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Research article

Forbidden subgraphs in reduced power graphs of finite groups

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Abstract: Let G be a finite group. The reduced power graph of G is the undirected graph whose vertex set consists of all elements of G, and two distinct vertices x and y are adjacent if either $\langle x\rangle \subset \langle y\rangle$ or $\langle y\rangle \subset \langle x\rangle$. In this paper, we show that the reduced power graph of G is perfect and characterize all finite groups whose reduced power graphs are split graphs, cographs, chordal graphs, and threshold graphs. We also give complete classifications in the case of abelian groups, dihedral groups, and generalized quaternion groups.

Keywords: reduced power graph; split graph; cograph; chordal graph; threshold graph; abelian group
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1. Introduction

Every graph considered in our paper is undirected, finite, and simple that has no multiple edges and loops. Also, a digraph means a finite and directed graph with no loops or multiple arcs. If Λ is a graph (resp. digraph), then denote by $V(\Lambda)$ and $E(\Lambda)$ the vertex set and the edge set (resp. the arc set) of Λ, respectively. A sequence of vertices $[x_1, x_2, \ldots, x_t]$ in a digraph is called a *cycle* if the sequence satisfies (x_t, x_1) is an arc of this digraph and for every index $i \in \{2, \ldots, t\}$, there exists an arc (x_{i-1}, x_i). If a digraph has no cycles, then this digraph is called *acyclic*. We follow book [28] for undefined notation and terminology.

The study of the subgroups of a group was a main impetus for the development of lattice theory. Since every subgroup of a cyclic group is cyclic, the cyclic subgroups of a group form a downset, and hence a meet subsemilattice, in the lattice of all subgroups. This paper is concerned with an expansion the comparability graph of this semilattice. Also, graph associated with a group has valuable applications (see [22, 18]) and is related to automata theory (see [19, 20]). Let G be a group. The *power graph* of G, denoted by $\mathcal{P}(G)$, is a graph whose vertex set is G and two distinct vertices are connected by an edge between if and only if one is a power of the other. In 2000, Kelarev and Quinn [21] first introduced the concept of a directed power graph. In 2009, Chakrabarty *et al.* [10] first introduced the concept of an undirected power graph. In the last decade, the study on directed and undirected power graphs has been growing (see [7, 8, 13, 24]). More results and some open problems on power graphs can be found in [2]. In recent years, many authors generalized the definition of a power graph, see, for example the proper power graph [25], the enhanced power graph [1], and the quotient power graph [6].

In a power graph, in order to avoid the complexity in all edges, Rajkumar and Anitha [26] first introduced the *reduced power graph* of G, which is denoted by $\mathcal{P}_R(G)$ and is the graph whose vertex set is G, where two distinct elements x and y are connected by an edge between if and only if either $\langle x\rangle \subset \langle y\rangle$ or $\langle y\rangle \subset \langle x\rangle$. Actually, it is easy to see that $\mathcal{P}_R(G)$ can be obtained by deleting the edges $\{x, y\}$, where $x, y \in G$ with $\langle x\rangle = \langle y\rangle$. In [26], the authors studied the interplay between a reduced power graph and a power graph. In 2019, Anitha and Rajkumar [4] classified the finite groups whose reduced power graphs are toroidal and projective-planar. Recently, Ma [23] investigated the perfect codes and total perfect codes in proper reduced power graph over a finite group, where the proper reduced power graph of a group is obtained by deleting the identity in the reduced power graph of this group. More results on reduced power graphs can be found in [3, 27].

A number of important graph classes, including perfect graphs, cographs, chordal graphs, split graphs, and threshold graphs, can be defined either structurally or in terms of forbidden induced subgraphs. Forbidden subgraphs of power graphs of groups have been studied by Doostabadi *et al.* [12] and Cameron *et al.* [9]. In this paper, we show that $\mathcal{P}_R(G)$ is perfect for each finite group G, and characterize the finite groups G such that $\mathcal{P}_R(G)$ is a split graph, a cograph, a chordal graph, and a threshold graph. We also give complete classifications in the case of abelian groups, dihedral groups, and generalized quaternion groups.

2. Preliminaries

In this section, we introduce some notation, terminology, and results in group theory.

Every group considered in our paper is finite. In this paper, G always denotes a finite group, and e denotes the identity element of G. Let $g \in G$. The *order* of g in G, denoted by $o(g)$, is the size of the cyclic subgroup $\langle g\rangle$ generated by g. Denote by $\pi_e(G)$ the set of the orders of all elements of G. A cyclic subgroup of G is called a *maximal cyclic subgroup* if the cyclic subgroup is not a proper subgroup of some cyclic subgroup of G. Let $\mathcal{M}_G$ denote the set consisting of the maximal cyclic subgroups of G. Notice that the number of all maximal cyclic subgroups is 1 if and only if G is a cyclic group. Denote by $\mathbb{Z}_n$ the cyclic group of order n. A group G is called *nilpotent* if G has an upper central series that terminates with G. Notice that a finite nilpotent group is the direct product of its Sylow p-subgroups, and both p-groups and abelian groups are nilpotent.

For a positive integer n at least 3, the dihedral group of order $2n$ is denote by D_{2n}. A presentation of D_{2n} is

$$D_{2n} = \langle a, b : a^n = e, b^2 = e, a^{-1} = b^{-1}ab\rangle. \tag{2.1}$$

For a positive integer m at least 2, in [17], Johnson gave the definition of a generalized quaternion group that is denoted by Q_{4m} and has order $4m$. Namely,

$$Q_{4m} = \langle x, y : x^m = y^2, y^4 = x^{2m} = e, x^{-1} = y^{-1}xy\rangle. \tag{2.2}$$

Remark that

$$o(x^m) = 2, \quad o(x^i y) = 4 \text{ for any } i \le 1 \le m \tag{2.3}$$

and

$$\mathcal{M}_{Q_{4m}} = \{\langle xy\rangle, \ldots, \langle x^m y\rangle, \langle x\rangle\}, \quad x^m \in \bigcap_{M \in \mathcal{M}_{Q_{4m}}} M. \tag{2.4}$$

A *P-group* [11] is a group whose every non-trivial element is of prime order. For example, for every odd prime q, D_{2q} is a P-group. A *CP-group* [16] is a group whose every non-trivial element is of prime power order. For example, a P-group is also a CP-group. Moreover, for an odd prime q and a positive integer n, the dihedral group D_{2q^n} is a CP-group.

The following result will be used in our proofs of main results.

Lemma 2.1. ([15, Theorem 5.4.10 (ii)]) *Suppose that p is prime. A p-group has a unique subgroup of order p if and only if the p-group is either a cyclic group or a generalized quaternion group.*

3. Perfect graphs

It is similar to the definition of a directed power graph, one can define the directed reduced power graph of a group (cf. [26]). The directed reduced power graph of G, denoted by $\overrightarrow{\mathcal{P}_R}(G)$, is a digraph with vertex set G, and for two distinct $x, y \in G$, there is an arc from x to y if $\langle y\rangle \subset \langle x\rangle$.

Let Γ be a graph. If O is a digraph such that $V(\Gamma) = V(O)$ and for every edge $\{x, y\} \in E(\Gamma)$, either $(x, y) \in E(O)$ or $(y, x) \in E(O)$, then O is called an *orientation* of Γ. A orientation O of Γ is called *transitive* if $(x, y), (y, z) \in E(O)$ implies $(x, z) \in E(O)$. A graph is a *comparability graph* if its edges can be oriented in such a way, that the resulting digraph is transitive and acyclic. In this section we show that $\mathcal{P}_R(G)$ is a perfect graph.

Theorem 3.1. $\mathcal{P}_R(G)$ *is perfect.*

Proof. By the definitions of $\mathcal{P}_R(G)$ and $\overrightarrow{\mathcal{P}_R}(G)$, it is easy to see that $\overrightarrow{\mathcal{P}_R}(G)$ is a transitive orientation of $\mathcal{P}_R(G)$. In the following, we claim that $\overrightarrow{\mathcal{P}_R}(G)$ is acyclic. In fact, if $[x_1, x_2, \ldots, x_t]$ is a cycle of $\overrightarrow{\mathcal{P}_R}(G)$, then $(x_1, x_t), (x_t, x_1) \in E(\overrightarrow{\mathcal{P}_R}(G))$, this contradicts the definition of $\overrightarrow{\mathcal{P}_R}(G)$. Thus, our claim is valid. It follows that $\mathcal{P}_R(G)$ is a comparability graph. Also, since it was noted in [5, Chapter V, Theorem 17] that every comparability graph is perfect, the desired result follows. □

4. Split graphs

A graph is called *split* if its vertex set is the disjoint union of two subsets A and B so that A induces a complete graph and B induces an empty graph. In this section we characterize the groups whose reduced power graphs are split (see Theorem 4.4). In particular, we completely classify abelian groups, dihedral groups, and generalized quaternion groups for which their reduced power graphs are split.

We first prove some lemmas required for the proofs of our main results.

Lemma 4.1. *$\mathcal{P}_R(G)$ is C_5-free.*

Proof. Since the clique number of C_5 is not equal to the chromatic number of C_5, it follows that a perfect graph has no induced subgraph isomorphic to C_5. Now Theorem 3.1 implies that $\mathcal{P}_R(G)$ is C_5-free. □

Lemma 4.2. *$\mathcal{P}_R(G)$ is C_4-free if and only if, for any non-trivial element $g \in G$, $o(g)$ is equal to 4 or a prime.*

Proof. We will use proof by contradiction to obtain the direct implication. Suppose $o(g)$ is either 4 or a prime for every g different from the identity. Also suppose $x - a - y - b$ is an induced 4-cycle. Without loss of generality, we may assume $\langle a \rangle \subset \langle y \rangle$. Then we must also have $\langle b \rangle \subset \langle y \rangle$. If $o(y) = 4$, then a and b must both have order 2. Since a cyclic group has at most one element of order 2, this forces $a = b$, a contradiction. Thus $o(y) \neq 4$, so $o(y)$ is prime. This forces both a and b to be the identity, again a contradiction. Hence there is no induced C_4.

For the converse, suppose there is an element of composite order other than 4. Then there is an element g of order $pq > 4$ where both p and q are prime. Therefore $\langle g \rangle = \langle g^{-1} \rangle$ and $g \neq g^{-1}$. Thus g and g^{-1} are not adjacent in $\mathcal{P}_R(G)$. Moreover, $\langle g^p \rangle$ and $\langle g^q \rangle$ are properly contained in $\langle g \rangle$, so g^p and g^q are adjacent to g and g^{-1}.

If $p \neq q$, then g^p and g^q are distinct and non-adjacent, so $g - g^p - g^{-1} - g^q$ is an induced C_4. If $p = q$, then since $pq \neq 4$, we must have $p = q \geq 3$. Thus $g^{-p} \neq g^p$ with $\langle g^{-p} \rangle = \langle g^p \rangle$, so g^{-p} and g^p are distinct and non-adjacent. Hence $g - g^p - g^{-1} - g^{-p}$ is an induced C_4. □

Lemma 4.3. *Let $\langle a \rangle$ and $\langle b \rangle$ be two distinct cyclic subgroups of G with $\langle a' \rangle \subset \langle a \rangle$ and $\langle b' \rangle \subset \langle b \rangle$, where $e \neq a'$, $e \neq a'$, and $a' \neq b'$. Then the induced subgraph of $\mathcal{P}_R(G)$ by the set $\{a, a', b, b'\}$ is isomorphic to $2K_2$ if and only if $a' \notin \langle b \rangle$ and $b' \notin \langle a \rangle$.*

Proof. The necessity is clear so we just need to prove the sufficiency. Suppose that $a' \notin \langle b \rangle$ and $b' \notin \langle a \rangle$. If $\langle a \rangle \subset \langle b \rangle$ or $\langle b \rangle \subset \langle a \rangle$, then $a' \in \langle b \rangle$ or $b' \in \langle a \rangle$, a contradiction. As a result, a and b are non-adjacent in $\mathcal{P}_R(G)$. If a' and b' are adjacent in $\mathcal{P}_R(G)$, then $\langle a' \rangle \subset \langle b' \rangle$ or $\langle b' \rangle \subset \langle a' \rangle$, which implies that $a' \in \langle b \rangle$ or $b' \in \langle a \rangle$, a contradiction. We conclude that a' and b' are non-adjacent in $\mathcal{P}_R(G)$. Moreover, if a' and b are adjacent in $\mathcal{P}_R(G)$, then $\langle a' \rangle \subset \langle b \rangle$ or $\langle b \rangle \subset \langle a' \rangle$, which implies that $a' \in \langle b \rangle$ or $b' \in \langle a \rangle$, also a contradiction. Therefore, a' and b are non-adjacent in $\mathcal{P}_R(G)$. Similarly, we also have that a and b' are non-adjacent in $\mathcal{P}_R(G)$. It follows that the induced subgraph of $\mathcal{P}_R(G)$ by the set $\{a, a', b, b'\}$ is isomorphic to $2K_2$, as desired. □

In [14], the authors proved that a graph is a split graph if and only if the graph contains no an induced subgraph isomorphic to C_4, C_5 and $2K_2$. Thus, by Lemmas 4.1–4.3, we have the following result which characterizes the groups whose reduced power graphs are split.

Theorem 4.4. *$\mathcal{P}_R(G)$ is split if and only if G satisfies the following two conditions:*
(a) *$\pi_e(G) \subseteq \{1, 4\} \cup \mathbf{P}$, where $\mathbf{P}$ is the set of all primes;*
(b) *If $\{x_1, x_2, \ldots, x_n\}$ is the set of all elements of order 4 in G, then $|\cap_{i=1}^{n} \langle x_i \rangle| \geq 2$.*

Example 4.5. *$\mathcal{P}_R(G)$ is split for each P-group G.*

For a prime p, the elementary abelian p-group of order p^n is denoted by $\mathbb{Z}_p^n$, that is, $\mathbb{Z}_p^n = \underbrace{\mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{n}$.

Theorem 4.6. *Let A be an abelian group. Then $\mathcal{P}_R(A)$ is split if and only if A is isomorphic to one of the following groups:*
(a) $\mathbb{Z}_p^n$*, where p is a prime and n is a positive integer;*
(b) $\mathbb{Z}_2^n \times \mathbb{Z}_4$*, where n is a positive integer;*
(c) $\mathbb{Z}_4$.

Proof. Clearly, if A is isomorphic to one of (a) and (c), then $\mathcal{P}_R(A)$ is split by Theorem 4.4. Now let $A = \mathbb{Z}_2^n \times \mathbb{Z}_4$ for some positive integer n. Then $\pi_e(A) = \{1, 2, 4\}$ and for every element $g \in A$ of order 4, g is either $(g_1, g_2, \ldots, g_n, 1)$ or $(g_1', g_2', \ldots, g_n', 3)$, where $g_i, g_i' \in \{0, 1\}$ for each $1 \le i \le n$. It follows that $2g = (0, 0, \ldots, 0, 2)$, which implies the intersection of all cyclic subgroups of order 4 in A has size 2. As a result, Theorem 4.4 implies that $\mathcal{P}_R(A)$ is split.

For the converse, suppose that $\mathcal{P}_R(A)$ is split. Notice that if A has an element x of order p and an element y of order q where p, q are distinct primes, then xy has order pq. Since $\pi_e(G) \subseteq \{1, 4\} \cup \mathbf{P}$ by Theorem 4.4, we conclude that A is a p-group. If A has no elements of order 4, then A is elementary abelian, and so A is isomorphic to $\mathbb{Z}_p^n$, as desired. In the following, we assume that A has elements of order 4. Then A is isomorphic to one of $\mathbb{Z}_2^n \times \mathbb{Z}_4^m$ and $\mathbb{Z}_4^m$, where m, n are two positive integers. It suffices to show that $A \ncong \mathbb{Z}_4^m$ and $A \ncong \mathbb{Z}_2^n \times \mathbb{Z}_4^m$ for some $m \ge 2$. We first prove that $A \ncong \mathbb{Z}_4^m$ for some $m \ge 2$. Suppose for a contradiction that $A \cong \mathbb{Z}_4^m$ for some $m \ge 2$. Then in A, both $g_1 = (0, 0, \ldots, 0, 1)$ and $g_2 = (1, 0, 0, \ldots, 0)$ are elements of order 4, but $2g_1 \ne 2g_2$. It follows that $|\langle g_1 \rangle \cap \langle g_2 \rangle| = 1$, contrary to (b) of Theorem 4.4. Similarly, we also can conclude that $A \ncong \mathbb{Z}_2^n \times \mathbb{Z}_4^m$ for some $m \ge 2$. □

Combining (2.3), (2.4) and Theorem 4.4, we have the following corollary.

Corollary 4.7. *Let D_{2n} and Q_{4m} be the dihedral group and the generalized quaternion group as presented in* (2.1) *and* (2.2)*, respectively. Then $\mathcal{P}_R(D_{2n})$ is split if and only if n is either a prime or* 4*, and $\mathcal{P}_R(Q_{4m})$ is split if and only if $m = 2$.*

5. Chordal graphs

A graph is called *chordal* if this graph contains no induced cycles of length greater than 3. In other words, a chordal graph is a graph in which every cycle of length at least 4 has a chord. Namely, if a chordal graph has an induced cycle, then the induced cycle is isomorphic to C_3.

In this section we characterize the groups whose reduced power graphs are chordal (see Theorem 5.1). In particular, we also classify abelian groups, dihedral groups, and generalized quaternion groups for which their reduced power graphs are chordal.

Theorem 5.1. *The following are equivalent for a group G:*
(I) $\mathcal{P}_R(G)$ *is chordal;*
(II) $\mathcal{P}_R(G)$ *is C_4-free;*
(III) $\pi_e(G) \subseteq \{1, 4\} \cup \mathbf{P}$.

Proof. By Lemma 4.2, (II) and (III) are equivalent. Thus, we only need to show that (I) and (II) are equivalent. Notice that it is obvious that (I) implies (II). It suffices to show that (II) implies (I).

Suppose now that $\mathcal{P}_R(G)$ is C_4-free. Then $\pi_e(G) \subseteq \{1, 4\} \cup \mathbf{P}$. Suppose for a contradiction that $\mathcal{P}_R(G)$ has an induced cycle of length greater than 4. It follows that $\mathcal{P}_R(G)$ has a four-vertex induced path, say, (x, y, z, w), where $\{x, y\}$, $\{y, z\}$, $\{z, w\} \in E(\mathcal{P}_R(G))$. As a result, we have that $\langle y\rangle \subset \langle z\rangle$ or $\langle z\rangle \subset \langle y\rangle$. Notice that every vertex of (x, y, z, w) is non-identity. Since $\pi_e(G) \subseteq \{1, 4\} \cup \mathbf{P}$, we deduce that $\{o(y), o(z)\} = \{2, 4\}$. Without loss of generality, we may set $o(y) = 2$ and $o(z) = 4$. Then $\langle w\rangle \subset \langle z\rangle$, and so $w, y \in \langle z\rangle$ with $o(w) = o(y) = 2$. It means $y = w$, a contradiction. We conclude that $\mathcal{P}_R(G)$ has no induced cycles of length greater than 4. Also, since $\mathcal{P}_R(G)$ is C_4-free, we deduce that $\mathcal{P}_R(G)$ is chordal, as required. □

The next corollary is obtained by applying Theorem 5.1 to P-groups and abelian groups.

Corollary 5.2. (1) *$\mathcal{P}_R(G)$ is chordal for each P-group G.*
(2) *Let A be an abelian group. Then $\mathcal{P}_R(A)$ is chordal if and only if A is isomorphic to one of the following:*

$$\mathbb{Z}_p^m,\ \ \mathbb{Z}_4^m,\ \ \mathbb{Z}_2^m \times \mathbb{Z}_4^n,$$

where p is a prime and $m, n \geq 1$.

By (2.3), (2.4) and Theorem 5.1, we end this section by determining all chordal reduced power graphs for dihedral groups and generalized quaternion groups.

Corollary 5.3. *Let D_{2n} and Q_{4m} be the dihedral group and the generalized quaternion group as presented in* (2.1) *and* (2.2)*, respectively. Then $\mathcal{P}_R(D_{2n})$ is chordal if and only if n is either a prime or* 4*, and $\mathcal{P}_R(Q_{4m})$ is chordal if and only if $m = 2$.*

6. Cographs

A graph is called a *cograph* if this graph has no induced subgraph isomorphic to the four-vertex path P_4. In this section we characterize the groups whose reduced power graphs are cographs (see Theorem 6.1). In particular, we also classify nilpotent groups, dihedral groups, and generalized quaternion groups for which their reduced power graphs are cographs.

For group G, let

$$\mathcal{S}(G) = \{g \in G : o(g) = pq, \text{where } p, q \text{ are primes}\}.$$

Theorem 6.1. *$\mathcal{P}_R(G)$ is a cograph if and only if G satisfies the following:*
(a) *For any non-trivial element $g \in G$, $o(g)$ is either a prime power or a product of two distinct primes;*
(b) *Let $a \in \mathcal{S}(G)$, and let $b \in G$ be an element whose order is a product of two distinct primes. If $\langle a\rangle \neq \langle b\rangle$, then $|\langle a\rangle \cap \langle b\rangle| = 1$.*

Proof. We first prove the sufficiency. Suppose that both (a) and (b) hold for a given group G. It suffices to show that $\mathcal{P}_R(G)$ has no induced subgraph isomorphic to P_4. Assume, to the contrary, that $\mathcal{P}_R(G)$ has an induced subgraph isomorphic to P_4, say, (x, y, z, w) where $\{x, y\}, \{y, z\}, \{z, w\} \in E(\mathcal{P}_R(G))$. Notice that every of $\{x, y, z, w\}$ is not the identity of G. We first claim that one of y and z must have order pq, where p, q are two distinct primes. In fact, if both y and z are prime powers, without loss of generality, we say that $\langle y\rangle \subset \langle z\rangle$ and $o(z) = p^t$ for some prime p and positive integer t at least 2, then it follows that $\langle w\rangle \subset \langle z\rangle$. Therefore $\langle w\rangle = \langle y\rangle$ since $\{w, y\} \notin E(\mathcal{P}_R(G))$. This means that w and x are adjacent in $\mathcal{P}_R(G)$, which is impossible. Thus, our claim is valid.

Now, without loss of generality, we say that y has order pq, where p, q are two distinct primes. Then $\{o(x), o(z)\} = \{p, q\}$. In fact, without loss of generality, we may let $o(x) = p$ and $o(z) = q$. Since $\{z, w\} \in E(\mathcal{P}_R(G))$, we have $\langle z \rangle \subset \langle w \rangle$. It follows that $o(w) = q^l$ or qr, where l is a positive integer at least 2 and r is a prime. If $o(w) = q^l$, then taking $w' \in \langle w \rangle$ with $o(w') = q^2$, we have $w' \in \mathcal{S}(G)$ and so $|\langle y \rangle \cap \langle w' \rangle| = q$, contrary to (b). We conclude that $o(w) = qr$, and so $w \in \mathcal{S}(G)$. If $\langle y \rangle = \langle w \rangle$, then x is adjacent to w, which is impossible. As a result, we have $\langle y \rangle \neq \langle w \rangle$, which implies that $|\langle y \rangle \cap \langle w \rangle| = q$, contrary to (b). This contradiction implies that $\mathcal{P}_R(G)$ has no induced subgraph isomorphic to P_4, and so $\mathcal{P}_R(G)$ is a cograph.

We next prove the necessity. Suppose that $\mathcal{P}_R(G)$ is a cograph. If G has an element a of order p^2q where p, q are two distinct primes, then the subgraph of $\mathcal{P}_R(G)$ induced by the vertices $a^q, a^{pq}, a^p, a^{p^2}$ is isomorphic to P_4, which is impossible. If G has an element b of order pqr where p, q, r are three distinct primes, then the subgraph of $\mathcal{P}_R(G)$ induced by the four vertices b^{qr}, b^r, b^{pr}, and a^p is isomorphic to P_4, also a contradiction. We conclude that (a) holds. In the following, we prove (b). Let $u \in \mathcal{S}(G)$, and let $v \in G$ be an element whose order is a product of two distinct primes. Also, let $\langle u \rangle \neq \langle v \rangle$. Suppose for a contradiction that $|\langle u \rangle \cap \langle v \rangle| > 1$. Let $\langle u \rangle \cap \langle v \rangle = \langle w \rangle$. Then $o(w)$ is a prime. Now set $w' \in \langle v \rangle$ with $o(w')o(w) = o(v)$. Since $\langle w \rangle \langle w' \rangle = \langle v \rangle$, it follows that $w' \notin \langle u \rangle$ as $\langle u \rangle \neq \langle v \rangle$. It follows that the subgraph of $\mathcal{P}_R(G)$ induced by the four vertices w', v, w, and u is isomorphic to P_4, a contradiction. Therefore, (b) holds. □

The next corollary is obtained by applying Theorem 6.1 to CP-groups.

Corollary 6.2. *$\mathcal{P}_R(G)$ is a cograph for each CP-group G.*

Applying Theorem 6.1 to nilpotent groups, we next classify all nilpotent groups whose reduced power graphs are cographs. Note that a finite nilpotent group is the direct product of its Sylow p-subgroups.

Theorem 6.3. *Let G be a nilpotent group. Then $\mathcal{P}_R(G)$ is a cograph if and only if G is either a p-group or $\mathbb{Z}_{pq}$.*

Proof. By Corollary 6.2, $\mathcal{P}_R(G)$ is a cograph for a p-group G. Also, by Theorem 6.1, it is straightforward that $\mathcal{P}_R(\mathbb{Z}_{pq})$ is a cograph. Thus, we only need to prove the necessity. Suppose that $\mathcal{P}_R(G)$ is a cograph. Note that if $x, y \in G$ with $o(x) = p^m$ and $o(y) = q^n$ where p, q are distinct primes, then xy has order $p^m q^n$. Applying Theorem 6.1 to nilpotent groups, we conclude that $|G|$ has at most two distinct prime divisors. If $|G|$ has a prime divisor, then G is a p-group, as desired. In the following, we assume that $|G|$ has precisely two distinct prime divisors, say, p and q. It follows that

$$\pi_e(G) = \{1, p, q, pq\}. \tag{6.1}$$

Let P and Q be Sylow p-subgroup and Sylow q-subgroup of G, respectively. Let $\langle c \rangle$ be a subgroup of order q in Q. We now claim that P has a unique subgroup of order p. In fact, if P has two distinct subgroups of order p, say, $\langle a \rangle$ and $\langle b \rangle$, since $\langle a, c \rangle = \langle ac \rangle$, $\langle b, c \rangle = \langle bc \rangle$, and $o(ac) = o(bc) = pq$, it follows that the subgraph of $\mathcal{P}_R(G)$ induced by the four vertices a, ac, c, and bc is isomorphic to P_4, this contradicts that $\mathcal{P}_R(G)$ is a cograph. Thus, our claim is valid, that is, P has a unique subgroup of order p. Combining Lemma 2.1, (2.3) and (6.1), we conclude that P is isomorphic to $\mathbb{Z}_p$. Similarly, we also can obtain that Q is isomorphic to $\mathbb{Z}_q$. It follows that G is isomorphic to $\mathbb{Z}_{pq}$, as desired. □

Combining Theorem 6.1 and Corollary 6.2, we can obtain easily the following result.

Corollary 6.4. *Let D_{2n} be the dihedral group as presented in* (2.1)*. Then $\mathcal{P}_R(D_{2n})$ is a cograph if and only if n is either a prime power or a product of two distinct primes.*

We conclude the section by the following corollary to classify all generalized quaternion groups whose reduced power graphs are cographs.

Corollary 6.5. *Let Q_{4m} be the generalized quaternion group as presented in* (2.2)*. Then $\mathcal{P}_R(Q_{4m})$ is a cograph if and only if m is a power of* 2*.*

Proof. Clearly, if m is a power of 2, then Q_{4m} is a 2-group by (2.3), and it follows from Corollary 6.2 that $\mathcal{P}_R(Q_{4m})$ is a cograph, as desired.

Conversely, suppose that $\mathcal{P}_R(Q_{4m})$ is a cograph. By (2.3) and Theorem 6.1, we have that $2m$ is either a prime power or a product of two distinct primes. Now suppose for a contradiction that $2m$ is a product of two distinct primes. Then $m = q$ for some odd prime q. We conclude that $o(x) = 2q$, $o(y) = 4$, and $\langle x\rangle \cap \langle y\rangle = \langle x^q\rangle$ by (2.3) and (2.4), contrary to the condition (b) of Theorem 6.1. Thus, we deduce that $2m$ is a prime power, that is, m is a power of 2, as required. □

7. Threshold graphs

A graph is called a *threshold graph* if the graph has no induced subgraph isomorphic to P_4, K_4, or $2K_2$. In this section we characterize the groups whose reduced power graphs are threshold (see Theorem 7.3). In particular, we also classify abelian groups, dihedral groups, and generalized quaternion groups for which their reduced power graphs are threshold.

Clearly, every threshold graph is also a cograph. Thus, by Theorem 6.1, we first have the following result.

Lemma 7.1. *If $\mathcal{P}_R(G)$ is a threshold graph, then the following hold:*
(a) *For any non-trivial element $g \in G$, $o(g)$ is either a prime power or a product of two distinct primes;*
(b) *Let $a \in \mathcal{S}(G)$, and let $b \in G$ be an element whose order is a product of two distinct primes. If $\langle a\rangle \neq \langle b\rangle$, then $|\langle a\rangle \cap \langle b\rangle| = 1$.*

Given a positive integer n, let $\Omega(n)$ denote the number of all prime divisors of n counted with multiplicity. For example, $\Omega(2^3) = \Omega(30) = 3$.

Lemma 7.2. ([27, Corollary 2.1]) *Let $n \in \pi_e(G)$ be such that $\Omega(n)$ is maximum. Then the clique number of $\mathcal{P}_R(G)$ is $\Omega(n) + 1$.*

Theorem 7.3. *$\mathcal{P}_R(G)$ is a threshold graph if and only if G is isomorphic to one of the following groups:*
(I) *a P-group;*
(II) *a group G which has a unique cyclic subgroup of order pq and satisfies $\pi_e(G) \subseteq \{1, pq\} \cup \mathbf{P}$, where p, q are two distinct primes;*
(III) *a group G with $\{p^2\} \subseteq \pi_e(G) \subseteq \{1, p^2\} \cup \mathbf{P}$ where p is a prime, and the intersection of each two distinct cyclic subgroups of p^2 has size p.*

Proof. Clearly, $\mathcal{P}_R(G)$ is a star if G is a P-group. Thus, it is easy to see that if G is a P-group, then $\mathcal{P}_R(G)$ is a threshold graph. Now let G be a group satisfying (II). Then from Theorem 6.1, it follows that $\mathcal{P}_R(G)$ is P_4-free. Also, Lemma 7.2 implies that the clique number of $\mathcal{P}_R(G)$ is 3, and so $\mathcal{P}_R(G)$ is

K_4-free. Finally, it is easy to see that $\mathcal{P}_R(G)$ is $2K_2$-free from Lemma 4.3. We conclude that $\mathcal{P}_R(G)$ is a threshold graph. Similarly, we can obtain that $\mathcal{P}_R(G)$ is a threshold graph if G is a group satisfying (III).

For the converse, suppose that $\mathcal{P}_R(G)$ is a threshold graph. Then G satisfies the two conditions (a) and (b) of Lemma 7.1. Since $\mathcal{P}_R(G)$ is K_4-free, the clique number of $\mathcal{P}_R(G)$ is at most 3. It follows from Lemma 7.2 that for any non-trivial element $g \in G$, if $o(g)$ is not a prime, then $o(g)$ is either a square of some prime or a product of two distinct primes. If every element of G has prime order, then G is a P-group, as desired.

Suppose now that G has an element x of order pq, where p, q are distinct primes. Let $y \in G\setminus\{x\}$ such that $o(y)$ is either a square of some prime or a product of two distinct primes. Assume, to the contrary, that $\langle x\rangle \neq \langle y\rangle$. By (b) of Lemma 7.1, we have $|\langle x\rangle \cap \langle y\rangle| = 1$. Let $y' \in \langle y\rangle$ such that $o(y')$ is a prime. Then Lemma 4.3 implies that the induced subgraph of $\mathcal{P}_R(G)$ by the set $\{x, x^p, y, y'\}$ is isomorphic to $2K_2$, a contradiction. We conclude that $\langle x\rangle = \langle y\rangle$, and so G has a unique cyclic subgroup $\langle x\rangle$ of order pq and has no element whose order is a square of some prime. Therefore, G belongs to a group in (II), as desired.

Suppose that G has no element whose order is a product of two distinct primes, and has an element x with $o(x) = p^2$ where p is a prime. If G has an element y with $o(y) = q^2$ where $q \neq p$ is a prime, then it follows from Lemma 4.3 that the induced subgraph of $\mathcal{P}_R(G)$ by $\{x, x^p, y, y^q\}$ is isomorphic to $2K_2$, which is impossible. Thus, if $z \in G \setminus \{x\}$ such that $o(z)$ is not a prime, then $o(z) = p^2$. Assume, to the contrary, that $|\langle x\rangle \cap \langle z\rangle| = 1$. Then by Lemma 4.3, the induced subgraph of $\mathcal{P}_R(G)$ by $\{x, x^p, z, z^p\}$ is isomorphic to $2K_2$, a contradiction. It follows that $|\langle x\rangle \cap \langle z\rangle| \geq p$, which implies that G is a group in (III), as desired. □

Applying Theorem 7.3 to abelian groups, we have the following result which classifies all threshold reduced power graphs for abelian groups.

Corollary 7.4. *Let A be an abelian group. Then $\mathcal{P}_R(A)$ is a threshold graph if and only if A is isomorphic to one of the following groups:*
(a) $\mathbb{Z}_p^n$, *where p is a prime and n is a positive integer;*
(b) $\mathbb{Z}_p^n \times \mathbb{Z}_{p^2}$, *where p is a prime and n is a positive integer;*
(c) $\mathbb{Z}_{p^2}$, *where p is a prime;*
(d) $\mathbb{Z}_{pq}$, *where p, q are distinct primes.*

We conclude this paper by determining all threshold reduced power graphs for dihedral groups and generalized quaternion groups, which can be obtained easily from (2.3), (2.4) and Theorem 7.3.

Corollary 7.5. *Let D_{2n} and Q_{4m} be the dihedral group and the generalized quaternion group as presented in* (2.1) *and* (2.2)*, respectively. Then $\mathcal{P}_R(D_{2n})$ is threshold if and only if $n = p$, p^2, or pq, where p, q are distinct primes. Moreover, $\mathcal{P}_R(Q_{4m})$ is threshold if and only if $m = 2$.*

8. Conclusions

In this paper we showed that the reduced power graph of a finite group is perfect and characterized all finite groups whose reduced power graphs are split graphs, cographs, chordal graphs, and threshold graphs. We also gave complete classifications in the case of abelian groups, dihedral groups, and generalized quaternion groups.

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Conflict of interest

The authors declared that they have no conflicts of interest to this work.

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