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Research article

Study of time fractional order problems with proportional delay and controllability term via fixed point approach

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Abstract: In the current manuscript, we are tying to study one of the important class of differential equations known is evolution equations. Here, we considered the problem under controllability term and with proportional delay. Before going to numerical or analytical solution it is important to check the existence and uniqueness of the solution. So, we will consider our problem for qualitative theory using fixed point theorems of Banach's and Krasnoselskii's type. For numerical solution the stability is important, hence the problem is also studied for Ulam-Hyer's type stability. At the end an example is constructed to ensure the establish results.

Keywords: time fractional order problems with proportional delay; controllability term; fixed point approach
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1. Introduction

The concept of differential equations (DEs) of any real order is a progressive area of consideration. Currently, the subject has been shown that, it can explain large number of problems in various branches of science such as physics, chemistry, biology, information processing system networking etc. DEs of any order can explain complex problems like memory and inherited properties of materials and processes. Thus, we can claim DEs of real order have widespread developments and significant results have reported in the ongoing time [1–5].

In the fields of dynamical systems and control theory, a fractional-order system is a dynamical system that can be modeled by a fractional differential equation containing derivatives of non-integer order. Such systems are said to have fractional dynamics. Derivatives and integrals of fractional orders are used to describe objects that can be characterized by power-law non-locality, power-law long-range dependence or fractal properties. Fractional-order systems are useful in studying the anomalous behavior of dynamical systems in physics, electrochemistry, biology, visco-elasticity and chaotic systems. Different important work can be found regarding fractional system in [6–9].

The area which got considerable attention from scholar's is the theory of existence of solution of fractional DEs. The stated field has very rich literature for ordinary DEs. However, for fractional order, the area is in progress and need further study. Different researchers studied DEs of arbitrary order in different aspects; one may see [10, 11] and references therein.

Another area which has currently allured more attention is the stability theory of DEs. Among various form of stability, Ulam-Hyer's stability is very important and interesting. In 1940, Ulam [12] introduce the stated stability, which was further extended by Hyer's [13]. Later on Rassias generalized the stability to Ulam-Hyer's Rassias stability [14]. Obloza for the first time studied such type of stability for DEs. Now tremendous work can be found about Ulam-Hyer's stability and its various form in the literature [15–17].

The important type of DEs in which delay parameter is involved is known is pantograph equations. Such type of DEs was found for the collection of electric current from overhead wire of electric vehicle [18]. The stated type of DEs has plenty of applications in different scientific disciplines for detail see [19–21] and references there in. Evolution equations are special type of DEs which explain laws of differential for development of system. Further, it can also be treated the behavior of positive quantities, like concentration of species, distribution temperature etc [22, 23]. Since the application of pantograph equation is very wide, so it is important to study evolution equations under pantograph equations. Here we mention that, Balachandran, et al. in [24] studied the following evolution equation with impulsive conditions for existence and uniqueness with the help of fixed point theory:

$$\begin{cases} {}^{c}\mathrm{D}_{0^+}^{\omega}\mathcal{U}(v) = \mathbf{A}(v,\mathcal{U}(v))\mathcal{U}(v) + \Phi\Big(v,\mathcal{U}(v),\int_0^v \Psi(v,x,\mathcal{U}(x))\mathrm{d}x\Big), \quad v \in \mathbf{I} = [0,\theta], \\ \quad \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{1.1}$$

where $0 < \omega \leq 1$, $\mathbf{A}(v,\mathcal{U}(v))$ is a bounded linear operator on Banach space $\mathrm{C}[0,\theta]$ and $\Phi \in \mathrm{C}[\mathbf{I}\times\mathbb{R}\times\mathbb{R},\mathbb{R}]$, $\Psi \in \mathrm{C}[\mathbf{I}\times\mathbf{I}\times\mathbb{R},\mathbb{R}]$.

In the current work, we consider the above controllability evolution problem under pantograph equation, i.e., with proportional delay to study existence and uniqueness and Ulam-Hyer's type stability of the proposed model:

$$\begin{cases} {}^{c}\mathrm{D}_{0^+}^{\omega}\mathcal{U}(v) = \mathbf{A}(v,\mathcal{U}(v))\mathcal{U}(v) + \Phi(v,\mathcal{U}(v),\mathbf{Z}\mathcal{U}(\lambda v)), \quad v \in \mathbf{I} = [0,\theta], \\ \quad \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{1.2}$$

where $\mathbf{Z}\mathcal{U}(\lambda v) = \int_0^v \Psi(v,x,\mathcal{U}(\lambda x))\mathrm{d}x$, is the controllability term and $0 < \omega \leq 1$, $0 < \lambda < 1$, $\mathbf{A}(v,\mathcal{U}(v))$ is a bounded linear operator and $\Phi \in \mathrm{C}[\mathbf{I}\times\mathbb{R}\times\mathbb{R},\mathbb{R}]$, $\Psi \in \mathrm{C}[\mathbf{I}\times\mathbf{I}\times\mathbb{R},\mathbb{R}]$.

From the above discussion, we are going to study problem (1.2) under proportional delay term for existence and uniqueness using Krasnoselskii's and Banach's fixed point theorems. In the same line the model should also be consider for different Ulam-Hyer's type stability by using tools of nonlinear analysis.

2. Preliminaries

Here, we provide some fundamental materials which are key to our study.

Definition 1. [5] Let $\mathcal{U} \in \mathrm{L}^1(\mathbf{I}, \mathbb{R})$ then fractional order integral is define as

$$\mathrm{I}_{0+}^{\omega}\mathcal{U}(v) = \frac{1}{\Gamma(\omega)}\int_0^v (v-x)^{\omega-1}\mathcal{U}(x)\,dx. \tag{2.1}$$

Definition 2. [5] For a function $\mathcal{U}$ Caputo fractional derivative on $\mathbf{I}$ is define as

$$^{\mathrm{c}}\mathrm{D}_{0+}^{\omega}\mathcal{U}(\mathrm{v}) = \frac{1}{\Gamma(\mathrm{v}-\omega)}\int_0^{\mathrm{v}} (\mathrm{v}-\mathrm{x})^{\mathrm{v}-\omega-1}\mathcal{U}^{(\mathrm{v})}(\mathrm{x})\,\mathrm{dx}, \tag{2.2}$$

here $v = [\omega] + 1$ and $[\omega]$ is the integer part of ω.

Lemma 1. *[5] The fractional DEs* $^{\mathrm{c}}\mathrm{D}_{0+}^{\omega}\mathcal{U}(v) = y(v)$, $\mathrm{n} - 1 < \omega \leq \mathrm{n}$, *has the solution as:*

$$\mathcal{U}(v) = \mathrm{I}_{+0}^{\omega}[y(v)] + \sum_{i=0}^{v-1}\chi_i v^i, \tag{2.3}$$

where $\chi_i \in \mathbb{R}$, $\mathrm{i} = 0, 1, 2, ..., \mathrm{n} - 1$.

Let $\mathbf{X} = \mathrm{C}[\mathbf{I}, \mathbb{R}]$ be Banach space and define norm on $\mathbf{X}$ as $\|\mathcal{U}\| = \sup\{|\mathcal{U}(v)|, \ \ v \in \mathbf{I}\}$.

Theorem 1. *[25]* (**Krasnoselskii′s fixedpoint theorem**) : *Let* $\mathbf{E} \subset X$ *be closed, convex non empty subset of* X *and there exist two operators* $\mathbf{F}$ *and* $\mathbf{G}$ *such that*

(1) $\mathbf{F}\mathcal{U} + \mathbf{G}\mathcal{U} \in \mathbf{E}\ \forall\ \mathcal{U} \in \mathbf{E}$.
(2) *Map* $\mathbf{F}$ *to be contraction and map* $\mathbf{G}$ *to be continuous and compact.*

Then there exist at least one solution $\mathcal{U} \in \mathbf{E}$*, such that*

$$\mathbf{F}\mathcal{U} + \mathbf{G}\mathcal{U} = \mathcal{U}.$$

Theorem 2. *[26] Let B be a compact set in $\mathbb{R}^n$ where $n \geq 1$, then a set $S \subset C(B)$ is relatively compact in $C(B)$ if and only if the functions in S are uniformly bounded and equi-continuous on B.*

3. Main results

In the current section, we are trying to analysis on the existence of at least one solution to the proposed problem. The suggested approach is based on "Banach's and Krasnoselskii's fixed point theorems". With the help of Lemma 1 problem (1.2) can be transform to the integral equation as:

$$\mathcal{U}(v) = \mathcal{U}_0 + \int_0^v \frac{(v-x)^{\omega-1}}{\Gamma(\omega)}\mathbf{A}(x, \mathcal{U}(x))\mathcal{U}(x)\mathrm{d}x + \int_0^v \frac{(v-x)^{\omega-1}}{\Gamma(\omega)}\Phi(x, \mathcal{U}(x), \mathbf{Z}\mathcal{U}(\lambda x))\mathrm{d}x. \tag{3.1}$$

Before going to the main results, we need some assumption as:

(φ_1) There exist a constant $L_{\mathbf{A}} \geq 0$, such that

$$\|\mathbf{A}(v, \bar{\mathcal{U}}) - \mathbf{A}(v, \mathcal{U})\| \leq L_{\mathbf{A}}\|\bar{\mathcal{U}} - \mathcal{U}\|,$$

$\forall\ (v, \mathcal{U}),\ (v, \bar{\mathcal{U}}) \in \mathbf{I} \times \mathbb{R}$.

(φ_2) There exist $c,\ d \geq 0$, such that

$$\|\Phi(v, \mathcal{U}, \bar{\mathcal{U}})\| \leq c\left[\|\mathcal{U}\| + \|\bar{\mathcal{U}}\|\right] + d,$$

$\forall\ (v, \mathcal{U}, \bar{\mathcal{U}}) \in \mathbf{I} \times \mathbb{R} \times \mathbb{R}$.

(φ_3) There exist $e,\ p \geq 0$, such that

$$\|\Psi(v, x, \mathcal{U})\| \leq e\|\mathcal{U}\| + p,$$

$\forall (v, x, \mathcal{U}) \in \mathbf{I} \times \mathbf{I} \times \mathbb{R}$.

Let us split the integral Eq (3.1) to the following operators as:

$$\mathbf{F}\mathcal{U} = \mathcal{U}_0 + \int_0^v \frac{(v - x)^{\omega - 1}}{\Gamma(\omega)} \mathbf{A}(x, \mathcal{U}(x))\mathcal{U}(x)\mathrm{d}x, \tag{3.2}$$

$$\mathbf{G}\mathcal{U} = \int_0^v \frac{(v - x)^{\omega - 1}}{\Gamma(\omega)} \Phi(x, \mathcal{U}(x), \mathbf{Z}\mathcal{U}(\lambda x))\mathrm{d}x, \tag{3.3}$$

and

$$\mathbf{H}\mathcal{U} = \mathbf{F}\mathcal{U} + \mathbf{G}\mathcal{U}. \tag{3.4}$$

Since $\mathbf{A}(v, \mathcal{U})$ is bounded so there exist $\mathcal{K}' \geq 0$ such that $|\mathbf{A}(v, \mathcal{U})| \leq \mathcal{K}'$.

Theorem 3. *If the conditions* (φ_1) − (φ_3) *hold, then the equation* $\mathbf{H}\mathcal{U} = \mathbf{F}\mathcal{U} + \mathbf{G}\mathcal{U}$ *has at least one solution if* $\frac{(\mathcal{K}' + \mathrm{s}L_{\mathbf{A}})\theta^{\omega}}{\Gamma(\omega+1)} < 1$.

Proof. We will prove our main result by following several steps.

StepI : First, we need **F** is contraction. Let $\mathbf{E} = \{\mathcal{U} \in \mathbf{X} : \|\mathcal{U}\| \leq s \text{ for some } s > 0\}$. Clearly **F** is continuous. Let $\mathcal{U}, \bar{\mathcal{U}} \in \mathbf{E}$, from (3.2), one has

$$\begin{aligned}
\|\mathbf{F}\bar{\mathcal{U}} - \mathbf{F}\mathcal{U}\| &\leq \sup_{v \in \mathbf{I}} \left\{ \int_0^v \left| \frac{(v - x)^{\omega - 1}}{\Gamma(\omega)} \right| |\mathbf{A}(v, \bar{\mathcal{U}})\bar{\mathcal{U}} - \mathbf{A}(v, \mathcal{U})\mathcal{U}| \mathrm{d}x \right\}, \\
&\leq \sup_{v \in \mathbf{I}} \left\{ \int_0^v \left| \frac{(v - x)^{\omega - 1}}{\Gamma(\omega)} \right| \Big(|\mathbf{A}(v, \bar{\mathcal{U}})||\bar{\mathcal{U}} - \mathcal{U}| + |\mathbf{A}(v, \bar{\mathcal{U}}) - \mathbf{A}(v, \mathcal{U})||\mathcal{U}| \Big) \mathrm{d}x \right\}, \\
&\leq (\mathcal{K}' + \mathrm{s}L_{\mathbf{A}})\|\bar{\mathcal{U}} - \mathcal{U}\| \sup_{v \in \mathbf{I}} \left\{ \int_0^v \left| \frac{(v - x)^{\omega - 1}}{\Gamma(\omega)} \right| \mathrm{d}x \right\}, \\
&\leq L \left\| \bar{\mathcal{U}} - \mathcal{U} \right\|, \text{ where } L = \frac{(\mathcal{K}' + \mathrm{s}L_{\mathbf{A}})\theta^{\omega}}{\Gamma(\omega + 1)}.
\end{aligned}$$

Thus **F** is contraction.

StepII : **G** is bounded. Let for any $\mathcal{U} \in \mathbf{E}$, we have

$$\begin{aligned}
\|\mathbf{G}\mathcal{U}\| &= \sup_{v\in\mathbf{I}}\left|\int_0^v \frac{(v-x)^{\omega-1}}{\Gamma(\omega)}\Phi(x,\mathcal{U}(x),\mathbf{Z}\mathcal{U}(\lambda x))\mathrm{d}x\right| \\
&\leq \sup_{v\in\mathbf{I}}\int_0^v \left|\frac{(v-x)^{\omega-1}}{\Gamma(\omega)}\right||\Phi(x,\mathcal{U}(x),\mathbf{Z}\mathcal{U}(\lambda x))|\mathrm{d}x \\
&\leq c'\|\mathcal{U}\| \ + \ d', \text{ where } c' \ = \ \frac{c(1+\theta e)\theta^{\omega}}{\Gamma(\omega+1)} \text{ and } d' \ = \ \frac{(cp\theta+d)\theta^{\omega}}{\Gamma(\omega+1)}, \\
&\leq c'(s) \ + \ d'.
\end{aligned}$$

So, $\mathbf{G}(S)$ is bounded. For continuity of $\mathbf{G}$, let $\mathrm{v}_1,\ \mathrm{v}_2\ \in\ [0,\theta]$, such that $v_1 > v_2$, then

$$\begin{aligned}
|(\mathbf{G}\mathcal{U})(v_1) \ - \ (\mathbf{G}\mathcal{U})(v_2)| &\leq \left|\int_0^{\mathrm{v}_1}\frac{(\mathrm{v}_1-x)^{\omega-1}}{\Gamma(\omega)}\mathrm{d}x - \int_0^{\mathrm{v}_2}\frac{(\mathrm{v}_2-x)^{\omega-1}}{\Gamma(\omega)}\mathrm{d}x\right||\Phi(x,\mathcal{U}(x),\mathbf{Z}\mathcal{U}(\lambda x))| \\
&\leq \frac{(\mathrm{c}[\|\mathcal{U}\|+\|\mathrm{G}\mathcal{U}\|]+\mathrm{d})}{\Gamma(\omega+1)}\,[v_1^{\omega}-v_2^{\omega}] \to 0 \text{ as } v_1 \to v_2.
\end{aligned}$$

Thus **G** is continuous. Thus, Arzel$\acute{a}$-Ascoli theorem assure $\mathbf{G}(S)$ is compact relatively. Hence, our problem (1.2) possess at least one solution .

□

Further if:

(φ_4) There exist constants $L_\Psi > 0$, such that

$$\|\Psi(v,x,\bar{\mathcal{U}}) - \Psi(v,x,\mathcal{U})\| \leq L_\Psi\|\bar{\mathcal{U}} - \mathcal{U}\|.$$

(φ_5) There exist constants L_v, $L_\Phi > 0$ such that

$$\begin{aligned}
\|\Phi(v,\bar{\mathcal{U}},\mathrm{G}\bar{\mathcal{U}}) - \Phi(v,\mathcal{U},\mathrm{G}\mathcal{U})\| &\leq L_v\Big[\|\bar{\mathcal{U}}-\mathcal{U}\| + \|\mathrm{G}\bar{\mathcal{U}} - \mathrm{G}\mathcal{U}\|\Big] \\
&\leq L_\Phi\|\bar{\mathcal{U}}-\mathcal{U}\|, \text{ where } L_\Phi = L_v(1+L_\Psi\theta),
\end{aligned}$$

for each $v,x\in\mathbf{I}$, $0<\lambda<1$ and $\forall\,\bar{\mathcal{U}},\ \mathcal{U}\in\mathbb{R}$.

Theorem 4. *In addition to the assumption* $(\varphi_1)-(\varphi_5)$*, a constant* $\lambda > 0$ *such that*

$$\delta = (\mathcal{K}' + \mathrm{s}L_{\mathbf{A}} + L_\Phi)\,\frac{\theta^\omega}{\Gamma(\omega+1)} < 1, \tag{3.5}$$

then (1.2) *has one solution at most.*

Proof. Utilizing Banach contraction principle, for $\bar{\mathcal{U}}$, $\mathcal{U}\in\mathbf{E}$, then

$$\begin{aligned}
\|\mathbf{H}\bar{\mathcal{U}} - \mathbf{H}\mathcal{U}\| &\leq \|\mathbf{F}\bar{\mathcal{U}} - \mathbf{F}\mathcal{U}\| + \|\mathbf{G}\bar{\mathcal{U}} - \mathbf{G}\mathcal{U}\| \\
&\leq (\mathcal{K}' + \mathrm{s}L_{\mathbf{A}} + L_\Phi)\,\frac{\theta^\omega}{\Gamma(\omega+1)}\|\bar{\mathcal{U}}-\mathcal{U}\| \\
&= \delta\|\bar{\mathcal{U}}-\mathcal{U}\|.
\end{aligned}$$

□

Hence, problem (1.2) has unique solution.

4. Stability results

Before going to main results here, we recall definitions of Ulam-Hyer's (UH), generalized UH (GUH), UH Rassias (UHR) and generalized UHR (GUHR) stability from [10].

Definition 3. Eq (1.2) is UH type stable, if there exist $\epsilon > 0$ and $C_{\mathrm{q}} \in \mathbb{R}^{+}$. Further if any solution $\mathcal{U} \in \mathbf{X}$ of

$$|{}^{\mathrm{c}}\mathrm{D}_{0+}^{\omega}\mathcal{U}(v) - (\mathbf{A}(v,\mathcal{U}(v))\mathcal{U}(v) + \Phi(v,\mathcal{U}(v),\mathbf{Z}\mathcal{U}(\lambda x))\mathrm{d}x)| \leq \epsilon, \quad \forall \quad v \in \mathbf{I} = [0,\theta], \tag{4.1}$$

there exist at most one solution of $\bar{\mathcal{U}} \in \mathbf{X}$ of (1.2) such that

$$|\mathcal{U} - \bar{\mathcal{U}}| \leq C_{\mathrm{q}}\epsilon, \ \ \forall \ \ v \in \mathbf{I}.$$

Definition 4. Eq (1.2) is GUH type stable if there exist $\Psi \in \mathrm{C}(\mathbb{R}^{+},\mathbb{R}^{+})$, $\Psi(0) = 0$, such that any solution $\mathcal{U} \in \mathbf{X}$ of (4.1) there exist at most one solution of $\bar{\mathcal{U}} \in \mathbf{X}$ of (1.2) such that

$$|\mathcal{U} - \bar{\mathcal{U}}| \leq \Psi(\epsilon), \ \ \forall \ \ v \in \mathbf{I}.$$

Definition 5. Eq (1.2) is UHR type stable for $\varphi \in \mathrm{C}[\mathbf{I},\mathbb{R}^{+}]$ if there exist $C_{q} \in \mathbb{R}^{+}$ such that for $\epsilon > 0$ and let $\mathcal{U} \in \mathbf{X}$ represent any solution of

$$|{}^{\mathrm{c}}\mathrm{D}_{0+}^{\omega}\mathcal{U}(v) - (\mathbf{A}(v,\mathcal{U}(v))\mathcal{U}(v) + \Phi(v,\mathcal{U}(v),\mathbf{Z}\mathcal{U}(\lambda x))\mathrm{d}x)| \leq \varphi(v)\epsilon, \quad \forall \quad v \in \mathbf{I}, \tag{4.2}$$

there exist at most one solution $\bar{\mathcal{U}} \in \mathbf{X}$ of (1.2) such that

$$|\mathcal{U} - \bar{\mathcal{U}}| \leq C_{q}\epsilon\varphi(v), \ \ \forall \ \ v \in \mathbf{I}.$$

Definition 6. Eq (1.2) is GUHR type stable for $\varphi \in \mathrm{C}[\mathbf{I},\mathbb{R}^{+}]$ if there exist $C_{q,\varphi} \in \mathbb{R}^{+}$ such that for $\epsilon > 0$ and any solution $\mathcal{U} \in \mathbf{X}$ of (4.2), there exist at most one solution $\bar{\mathcal{U}} \in \mathbf{X}$ of (1.2) such that

$$|\mathcal{U} - \bar{\mathcal{U}}| \leq C_{q,\varphi}\varphi(v), \ \ \forall \ \ v \in \mathbf{I}.$$

Remark 1. *The map* $\bar{\mathcal{U}} \in \mathbf{X}$ *represent solution of* (4.1) *if a map* $x(v) \in \mathrm{C}(\mathbf{I};\mathbb{R})$ *exist (dependent on* $\bar{\mathcal{U}}$*) such that*
(*i*) $|x(v)| \leq \epsilon, \quad \forall \quad v \in \mathbf{I}.$
(*ii*) ${}^{\mathrm{c}}\mathrm{D}^{\omega}\bar{\mathcal{U}}(v) = \mathbf{A}(v,\mathcal{U}(v))\mathcal{U}(v) + \Phi(v,\mathcal{U}(v),\mathbf{Z}\mathcal{U}(\lambda x))\mathrm{d}x + x(v), \quad \forall \quad v \in \mathbf{I}.$

Remark 2. *The map* $\bar{\mathcal{U}} \in \mathbf{X}$ *represent solution of* (4.2) *if a map* x(v) $\in \mathrm{C}(\mathbf{I};\mathbb{R})$ *exist (dependent on* $\bar{\mathcal{U}}$*) such that*
(*i*) $|x(v)| \leq \epsilon\varphi, \quad \forall \quad v \in \mathbf{I}.$
(*ii*) ${}^{\mathrm{c}}\mathrm{D}^{\omega}\bar{\mathcal{U}}(v) = \mathbf{A}(v,\mathcal{U}(v))\mathcal{U}(v) + \Phi(v,\mathcal{U}(v),\mathbf{Z}\mathcal{U}(\lambda x))\mathrm{d}x + x(v), \quad \forall \quad v \in \mathbf{I}.$

The following Lemma is key to our first stability result.

Lemma 2. *The perturb problem*

$$\begin{cases} {}^{c}\mathrm{D}^{\omega}\mathcal{U}(v) = \mathbf{A}(v,\mathcal{U}(v))\mathcal{U}(v) + \Phi(v,\mathcal{U}(v),\mathbf{Z}\mathcal{U}(\lambda x)) + h(v), \quad v \in \mathbf{I} = [0,\theta], \\ \quad \mathcal{U}(0) = \mathcal{U}_{0}, \end{cases} \tag{4.3}$$

satisfying the following

$$|\mathcal{U}(v) - \mathbf{H}\mathcal{U}| \leq \frac{\theta^{\omega}\epsilon}{\Gamma(\omega+1)}, \quad v \in \mathbf{I}.$$

Proof. Lemma 1 gives solution of (4.3) as:

$$\mathcal{U}(v) = \mathbf{H}\mathcal{U} + \int_0^v \frac{(v-x)^{\omega-1}}{\Gamma(\omega)} x(x)\mathrm{d}x.$$

On (*i*) of Remark (1):

$$\left|\mathcal{U}(v) - \mathbf{H}\mathcal{U}\right| \leq \int_0^v \left|\frac{(v-x)^{\omega-1}}{\Gamma(\omega)}\right| |x(x)|\mathrm{d}x \leq \frac{\theta^{\omega}\epsilon}{\Gamma(\omega+1)}.$$

□

Theorem 5. *The stated problem* (1.2) *is UH and GUH stable if* (φ_4),(φ_5)*, lemma* (2) *and* $\Gamma(\omega+1) \neq (\mathcal{K}' + \mathrm{s}L_{\mathbf{A}} + L_{\Phi})\,\theta^{\omega}$ *hold.*

Proof. Suppose unique solution $\mathcal{U} \in \mathbf{X}$ of (1.2) and any other solution $\bar{\mathcal{U}}$ of (4.3), then

$$\begin{aligned}
\|\bar{\mathcal{U}} - \mathcal{U}\| &= \left\|\bar{\mathcal{U}} - \mathbf{H}\mathcal{U}\right\| \leq \left\|\bar{\mathcal{U}} - \mathbf{H}\bar{\mathcal{U}}\right\| + \left\|\mathbf{H}\bar{\mathcal{U}} - \mathbf{H}\mathcal{U}\right\| \\
&\leq \left\|\bar{\mathcal{U}} - \mathbf{H}\bar{\mathcal{U}}\right\| + \left\|\mathbf{H}\bar{\mathcal{U}} - \mathbf{H}\mathcal{U}\right\| \\
&\leq \frac{\theta^{\omega}\epsilon}{\Gamma(\omega+1)} + (\mathcal{K}' + \mathrm{s}L_{\mathbf{A}} + L_{\Phi})\,\frac{\theta^{\omega}}{\Gamma(\omega+1)}\|\bar{\mathcal{U}} - \mathcal{U}\| \\
&= C_{\mathrm{q}}\epsilon, \quad \text{where} \quad C_q = \frac{\theta^{\omega}}{\Gamma(\omega+1) - (\mathcal{K}' + \mathrm{s}L_{\mathbf{A}} + L_{\Phi})\,\theta^{\omega}}
\end{aligned}$$

Hence Eq (1.2) is UH stable. If there exist a function $\nabla : (0,1) \to (0,\infty)$ (nondecreasing) such that $\nabla(\epsilon) = \epsilon$ and $\nabla(0) = 0$, we have

$$\|\bar{\mathcal{U}} - \mathcal{U}\| \leq C_{\mathrm{q}}\nabla(\epsilon).$$

Which shows Eq (1.2) is GUH stable. □

Lemma 3. *For problem* (4.3) *the inequality given below hold:*

$$\|\mathcal{U} - \mathbf{H}\mathcal{U}\| \leq \frac{\theta^{\omega}\varphi\epsilon}{\Gamma(\omega+1)}, \quad v \in \mathbf{I}.$$

Proof. The proof is similar to Lemma 2. □

Theorem 6. *Under assumption* (φ_4),(φ_5) *and Lemma 3, problem* (1.2) *is UHR and GUHR stable if* $\Gamma(\omega+1) \neq (\mathcal{K}' + \mathrm{s}L_{\mathbf{A}} + L_{\Phi})\,\theta^{\omega}$ *hold.*

Proof. The proof is same as Theorem 5. Using assumption (φ_4),(φ_5), Lemma 3 and Remark 2 one can easily prove the required results. □

5. Example

For justification of our results, we consider the following general example.

Example 1.

$$\begin{aligned} \mathrm{D}^{\frac{1}{2}}\mathcal{U}(v) &= \frac{1}{49}\sin(\mathcal{U}(v)) + \frac{1}{(v+7)^2}\frac{|\mathcal{U}(v)|}{1+|\mathcal{U}(v)|} + \frac{1}{49}\int_0^v e^{-\frac{1}{5}\mathcal{U}(\lambda x)}\mathrm{d}x \\ \mathcal{U}(0) &= 1, \end{aligned} \tag{5.1}$$

here $\mathbf{A}(v,\mathcal{U}) = \dfrac{1}{49}\sin(\mathcal{U}(v)),\ \mathrm{G}\mathcal{U}(\lambda v) = \displaystyle\int_0^v e^{-\frac{1}{5}\mathcal{U}(\lambda x)}\mathrm{d}x,\ \Psi(v,x,\mathcal{U}(\lambda x)) = e^{-\frac{1}{5}\mathcal{U}(\lambda x)}$

and $\Phi(v,\mathcal{U}(v),\mathrm{G}\mathcal{U}(\lambda v)) = \dfrac{1}{(v+7)^2}\dfrac{|\mathcal{U}(v)|}{1+|\mathcal{U}(v)|} + \dfrac{1}{49}\mathrm{G}\mathcal{U}(\lambda v).$

Now $\|\mathbf{A}(v,\bar{\mathcal{U}}) - \mathbf{A}(v,\mathcal{U})\| \leq \frac{1}{49}\|\bar{\mathcal{U}} - \mathcal{U}\|$.

$\|\Psi(v,x,\mathcal{U}(\lambda x))\| = \sup_{v\in\mathbf{I}}\left\{\left|\int_0^1 e^{-\frac{1}{5}\mathcal{U}(\lambda v)}dv\right|\right\} \leq \frac{1}{5}\|\mathcal{U}(\lambda x)\| \leq \frac{1}{5}\|\mathcal{U}(v)\|$.

$\|\Phi(v,\mathcal{U}(v),\mathrm{G}\mathcal{U}(\lambda v))\| \leq \sup_{v\in\mathbf{I}}\left\{\left|\frac{1}{(v+7)^2}\frac{|\mathcal{U}(v)|}{1+|\mathcal{U}(v)|} + \frac{1}{49}\mathrm{G}\mathcal{U}(\lambda v)\right|\right\} \leq \frac{6}{245}\|\mathcal{U}\|$.

Thus $\mathbf{A}$, Φ *and* Ψ *satisfy conditions* $(\varphi_1)-(\varphi_3)$ *for* $\omega = \frac{1}{2},\ \mathbf{I} = [0,1],\ L_{\mathbf{A}} = \frac{1}{49} = c,\ d = 0,\ e = \frac{1}{5},$ *and* $p = 0$. *Thus the considered problem* (5.1) *posses at least one solution Theorem 3*

$$\begin{aligned} \|\mathrm{G}\bar{\mathcal{U}} - \mathrm{G}\mathcal{U}\| &= \sup_{v\in\mathbf{I}}\left\{\left|\int_0^v e^{-\frac{1}{5}\bar{\mathcal{U}}(\lambda x)}\mathrm{d}x - \int_0^v e^{-\frac{1}{5}\mathcal{U}(\lambda x)}\mathrm{d}x\right|\right\} \\ &\leq \frac{1}{5}\|\bar{\mathcal{U}} - \mathcal{U}\|, \end{aligned}$$

also

$$\begin{aligned} \|\Phi(v,\bar{\mathcal{U}}(v),\mathrm{G}\bar{\mathcal{U}}(\lambda v)) - \Phi(v,\mathcal{U}(v),\mathrm{G}\mathcal{U}(\lambda v))\| &\leq \frac{1}{49}[\|\bar{\mathcal{U}} - \mathcal{U}\| + \|\mathrm{G}\bar{\mathcal{U}} - \mathrm{G}\mathcal{U}\|] \\ &\leq \frac{6}{245}\|\bar{\mathcal{U}} - \mathcal{U}\|. \end{aligned}$$

Thus Φ *and* Ψ *satisfy condition* (φ_4) *and* (φ_5) *with* $\mathcal{K}' = \frac{1}{49}$, $L_\Psi = \frac{1}{5}$, $L_\Phi = \frac{6}{245}$ *and let* $\mathrm{s} = 1$, *then*

$$v \approx 0.07369 < 1.$$

Hence, problem (5.1) *has unique solution by using Theorem 4.*
Since $\Gamma(\omega+1) \neq (\mathcal{K}' + \mathrm{s}L_{\mathbf{A}} + L_\Phi)\,\theta^\omega$ *for* $\mathcal{K}' = \frac{1}{49}$, $\mathrm{s} = 1$, $L_{\mathbf{A}} = \frac{1}{49}$, $L_\Phi = \frac{1}{25}$, $\theta = 1$, $\omega = \frac{1}{2}$, *which gives UH and GUH stability of* (5.1). *On the other hand if, we take* $\varphi(v) = v$ *for* $v \in (0,1)$., *then problem* (5.1) *is UHR and GUHR stable.*

6. Concluding remarks

In this article, we have successfully study evolution equation with proportional delay for existence and uniqueness and also discuss the Ulam type stability of the consider problem. We used fixed point theory to developed the desired results. Thus we can say that the tool we used is simple and easy to apply for nonlinear problems. At the end an example is constructed to justify the developed results.

Since in present time, the concept of Caputo-Fabrizo, ABC derivatives, Conformable fractional derivatives, etc., are increasingly used by researchers. Also, many researchers have used the new concept in modeling real-world problems, which are limited only to ordinary problems. Therefore, it is suggested for the young researchers to use the same methodology to the problems involving the stated derivatives for qualitative analysis.

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Conflict of interest

There is no competing interest regarding this work.

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