x_2) \geq 1$. So by induction on n, we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Now we apply the contractive condition 3.5. As $\mathcal{J}$ is a *generalized α-ψ-rational contraction*, so

$$\begin{aligned}
d_\theta(x_n, x_{n+1}) &\leq \alpha(x_{n-1}, x_n) d_\theta(x_n, x_{n+1}) \\
&= \alpha(x_{n-1}, x_n) d_\theta(\mathcal{J}x_{n-1}, \mathcal{J}x_n) \\
&\leq \psi\left(\max\left\{\frac{[1 + d_\theta(x_{n-1}, \mathcal{J}x_{n-1})]\, d_\theta(x_n, \mathcal{J}x_n)}{1 + d_\theta(x_{n-1}, x_n)}, d_\theta(x_{n-1}, x_n)\right\}\right) \\
&= \psi\left(\max\left\{\frac{[1 + d_\theta(x_{n-1}, x_n)]\, d_\theta(x_n, x_{n+1})}{1 + d_\theta(x_{n-1}, x_n)}, d_\theta(x_{n-1}, x_n)\right\}\right) \\
&= \psi\left(\max\left\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\right\}\right).
\end{aligned}$$

If $\max\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} = d_\theta(x_n, x_{n+1})$, then

$$d_\theta(x_n, x_{n+1}) \leq \psi\left(d_\theta(x_n, x_{n+1})\right) < d_\theta(x_n, x_{n+1})$$

which is a contradiction. Thus $\max\{d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} = d_\theta(x_{n-1}, x_n)$. Hence

$$d_\theta(x_n, x_{n+1}) \leq \psi\left(d_\theta(x_{n-1}, x_n)\right) \tag{3.6}$$

Similarly

$$\begin{aligned}
d_\theta(x_{n-1}, x_n) &\leq \alpha(x_{n-2}, x_{n-1}) d_\theta(x_{n-1}, x_n) \\
&= \alpha(x_{n-2}, x_{n-1}) d_\theta(\mathcal{J}x_{n-2}, \mathcal{J}x_{n-1}) \\
&\leq \psi\left(\max\left\{\frac{[1 + d_\theta(x_{n-2}, \mathcal{J}x_{n-2})]\, d_\theta(x_{n-1}, \mathcal{J}x_{n-1})}{1 + d_\theta(x_{n-2}, x_{n-1})}, d_\theta(x_{n-2}, x_{n-1})\right\}\right) \\
&= \psi\left(\max\left\{\frac{[1 + d_\theta(x_{n-2}, x_{n-1})]\, d_\theta(x_{n-1}, x_n)}{1 + d_\theta(x_{n-2}, x_{n-1})}, d_\theta(x_{n-2}, x_{n-1})\right\}\right) \\
&= \psi\left(\max\left\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-2}, x_{n-1})\right\}\right)
\end{aligned}$$

If $\psi\left(\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-2}, x_{n-1})\}\right) = d_\theta(x_{n-1}, x_n)$, then

$$d_\theta(x_{n-1}, x_n) \leq \psi\left(d_\theta(x_{n-1}, x_n)\right) < d_\theta(x_{n-1}, x_n)$$

which is a contradiction. Thus
$\psi\left(\max\{d_\theta(x_{n-1}, x_n), d_\theta(x_{n-2}, x_{n-1})\}\right) = d_\theta(x_{n-2}, x_{n-1})$. Hence

$$d_\theta(x_{n-1}, x_n) \leq \psi\left(d_\theta(x_{n-2}, x_{n-1})\right) \tag{3.7}$$

Thus by (3.6), (3.7) and induction, we have

$$\begin{aligned} d_\theta(x_n, x_{n+1}) &\leq \psi\left(d_\theta(x_{n-1}, x_n)\right) \\ &\leq \psi\left(d_\theta(x_{n-2}, x_{n-1})\right) \\ &\leq \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \psi^n(d_\theta(x_0, x_1)) \end{aligned}$$

for all $n \in \mathbb{N}$. Now for any $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d_\theta(x_n, x_m) &\leq \theta(x_n, x_m)[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_m)] \\ &= \theta(x_n, x_m)d_\theta(x_n, x_{n+1}) + \theta(x_n, x_m)d_\theta(x_{n+1}, x_m) \\ &\leq \theta(x_n, x_m)d_\theta(x_n, x_{n+1}) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)[d_\theta(x_{n+1}, x_{n+2}) + d_\theta(x_{n+2}, x_m)] \\ &\leq \theta(x_n, x_m)\psi^n(d_\theta(x_0, x_1)) + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\psi^{n+1}(d_\theta(x_0, x_1)) + ... \\ &+ \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)...\theta(x_{m-2}, x_m)\theta(x_{m-1}, x_m)\psi^{m-1}(d_\theta(x_0, x_1)) \\ &= \sum_{j=n}^{m-1} \psi^j\left(d_\theta(x_0, x_1)\right) \prod_{i=n}^{j} \theta(x_i, x_m) \\ &= \sum_{j=1}^{m-1} \psi^j\left(d_\theta(x_0, x_1)\right) \prod_{i=n}^{j} \theta(x_i, x_m) - \sum_{j=1}^{n-1} \psi^j\left(d_\theta(x_0, x_1)\right) \prod_{i=n}^{j} \theta(x_i, x_m) \\ &= S_{m-1} - S_{n-1} \end{aligned}$$

where $S_{m-1} = \sum_{j=1}^{m-1} \psi^j\left(d_\theta(x_0, x_1)\right) \prod_{i=n}^{j} \theta(x_i, x_m)$. Since, $\psi \in \Psi_s$, so $\lim_{n,m\to\infty}[S_{m-1} - S_{n-1}] = 0$. Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. The completeness of the space (X, d_θ) implies that the sequence is convergent to a point $x^* \in X$. Now, we show that x^* is a fixed point for $\mathcal{J}$. Since $\mathcal{J}$ is continuous, we have $\lim_{n\to\infty} \mathcal{J}x_n = \mathcal{J}x^*$ and so $\mathcal{J}x^* = x^*$.
Now as $\lim_{n\to\infty} x_n = x^*$, so by the second part of assumption (iv), we have $\alpha(x_n, x^*) \geq 1$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} d_\theta(x_{n+1}, \mathcal{J}x^*) &= d_\theta(Tx_n, \mathcal{J}x^*) \\ &\leq \alpha(x_n, x^*)d_\theta(Tx_n, \mathcal{J}x^*) \\ &\leq \psi\left(\max\left\{\frac{[1 + d_\theta(x_n, Tx_n)]\, d_\theta(x^*, \mathcal{J}x^*)}{1 + d_\theta(x_n, x^*)}, d_\theta(x_n, x^*)\right\}\right) \end{aligned}$$

Letting $n \to +\infty$, since ψ is continuous at $t = 0$, we obtain $d_\theta(x^*, \mathcal{J}x^*) = 0$, that is, $x^* = \mathcal{J}x^*$.Suppose that $\mathcal{J}$ has two fixed points $u, v \in X$.The last assumption of the theorem implies that $\alpha(u, z) \geq 1$ and $\alpha(v, z) \geq 1$. Since $\mathcal{J}$ is α-admissible, we have $\alpha(\mathcal{J}^n u; \mathcal{J}^n z) = \alpha(u, \mathcal{J}^n z) \geq 1$ and $\alpha(\mathcal{J}^n v; \mathcal{J}^n z) = \alpha(v, \mathcal{J}^n z) \geq 1$, for all $n \in \mathbb{N}$. Using (3.5), we get

$$d_\theta(u, \mathcal{J}^n z) = d_\theta(\mathcal{J}u, \mathcal{J}(\mathcal{J}^{n-1}z))$$

$$\leq \alpha(u,\mathcal{J}^{n-1}z)d_\theta(\mathcal{J}u,\mathcal{J}(\mathcal{J}^{n-1}z))$$
$$\leq \psi\left(\max\left\{\frac{[1+d_\theta(u,\mathcal{J}u)]\,d_\theta(\mathcal{J}^{n-1}z,\mathcal{J}(\mathcal{J}^{n-1}z))}{1+d_\theta(u,\mathcal{J}^{n-1}z)},d_\theta(u,\mathcal{J}^{n-1}z)\right\}\right).$$

If

$$\max\left\{\frac{[1+d_\theta(u,\mathcal{J}u)]\,d_\theta(\mathcal{J}^{n-1}z,\mathcal{J}(\mathcal{J}^{n-1}z))}{1+d_\theta(u,\mathcal{J}^{n-1}z)},d_\theta(u,\mathcal{J}^{n-1}z)\right\}=\frac{[1+d_\theta(u,\mathcal{J}u)]\,d_\theta(\mathcal{J}^{n-1}z,\mathcal{J}(\mathcal{J}^{n-1}z))}{1+d_\theta(u,\mathcal{J}^{n-1}z)}$$

then

$$d_\theta(u,\mathcal{J}^nz)\leq\psi\left(\frac{[1+d_\theta(u,\mathcal{J}u)]\,d_\theta(\mathcal{J}^{n-1}z,\mathcal{J}(\mathcal{J}^{n-1}z))}{1+d_\theta(u,\mathcal{J}^{n-1}z)}\right).$$

Taking the limit as $n\to\infty$, we get $\mathcal{J}^nz\to u$. Similarly, if

$$\max\left\{\frac{[1+d_\theta(u,\mathcal{J}u)]\,d_\theta(\mathcal{J}^{n-1}z,\mathcal{J}(\mathcal{J}^{n-1}z))}{1+d_\theta(u,\mathcal{J}^{n-1}z)},d_\theta(u,\mathcal{J}^{n-1}z)\right\}=d_\theta(u,\mathcal{J}^{n-1}z).$$

Then

$$d_\theta(u,\mathcal{J}^nz)\leq\psi\left(d_\theta(u,\mathcal{J}^{n-1}z)\right).$$

Taking the limit as $n\to\infty$, we get $\mathcal{J}^nz\to u$. Similarly, we can show that $\mathcal{J}^nz$ converges to v. The uniqueness of the limit implies that $u=v$. □

In Theorem 17, if we put $\alpha(x,y)=1$, then we obtain the following result.

Corollary 18. *Let (X,d_θ) be a complete extended b-metric space such that d_θ is continuous functional and let $\mathcal{J}:X\to X$ be a continuous self mapping such that*

$$d_\theta(\mathcal{J}x,\mathcal{J}y)\leq\psi\left(\max\left\{\frac{[1+d_\theta(x,\mathcal{J}x)]\,d_\theta(y,\mathcal{J}y)}{1+d_\theta(x,y)},d_\theta(x,y)\right\}\right).$$

Then $\mathcal{J}$ has a unique fixed point in X.

4. Fixed point theorem for graphic contraction

In 2008, Jachymski [19] introduced the notion of Banach G-contraction to generalize Banach contraction principle in the context of metric space. Very recently, Chifu [20] introduced Banach graphic contraction in the setting of extended b-metric space and obtained fixed and common fixed point theorems. Now we derive a fixed point theorem as an application of our result. Throughout this paper, we denote (X,d_θ) as extended b-metric space and Δ the diagonal of the Cartesian product $X\times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G)\supseteq\Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G),E(G))$. Moreover, we may treat G as a weighted graph (see [19]) by assigning to each edge the distance between its vertices. If x and y are vertices in a graph G, then a path in G from x to y of length N ($N\in\mathbb{N}$) is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices such that $x_0=x$, $x_N=y$ and $(x_{n-1},x_n)\in E(G)$ for $i=1,...,N$. A graph G is connected if there is a path between any two vertices. G is weakly connected if $\tilde{G}$ is connected(see for details [21–23]).

Definition 19. *[20] A mapping $\mathcal{J}$:X→X is a Banach graphic contraction or simply graphic contraction if $\mathcal{J}$ preserves edges of G, i.e.,*

$$\forall x\ , y \in X((x, y) \in E(G) \Rightarrow (\mathcal{J}x, \mathcal{J}y) \in E(G))$$

and $\mathcal{J}$ decreases weights of edges of G in the following way:

$$\exists \lambda \in [0, 1), \forall x\ , y \in X((x, y) \in E(G) \Rightarrow d_\theta(\mathcal{J}x, \mathcal{J}y) \leq \lambda d_\theta(x, y)).$$

Theorem 20. *Let (X,d_θ) be a complete extended b-metric space endowed with a graph G such that d_θ is continuous functional and let $\mathcal{J}$:X→X be a continuous self mapping such that*

Suppose the following assertions hold:
(i) $\forall x, y \in X$, $(x, y) \in E(G) \Longrightarrow (\mathcal{J}x, \mathcal{J}y) \in E(G)$,
(ii) there exists $x_0 \in X$ such that $(x_0, \mathcal{J}x_0) \in E(G)$,
(iii) there exists $\psi \in \Psi_\theta$ such that

$$d_\theta(\mathcal{J}x, \mathcal{J}y) \leq \psi\left(\max\left\{\frac{[1 + d_\theta(x, \mathcal{J}x)]\, d_\theta(y, \mathcal{J}y)}{1 + d_\theta(x, y)}, d_\theta(x, y)\right\}\right)$$

for all $x, y \in X$

(iv) $(x_n, x) \in E(G)$ for a sequence $\{x_n\}$ in X such that $(x_n, x_{n+1}) \in E(G)$ for all n$\in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$.

Then $\mathcal{J}$ has a fixed point.

Proof. Define $\alpha : X^2 \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 \text{ if } (x, y) \in E(G) \\ 0, \text{otherwise.} \end{cases}$$

First we prove that $\mathcal{J}$ is α-admissible mapping. Let $\alpha(x, y) \geq 1$, then $(x, y) \in E(G)$. From (i), we have $(\mathcal{J}x, \mathcal{J}y) \in E(G)$, that is, $\alpha(\mathcal{J}x, \mathcal{J}y) \geq 1$. Thus $\mathcal{J}$ is α-admissible mapping. From (ii), there exists $x_0 \in X$ such that $(x_0, \mathcal{J}x_0) \in E(G)$, that is, $\alpha(x_0, \mathcal{J}x_0) \geq 1$. If $(x, y) \in E(G)$, then $\alpha(x, y) = 1$. Thus from (iii), we have

$$\begin{aligned} \alpha(x, y)d_\theta(\mathcal{J}x, \mathcal{J}y) &= d_\theta(\mathcal{J}x, \mathcal{J}y) \\ &\leq \psi\left(\max\left\{\frac{[1 + d_\theta(x, \mathcal{J}x)]\, d_\theta(y, \mathcal{J}y)}{1 + d_\theta(x, y)}, d_\theta(x, y)\right\}\right) \end{aligned}$$

Condition (iv) implies condition (ii) of Theorem 17. Hence, all conditions of Theorem 17 are satisfied and $\mathcal{J}$ has a fixed point. □

5. Conclusions

In this paper, we have introduced some notions to study the existence and uniqueness of fixed points for locally α-ψ-contractive mapping and generalized $\alpha - \psi$ rational contraction in the context of complete extended b-metric spaces. The obtained results improved and unified some of the results in the literature. We also have provided an example to support the new theorem. Our results are new and significantly contribute to the existing literature in the fixed point theory. In this area, our future work will focus on studying the fixed points of multi-valued and fuzzy mappings in extended b-metric spaces, with fractional differential inclusion problems as applications.

Conflict of interest

The authors declares that they have no competing interests.

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