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Research article

Averaging principle on infinite intervals for stochastic ordinary differential equations with Lévy noise

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Abstract: In this paper, we establish an averaging principle on the infinite time intervals for semilinear stochastic ordinary differential equations with Lévy noise. In particular, under suitable conditions we prove that if the coefficients are Poisson stable (including periodic, quasi-periodic, almost periodic, almost automorphic etc), then there exists a unique $\mathcal{L}^2$-bounded solution of the original equation, which inherits the recurrence property of the coefficients, and the recurrent solution uniformly converges to the stationary solution of the averaged equation on the whole real axis in distribution sense.

Keywords: averaging principle; stochastic differential equations; Lévy noise; periodic solution; quasi-periodic solution; almost periodic solution; Poisson stable solution
Mathematics Subject Classification: 34C29, 60H10, 60G51, 37B20, 34C27

1. Introduction

The core idea of the averaging method is to simplify the original system so as to get an effective one which can in some sense reflect the dynamics of the original one. It was first introduced in perturbation theory by Clairaut, Laplace and Lagrange. Some rigorous results on averaging principles can date back to Krylov and Bogolyubov's work [1], now called Krylov-Bogolyubov-Mitropolsky method [2–4]. Driven by applications the averaging principle has been developed in mechanics, mathematics, control and other areas. There are lots of works on averaging for deterministic systems which we will not mention here. Stochastic averaging principle is proposed by Stratonovich [5] for nonlinear oscillation problems with random noise. For averaging principles of stochastic differential equations among others [6–14].

In previous studies for stochastic averaging, researchers usually consider Gaussian noise. Although it is an ideal noise, we do agree on another fact: Gaussian noise cannot describe discontinuous situations and cannot simulate large fluctuations. It is evident that random noises in practise are more likely to be non-Gaussian. Lévy processes are essentially stochastic processes with stationary and independent increments, and they are viewed as an important and useful class of non-Gaussian processes since they are the simplest examples of random motions whose sample paths are right-continuous and have a number of (at most countable) random jump discontinuities occurring at random times, on each finite time interval. The method of averaging has been applied to stochastic differential equations with Lévy noise, but in general to the initial problem on a finite interval, such as [15, 16]. As for the averaging method on infinite intervals to deterministic equations, the book of Burd [17] provides a detail description.

Motivated by the work of Cheban and Liu [18], we investigate the averaging principle on infinite intervals for linear and semilinear stochastic differential equations based on Lévy noise with Poisson stable (including stationary, periodic, quasi-periodic, almost periodic, almost automorphic, Birkhoff recurrent, almost recurrent in the sense of Bebutov, Levitan almost periodic, pseudo-periodic, pseudo-recurrent and Poisson stable) coefficients. Under some suitable conditions, the original equation has a unique bounded solution with the same recurrent property as the coefficients, see [19–21] for details. Besides we show that this recurrent solution uniformly converges to the unique stationary solution of the averaged equation on the whole real axis in distribution when the time scale goes to zero.

The paper is organized as follows. Section 2 begins with definitions of Poisson stable functions, Lévy processes and their basic properties. We simply review Lévy-Itô decomposition, B. A. Shcherbakov's comparable method by character of recurrence, and the existence of recurrent solutions for stochastic differential equations. In the third and fourth sections, we respectively investigate the averaging principles for the following equations

$$\begin{aligned}\mathrm{d}Y(t) &= \varepsilon(\mathcal{A}(t)Y(t) + f(t))\mathrm{d}t + \sqrt{\varepsilon}g(t)\mathrm{d}W(t)\\ &\quad + \sqrt{\varepsilon}\int_{|x|_U<1} F(t,x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \sqrt{\varepsilon}\int_{|x|_U\geq 1} G(t,x)N(\mathrm{d}t,\mathrm{d}x)\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}Y(t) &= \varepsilon\left(\mathcal{A}(t)Y(t) + f(t,Y(t))\right)\mathrm{d}t + \sqrt{\varepsilon}g(t,Y(t))\mathrm{d}W(t)\\ &\quad + \sqrt{\varepsilon}\int_{|x|_U<1} F(t,Y(t-),x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \sqrt{\varepsilon}\int_{|x|_U\geq 1} G(t,Y(t-),x)N(\mathrm{d}t,\mathrm{d}x),\end{aligned}$$

with ε a small positive parameter, operator $\mathcal{A}$ non-stationary but bounded, coefficients f, g, F, G Poisson stable in time, $t \in \mathbb{R}$ an infinite time interval.

2. Preliminaries

2.1. *Poisson stable functions*

Let $(\mathcal{Y}, \rho)$ be a complete metric space. Denote by $C(\mathbb{R}, \mathcal{Y})$ the space of all continuous functions $\varphi : \mathbb{R} \to \mathcal{Y}$ equipped with the metric

$$d(\varphi,\psi) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(\varphi,\psi)}{1 + d_n(\varphi,\psi)}$$

for any $\varphi, \psi \in C(\mathbb{R}, \mathcal{Y})$, where $d_n(\varphi,\psi) := \sup_{|t|\leq n} \rho(\varphi(t), \psi(t))$. Note that the metric d generates the compact-open topology on $C(\mathbb{R}, \mathcal{Y})$ and the space $(C(\mathbb{R}, \mathcal{Y}), d)$ is a complete metric space [22–25].

Remark 2.1. Let $\varphi, \varphi_n \in C(\mathbb{R}, \mathcal{Y})$ $(n \in \mathbb{N})$. Then the following statements are equivalent:

(i) $d(\varphi_n, \varphi) \to 0$ as $n \to \infty$;
(ii) for each $l > 0$, $\lim_{n\to\infty} \max_{|t|\le l} \rho(\varphi_n(t), \varphi(t)) = 0$;
(iii) there exists a sequence $\{l_n\} \to +\infty$ such that $\lim_{n\to\infty} \max_{|t|\le l_n} \rho(\varphi_n(t), \varphi(t)) = 0$.

For given $\varphi \in C(\mathbb{R}, \mathcal{Y})$, we use φ^h to denote the *h-translation of* φ, where $\varphi^h(t) := \varphi(h+t)$ for $t \in \mathbb{R}$. The hull of φ, denoted by $H(\varphi)$, is the set of all the limits of φ^{h_n} in $C(\mathbb{R}, \mathcal{Y})$, i.e. $H(\varphi) := \{\psi \in C(\mathbb{R}, \mathcal{Y}) : \psi = \lim_{n\to\infty} \varphi^{h_n}$ for some sequence $\{h_n\} \subset \mathbb{R}\}$.

Remark 2.2. The mapping $\pi : \mathbb{R} \times C(\mathbb{R}, \mathcal{Y}) \to C(\mathbb{R}, \mathcal{Y})$ defined by $\pi(h, \varphi) = \varphi^h$ is a dynamical system, i.e. $\pi(0, \varphi) = \varphi$, $\pi(h_1 + h_2, \varphi) = \pi(h_2, \pi(h_1, \varphi))$ and the mapping π is continuous (see [22, 26]). In particular, the mapping π restricted to $\mathbb{R} \times H(\varphi)$ is a dynamical system.

Now we recall the types of Poisson stable functions to be studied in this paper and the relations among them; see [22–25] for further details.

Definition 2.3. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *stationary* (respectively, *τ-periodic*) if $\varphi(t) = \varphi(0)$ (respectively, $\varphi(t + \tau) = \varphi(t)$) for all $t \in \mathbb{R}$.

Definition 2.4. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *quasi-periodic with the spectrum of frequencies* $\nu_1, \nu_2, \dots, \nu_m$ if the following conditions are fulfilled:

(i) the numbers $\nu_1, \nu_2, \dots, \nu_m$ are rationally independent;
(ii) there exists a continuous function $\Phi : \mathbb{R}^m \to \mathcal{Y}$ such that $\Phi(t_1 + 2\pi, t_2 + 2\pi, \dots, t_m + 2\pi) = \Phi(t_1, t_2, \dots, t_m)$ for all $(t_1, t_2, \dots, t_m) \in \mathbb{R}^m$;
(iii) $\varphi(t) = \Phi(\nu_1 t, \nu_2 t, \dots, \nu_m t)$ for all $t \in \mathbb{R}$.

Definition 2.5. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is said to be *almost periodic* if for each $\varepsilon > 0$, the set $\mathcal{T}(\varphi, \varepsilon) := \{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}} \rho(\varphi(t + \tau), \varphi(t)) < \varepsilon\}$ is *relatively dense* on $\mathbb{R}$, i.e. there exists $l = l(\varepsilon) > 0$ such that $\mathcal{T}(\varphi, \varepsilon) \cap [a, a + l] \neq \emptyset$ for any $a \in \mathbb{R}$. The set $\mathcal{T}(\varphi, \varepsilon)$ is called the set of *ε-almost period* of the function φ.

Definition 2.6. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is said to be *pseudo-periodic in the positive* (respectively, *negative*) *direction* if for each $\varepsilon > 0$ and $l > 0$ there exists an ε-almost period $\tau > l$ (respectively, $\tau < -l$) of the function φ. The function φ is called pseudo-periodic if it is pseudo-periodic in both directions.

Remark 2.7. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is pseudo-periodic in the positive (respectively, negative) direction if and only if there is a sequence $\{t_n\} \to +\infty$ (respectively, $\{t_n\} \to -\infty$) such that φ^{t_n} converges to φ uniformly with respect to (w.r.t.) $t \in \mathbb{R}$ as $n \to \infty$.

Definition 2.8. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *almost automorphic* if and only if for any sequence $\{t'_n\} \subset \mathbb{R}$ there are a subsequence $\{t_n\}$ and some function $\psi : \mathbb{R} \to \mathcal{Y}$ such that

$$\varphi(t + t_n) \to \psi(t) \ \text{ and } \ \psi(t - t_n) \to \varphi(t)$$

uniformly in t on every compact subset from $\mathbb{R}$.

Definition 2.9. (i) A number $\tau \in \mathbb{R}$ is said to be *ε-shift* for $\varphi \in C(\mathbb{R}, \mathcal{Y})$ if $d(\varphi^\tau, \varphi) < \varepsilon$; a function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *almost recurrent (in the sense of Bebutov)* if for every $\varepsilon > 0$ the set $\{\tau : d(\varphi^\tau, \varphi) < \varepsilon\}$ is relatively dense.

(ii) A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *Lagrange stable* if $\{\varphi^h : h \in \mathbb{R}\}$ is a relatively compact subset of $C(\mathbb{R}, \mathcal{Y})$.

(iii) A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *Birkhoff recurrent* if it is almost recurrent and Lagrange stable.

In what follows, we denote by $(\mathcal{X}, \gamma)$ a complete metric space.

Definition 2.10. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *Levitan almost periodic* if there exists an almost periodic function $\psi \in C(\mathbb{R}, \mathcal{X})$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $d(\varphi^\tau, \varphi) < \varepsilon$ for all $\tau \in \mathcal{T}(\psi, \delta)$.

Remark 2.11. ([27, ChIV])

(i) Every almost periodic function is Levitan almost periodic but the inverse statement is not true in general.

(ii) A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is said to be almost automorphic if it is Levitan almost periodic and Lagrange stable.

Definition 2.12. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *pseudo-recurrent* if for any $\varepsilon > 0$ and $l \in \mathbb{R}$ there exists a constant $L \geq l$ such that for any $\tau_0 \in \mathbb{R}$ there exists a number $\tau \in [l, L]$ satisfying

$$\sup_{|t| \leq \varepsilon^{-1}} \rho(\varphi(t + \tau_0 + \tau), \varphi(t + \tau_0)) \leq \varepsilon.$$

Definition 2.13. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is called *Poisson stable in the positive (respectively, negative) direction* if for every $\varepsilon > 0$ and $l > 0$ there exists $\tau > l$ (respectively, $\tau < -l$) such that $d(\varphi^\tau, \varphi) < \varepsilon$. The function φ is called *Poisson stable* provided it is Poisson stable in both directions.

Remark 2.14. ([23–25, 28])

(i) Every Birkhoff recurrent function is pseudo-recurrent, but the inverse is not always true.

(ii) If the function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is pseudo-recurrent, then every function $\psi \in H(\varphi)$ is pseudo-recurrent.

(iii) If the function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is Lagrange stable and every function $\psi \in H(\varphi)$ is Poisson stable, then φ is pseudo-recurrent.

Finally, we remark that a Lagrange stable function is not Poisson stable in general, but all other types of functions introduced above are Poisson stable.

2.2. Shcherbakov's comparability method by character of recurrence

Let $\varphi \in C(\mathbb{R}, \mathcal{Y})$. Denote by $\mathfrak{N}_\varphi$ (respectively, $\mathfrak{M}_\varphi$) the family of all sequences $\{t_n\} \subset \mathbb{R}$ such that $\varphi^{t_n} \to \varphi$ (respectively, $\{\varphi^{t_n}\}$ converges) in $C(\mathbb{R}, \mathcal{Y})$ as $n \to \infty$. We denote by $\mathfrak{N}_\varphi^u$ (respectively, $\mathfrak{M}_\varphi^u$) the family of sequences $\{t_n\} \in \mathfrak{N}_\varphi$ (respectively, $\{t_n\} \in \mathfrak{M}_\varphi$) such that φ^{t_n} converges to φ (respectively, $\{\varphi^{t_n}\}$ converges) uniformly w.r.t. $t \in \mathbb{R}$ as $n \to \infty$.

Definition 2.15. A function $\varphi \in C(\mathbb{R}, \mathcal{Y})$ is said to be *comparable (by character of recurrence)* with $\psi \in C(\mathbb{R}, \mathcal{X})$ if $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$; φ is said to be *strongly comparable (by character of recurrence)* with ψ if $\mathfrak{M}_\psi \subseteq \mathfrak{M}_\varphi$.

Theorem 2.16. ([23, 29, Chapter II]) *Let $\varphi \in C(\mathbb{R}, \mathcal{Y})$, $\psi \in C(\mathbb{R}, \mathcal{X})$. Then the following statements hold.*

(i) $\mathfrak{M}_\psi \subseteq \mathfrak{M}_\varphi$ implies $\mathfrak{N}_\psi \subseteq \mathfrak{N}_\varphi$, and hence strong comparability implies comparability.
(ii) Let φ be comparable with ψ. If the function ψ is stationary (respectively, τ-periodic, Levitan almost periodic, almost recurrent, Poisson stable), then so is φ.
(iii) Let φ be strongly comparable with ψ. If the function ψ is quasi-periodic with the spectrum of frequencies $\nu_1, \nu_2, \dots, \nu_m$ (respectively, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable), then so is φ.
(iv) Let φ be strongly comparable with ψ and ψ be Lagrange stable. If ψ is pseudo-periodic (respectively, pseudo-recurrent), then so is φ.

Lemma 2.17. ([19]) *Let $\varphi \in C(\mathbb{R}, \mathcal{Y})$, $\psi \in C(\mathbb{R}, \mathcal{X})$. The following statements hold:*

(i) If $\mathfrak{M}^u_\psi \subseteq \mathfrak{M}^u_\varphi$, then $\mathfrak{N}^u_\psi \subseteq \mathfrak{N}^u_\varphi$.
(ii) If $\mathfrak{M}^u_\psi \subseteq \mathfrak{M}^u_\varphi$ and the function ψ is almost periodic, then so is φ.
(iii) If $\mathfrak{N}^u_\psi \subseteq \mathfrak{N}^u_\varphi$ and the function ψ is pseudo periodic, then so is φ.

Denote by $BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X})$ the space of all functions $f : \mathbb{R} \times \mathcal{Y} \to \mathcal{X}$ which are continuous in t uniformly w.r.t. y on every bounded subset $Q \subseteq \mathcal{Y}$ and bounded on every bounded subset from $\mathbb{R} \times \mathcal{Y}$. We endow the function space with the metric

$$d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f, g)}{1 + d_n(f, g)}, \tag{2.1}$$

where

$$d_n(f, g) := \sup_{|t| \le n,\ y \in Q_n} \gamma(f(t, y), g(t, y))$$

with $Q_n \subset \mathcal{Y}$ being bounded, $Q_n \subset Q_{n+1}$ and $\mathcal{Y} = \bigcup_{n \ge 1} Q_n$. Note that $(BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X}), d)$ is a complete metric space and $d(f_n, f) \to 0$ if and only if $f_n(t, y) \to f(t, y)$ uniformly w.r.t. (t, y) on every bounded subset from $\mathbb{R} \times \mathcal{Y}$. For given $f \in BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X})$ and $\tau \in \mathbb{R}$, denote the *translation of f* by f^τ, i.e. $f^\tau(t, y) := f(t + \tau, y)$ for $(t, y) \in \mathbb{R} \times \mathcal{Y}$, and the *hull of f* by $H(f) := \overline{\{f^\tau : \tau \in \mathbb{R}\}}$ with the closure being taken under the metric d given by (2.1). Note that the mapping $\pi : \mathbb{R} \times BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X}) \to BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X})$ defined by $\pi(\tau, f) := f^\tau$ is a dynamical system, i.e. $\pi(0, f) = f$, $\pi(\tau_1 + \tau_2, f) = \pi(\tau_2, \pi(\tau_1, f))$ and the mapping π is continuous. See [22, 26] or [§1.1] for details.

We use $BC(\mathcal{Y}, \mathcal{X})$ to denote the space of all continuous functions $f : \mathcal{Y} \to \mathcal{X}$ which are bounded on every bounded subset $Q \subset \mathcal{Y}$ and equip the space with the metric

$$d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(f, g)}{1 + d_n(f, g)},$$

where $d_n(f, g) := \sup_{y \in Q_n} \gamma(f(y), g(y))$ with Q_n the same as (2.1). It is immediate to check that $(BC(\mathcal{Y}, \mathcal{X}), d)$ is a complete metric space.

Remark 2.18. Let $F \in BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X})$ and $f : \mathbb{R} \to BC(\mathcal{Y}, \mathcal{X})$ be a mapping defined by $f(t) := F(t, \cdot)$. Note that for any $F \in BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X})$, we have $\mathfrak{M}_F = \mathfrak{M}_f$ and $\mathfrak{M}^u_F = \mathfrak{M}^u_f$. Here $\mathfrak{M}_F$ is the set of all sequences $\{t_n\}$ such that $\{F^{t_n}\}$ converges in the space $BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X})$; $\mathfrak{M}^u_F$ is the set of all sequences $\{t_n\}$ such that $\{F^{t_n}\}$ converges in the space $BUC(\mathbb{R} \times \mathcal{Y}, \mathcal{X})$ uniformly w.r.t. $t \in \mathbb{R}$.

2.3. Semilinear stochastic differential equations with Lévy noise

Throughout the paper, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, two real separable Hilbert spaces $(\mathbb{H}, |\cdot|)$ and $(U, |\cdot|_U)$, and a real separable Banach space $(\mathfrak{B}, |\cdot|_{\mathfrak{B}})$. Denote by $L(\mathbb{H})$ (respectively, $L(\mathfrak{B})$) the Hilbert (respectively, Banach) space of all bounded linear operators on $\mathbb{H}$ (respectively, $\mathfrak{B}$) endowed with operator norm $\|\cdot\|$ (respectively, $\|\cdot\|_{\mathfrak{B}}$). Denote by $L(U, \mathbb{H})$ the Banach space of all bounded linear operators from U to $\mathbb{H}$ with the norm $\|\cdot\|_{L(U,\mathbb{H})}$.

We now review the definition of Lévy processes and the important Lévy-Itô decomposition theorem (see [30, 31]). In this paper, the Lévy processes we consider are U-valued.

Definition 2.19. A U-valued stochastic process $L = (L(t), t \geq 0)$ is called *Lévy process* if it has the following three properties:

(i) $L(0) = 0$ almost surely.
(ii) L has stationary and independent increments, i.e. the law of $L(t+h) - L(t)$ does not depend on t and for all $0 \leq t_0 < t_1 < t_2 < ... < t_n < \infty$, random variables $L(t_1) - L(t_0)$, $L(t_2) - L(t_1)$,..., $L(t_n) - L(t_{n-1})$ are independent.
(iii) L is *stochastically continuous*, i.e. for all $\epsilon > 0$ and for all $s > 0$

$$\lim_{t \to s} \mathbf{P}(|L(t) - L(s)|_U > \epsilon) = 0.$$

Since a Lévy process L is càdlàg, the associated *jump process* $\Delta L = (\Delta L(t), t \geq 0)$ is given by $\Delta L(t) = L(t) - L(t-)$. Let $\mathcal{B}(U - \{0\})$ be the Borel field of $U - \{0\}$ and $B \in \mathcal{B}(U - \{0\})$. Define the random counting measure

$$N(t, B)(\omega) := \sharp\{0 \leq s \leq t : \Delta L(s)(\omega) \in B\} = \sum_{0 \leq s \leq t} \chi_B(\Delta L(s)(\omega)),$$

where χ_B is the indicator function of B. We call $\nu(\cdot) := \mathbb{E}(N(1, \cdot))$ the *intensity measure* of L. We say that $B \in \mathcal{B}(U - \{0\})$ is *bounded below* if $0 \notin \overline{B}$, the closure of B. Note that if B is bounded below, then $N(t, B) < \infty$ holds almost surely for all $t \geq 0$ and $(N(t, B), t \geq 0)$ is a Poisson process with intensity $\nu(B)$. N is called *Poisson random measure*. For each $t \geq 0$ and B bounded below, define the *compensated Poisson random measure* by

$$\widetilde{N}(t, B) = N(t, B) - t\nu(B).$$

Proposition 2.20 (Lévy-Itô decomposition). *If L is a Lévy process in U, then there exists $a \in U$, a U-valued Q-Wiener process W and an independent Poisson random measure N on $\mathbb{R}^+ \times (U - \{0\})$ with intensity measure ν such that for any $t \geq 0$,*

$$L(t) = at + W(t) + \int_{|x|_U < 1} x\widetilde{N}(t, \mathrm{d}x) + \int_{|x|_U \geq 1} xN(t, \mathrm{d}x). \tag{2.2}$$

Here the intensity measure ν satisfies

$$\int_U (|x|_U^2 \wedge 1)\nu(\mathrm{d}x) < \infty \tag{2.3}$$

and $\widetilde{N}$ is the compensated Poisson random measure of N.

As for Q-Wiener processes and the stochastic integral based on them, the monograph [32] provides a thorough description. Assume that L_1 and L_2 are two independent, identically distributed Lévy processes with decompositions as in Proposition 2.20 and let

$$L(t) = \begin{cases} L_1(t), & \text{for } t \geq 0, \\ -L_2(-t), & \text{for } t < 0. \end{cases}$$

Then L is a two-sided Lévy process. In this paper, we consider two-sided Lévy process L which is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, (\mathcal{F}_t)_{t\in\mathbb{R}})$ and suppose that the covariance operator Q of W is of trace class, i.e. $\mathrm{Tr}Q < \infty$.

Remark 2.21. It follows from (2.3) that $\int_{|x|_U \geq 1} \nu(\mathrm{d}x) < \infty$. For convenience, we set hereafter

$$b := \int_{|x|_U \geq 1} \nu(\mathrm{d}x).$$

Remark 2.22. Note that the stochastic process $\widetilde{L} = (\widetilde{L}(t), t \in \mathbb{R})$ given by $\widetilde{L}(t) := L(t+s) - L(s)$ for some $s \in \mathbb{R}$ is also a two-sided Lévy process which shares the same law as L. In particular, when $s \in \mathbb{R}^+$, the similar conclusion holds for one-sided Lévy processes.

Consider the linear homogeneous equation

$$\dot{y} = \mathcal{A}(t)y \tag{2.4}$$

on the space $\mathfrak{B}$, where $\mathcal{A} \in C(\mathbb{R}, L(\mathfrak{B}))$. Denote by $T_{\mathcal{A}}(t,\tau)$ the evolution (solving) operator of Eq (2.4), where $T_{\mathcal{A}}(t,\tau) := U_{\mathcal{A}}(t)U_{\mathcal{A}}^{-1}(\tau)$ with $U_{\mathcal{A}}(t)$ the Cauchy operator of Eq (2.4) (see [33]).

Definition 2.23. Eq (2.4) is said to be *uniformly asymptotically stable* if there are positive constants K and ω such that

$$\|T_{\mathcal{A}}(t,\tau)\|_{\mathfrak{B}} \leq Ke^{-\omega(t-\tau)} \quad \text{for any } t \geq \tau \ (t,\tau \in \mathbb{R}). \tag{2.5}$$

Lemma 2.24. ([34, Chapter III]) *Suppose that Eq (2.4) is uniformly asymptotically stable such that Eq (2.5) holds. Then for any $t \geq \tau$ $(t,\tau \in \mathbb{R})$ and $\tilde{\mathcal{A}} \in H(\mathcal{A})$*

$$\|T_{\tilde{\mathcal{A}}}(t,\tau)\|_{\mathfrak{B}} \leq Ke^{-\omega(t-\tau)},$$

where $H(\mathcal{A})$ denotes the closure in the space $C(\mathbb{R}, L(\mathfrak{B}))$ of all translations $\{\mathcal{A}^h : h \in \mathbb{R}\}$ with $\mathcal{A}^h(t) := \mathcal{A}(t+h)$ for $t \in \mathbb{R}$.

We now consider the stochastic differential equation driven by Lévy noise

$$\mathrm{d}Y(t) = (\mathcal{A}(t)Y(t) + f(t, Y(t)))\mathrm{d}t + g(t, Y(t))\mathrm{d}L(t), \tag{2.6}$$

where $\mathcal{A} : \mathbb{R} \to L(\mathbb{H})$, $f : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$, $g : \mathbb{R} \times \mathbb{H} \to L(U, \mathbb{H})$; L is a U-valued Lévy process. By Lévy-Itô decomposition (2.2), Eq (2.6) reads

$$\begin{aligned} \mathrm{d}Y(t) =& (\mathcal{A}(t)Y(t) + f(t, Y(t)))\mathrm{d}t + g(t, Y(t))\mathrm{d}W(t) \\ &+ \int_{|x|_U<1} F(t, Y(t-), x)\widetilde{N}(\mathrm{d}t, \mathrm{d}x) + \int_{|x|_U\geq 1} G(t, Y(t-), x)N(\mathrm{d}t, \mathrm{d}x), \end{aligned} \tag{2.7}$$

where F and G are $\mathbb{H}$-valued. It allows us to study large jumps with considerable probability. We set $\mathcal{F}_t := \sigma\{L(u) : u \leq t\}$.

Definition 2.25. An $\mathcal{F}_t$-adapted process $\{Y(t)\}_{t\in\mathbb{R}}$ is called a *mild solution* of Eq (2.7) if it satisfies the corresponding stochastic integral equation

$$\begin{aligned} Y(t) = {} & T_{\mathcal{A}}(t,r)Y(r) + \int_r^t T_{\mathcal{A}}(t,s)f(s,Y(s))\mathrm{d}s + \int_r^t T_{\mathcal{A}}(t,s)g(s,Y(s))\mathrm{d}W(s) \\ & + \int_r^t \int_{|x|_U<1} T_{\mathcal{A}}(t,s)F(s,Y(s-),x)\widetilde{N}(\mathrm{d}s,\mathrm{d}x) \\ & + \int_r^t \int_{|x|_U\geq 1} T_{\mathcal{A}}(t,s)G(s,Y(s-),x)N(\mathrm{d}s,\mathrm{d}x), \end{aligned} \tag{2.8}$$

for all $t \geq r$ and each $r \in \mathbb{R}$.

Let $\mathcal{P}(\mathfrak{B})$ be the space of all Borel probability measures on the space $\mathfrak{B}$ endowed with the β metric:

$$\beta(\mu,\nu) := \sup\left\{\left|\int f\mathrm{d}\mu - \int f\mathrm{d}\nu\right|_{\mathfrak{B}} : \|f\|_{BL} \leq 1\right\}, \quad \text{for } \mu,\nu \in \mathcal{P}(\mathfrak{B}).$$

Here f varies in the space of bounded Lipschitz continuous real-valued functions on the space $\mathfrak{B}$ with the norm

$$\|f\|_{BL} = Lip(f) + \|f\|_\infty,$$

where

$$Lip(f) = \sup_{x\neq y}\frac{|f(x)-f(y)|_{\mathfrak{B}}}{|x-y|_{\mathfrak{B}}}, \quad \|f\|_\infty = \sup_{x\in\mathfrak{B}}|f(x)|_{\mathfrak{B}}.$$

A sequence $\{\mu_n\} \subset \mathcal{P}(\mathfrak{B})$ is said to *weakly converge* to μ if $\int f\mathrm{d}\mu_n \to \int f\mathrm{d}\mu$ for all $f \in C_b(\mathfrak{B})$, where $C_b(\mathfrak{B})$ is the space of all bounded continuous real-valued functions on the space $\mathfrak{B}$. As we know that $(\mathcal{P}(\mathfrak{B}),\beta)$ is a separable complete metric space and that a sequence $\{\mu_n\}$ weakly converges to μ if and only if $\beta(\mu_n,\mu) \to 0$ as $n \to \infty$. See [35, §11.3] for β metric and related properties.

Definition 2.26. A sequence of random variables $\{Y_n\}$ is said to *converge in distribution* to the random variable Y if the corresponding laws $\{\mu_n\}$ of $\{Y_n\}$ weakly converge to the law μ of Y, i.e. $\beta(\mu_n,\mu) \to 0$.

Definition 2.27. Let $\{\varphi(t)\}_{t\in\mathbb{R}}$ be a mild solution of Eq (2.7). Then φ is called *compatible* (respectively, *strongly compatible*) *in distribution* if $\mathfrak{N}_{(\mathcal{A},f,g,F,G)} \subseteq \tilde{\mathfrak{N}}_\varphi$ (respectively, $\mathfrak{M}_{(\mathcal{A},f,g,F,G)} \subseteq \tilde{\mathfrak{M}}_\varphi$), where $\tilde{\mathfrak{N}}_\varphi$ (respectively, $\tilde{\mathfrak{M}}_\varphi$) means the set of all sequences $\{t_n\} \subset \mathbb{R}$ such that the sequence $\{\varphi(\cdot + t_n)\}$ converges to $\varphi(\cdot)$ (respectively, $\{\varphi(\cdot + t_n)\}$ converges) in distribution uniformly on any compact interval.

2.4. Some results of linear and semilinear stochastic differential equations with Lévy noise

Define for $p \geq 2$

$$\mathcal{L}^p(\mathbf{P};\mathbb{H}) := \mathcal{L}^p(\Omega,\mathcal{F},\mathbf{P};\mathbb{H}) = \left\{Y : \Omega \to \mathbb{H}\middle|\mathbb{E}|Y|^p = \int_\Omega |Y|^p\mathrm{d}\mathbf{P} < \infty\right\}$$

with the norm

$$\|Y\|_{\mathcal{L}^p(\mathbf{P};\mathbb{H})} := \left(\int_\Omega |Y|^p\mathrm{d}\mathbf{P}\right)^{\frac{1}{p}}.$$

Then $(\mathcal{L}^p(\mathbf{P};\mathbb{H}), \|\cdot\|_{\mathcal{L}^p(\mathbf{P};\mathbb{H})})$ is a Banach space. Set

$$\begin{aligned}\mathcal{L}^2(\mathbf{P};L(U,\mathbb{H})) &:= \mathcal{L}^2(\Omega,\mathcal{F},\mathbf{P};L(U,\mathbb{H}))\\ &= \left\{Y:\Omega\to L(U,\mathbb{H})\middle|\mathbb{E}\|Y\|^2_{L(U,\mathbb{H})} = \int_\Omega \|Y\|^2_{L(U,\mathbb{H})}\mathrm{d}\mathbf{P}<\infty\right\}\end{aligned}$$

and define a norm by

$$\|Y\|_{\mathcal{L}^2(\mathbf{P};L(U,\mathbb{H}))} := \left(\int_\Omega \|Y\|^2_{L(U,\mathbb{H})}\mathrm{d}\mathbf{P}\right)^{\frac{1}{2}}.$$

Note that $(\mathcal{L}^2(\mathbf{P};L(U,\mathbb{H})), \|\cdot\|_{\mathcal{L}^2(\mathbf{P};L(U,\mathbb{H}))})$ is a Banach space.

Remark 2.28. If the operator $Q\in L(U)$, the space of bounded linear operators from U to U, is nonnegative, symmetric and $\mathrm{Tr}Q<\infty$, then $LQ^{\frac{1}{2}}\in L_2(U,\mathbb{H})$ for all $L\in L(U,\mathbb{H})$, where the space $L_2(U,\mathbb{H})$ is a separable Hilbert space that consists of all Hilbert-Schmidt operators from U to $\mathbb{H}$ with inner product $\langle A,B\rangle_{L_2(U,\mathbb{H})} := \sum\limits_{k\in\mathbb{N}}\langle Ae_k,Be_k\rangle$ and $\{e_k\}_{k\in\mathbb{N}}$ an orthonormal basis of U. For $g\in\mathcal{L}^2(\mathbf{P};L(U,\mathbb{H}))$, we have $gQ^{\frac{1}{2}}\in\mathcal{L}^2(\mathbf{P};L_2(U,\mathbb{H}))$ and denote $\|gQ^{\frac{1}{2}}\|_{\mathcal{L}^2(\mathbf{P};L_2(U,\mathbb{H}))} = \left(\mathbb{E}\|gQ^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})}\right)^{\frac{1}{2}}$.

Define

$$\begin{aligned}\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H}) &:= \mathcal{L}^2(\Omega\times U,\mathcal{P}_U,\mathbf{P}_\nu;\mathbb{H})\\ &= \left\{Y:\Omega\times U\to\mathbb{H}\middle|\int_{\Omega\times U}|Y|^2\mathrm{d}\mathbf{P}_\nu = \int_U \mathbb{E}|Y|^2\nu(\mathrm{d}x)<\infty\right\},\end{aligned}$$

where $\mathcal{P}_U$ is the product σ-algebra on $\Omega\times U$ and $\mathbf{P}_\nu = \mathbf{P}\otimes\nu$. For $Y\in\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H})$, let

$$\|Y\|_{\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H})} := \left(\int_U \mathbb{E}|Y|^2\nu(\mathrm{d}x)\right)^{\frac{1}{2}}.$$

Then $\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H})$ is a Hilbert space equipped with the norm $\|\cdot\|_{\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H})}$.

Denote by $C_b(\mathbb{R},\mathfrak{B})$ the Banach space of all continuous and bounded mappings $\varphi:\mathbb{R}\to\mathfrak{B}$ equipped with the norm $\|\varphi\|_\infty := \sup\{|\varphi(t)|_{\mathfrak{B}} : t\in\mathbb{R}\}$. Note that if $f\in C_b(\mathbb{R},\mathfrak{B})$ and $\tilde{f}\in H(f)$, the hull of f, then $\|\tilde{f}\|_\infty\le\|f\|_\infty$.

Theorem 2.29. *Consider the linear stochastic differential equation*

$$\begin{aligned}\mathrm{d}Y(t) =&(\mathcal{A}(t)Y(t)+f(t))\mathrm{d}t + g(t)\mathrm{d}W(t)\\ &+\int_{|x|_U<1}F(t,x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x)+\int_{|x|_U\ge1}G(t,x)N(\mathrm{d}t,\mathrm{d}x).\end{aligned}\tag{2.9}$$

Assume that $\mathcal{A}\in C_b(\mathbb{R},L(\mathbb{H}))$; Eq (2.4) is uniformly asymptotically stable such that (2.5) holds; $f\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$, $g\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};L(U,\mathbb{H})))$, $F,G\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H}))$. Suppose that W and N are the same as in Section 2.3. Then Eq (2.9) has a unique mild solution $\varphi\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ which satisfies

$$\varphi(t) = \int_{-\infty}^t T_{\mathcal{A}}(t,\tau)f(\tau)\mathrm{d}\tau + \int_{-\infty}^t T_{\mathcal{A}}(t,\tau)g(\tau)\mathrm{d}W(\tau)$$

$$+\int_{-\infty}^{t}\int_{|x|_U<1}T_{\mathcal{A}}(t,\tau)F(\tau,x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)+\int_{-\infty}^{t}\int_{|x|_U\geq 1}T_{\mathcal{A}}(t,\tau)G(\tau,x)N(\mathrm{d}\tau,\mathrm{d}x),$$

and this unique $\mathcal{L}^2$-bounded solution is strongly compatible in distribution (i.e. $\mathfrak{M}_{(\mathcal{A},f,g,F,G)}\subseteq\tilde{\mathfrak{M}}_{\varphi_\varepsilon}$). Furthermore, $\mathfrak{M}^u_{(\mathcal{A},f,g,F,G)}\subseteq\tilde{\mathfrak{M}}^u_{\varphi}$, where $\tilde{\mathfrak{M}}^u_{\varphi}$ is the set of all sequences $\{t_n\}$ such that the sequence $\{\varphi(t+t_n)\}$ converges in distribution uniformly in $t\in\mathbb{R}$.

Proof. The proof is analogous to Theorem 3.3 in [21]. □

Define

$$\mathcal{L}^2(\nu;\mathbb{H}):=\mathcal{L}^2(U,\mathcal{B}_U,\nu;\mathbb{H})=\left\{Y:U\to\mathbb{H}\,\middle|\,\int_U|Y|^2\nu(\mathrm{d}x)<\infty\right\},$$

with $\mathcal{B}_U$ being the Borel σ-algebra on U. For $Y\in\mathcal{L}^2(\nu;\mathbb{H})$, let

$$\|Y\|_{\mathcal{L}^2(\nu;\mathbb{H})}:=\left(\int_U|Y|^2\nu(\mathrm{d}x)\right)^{\frac{1}{2}}.$$

Denote by $C(\mathbb{R}\times\mathcal{Y},\mathcal{X})$ the space of all continuous functions $f:\mathbb{R}\times\mathcal{Y}\to\mathcal{X}$, recalling that $(\mathcal{Y},\rho)$ and $(\mathcal{X},\gamma)$ are complete metric spaces.

Consider Eq (2.7). Like in [21], assume that $f\in C(\mathbb{R}\times\mathbb{H},\mathbb{H})$, $g\in C(\mathbb{R}\times\mathbb{H},L(U,\mathbb{H}))$, $F,G\in C(\mathbb{R}\times\mathbb{H},\mathcal{L}^2(\nu;\mathbb{H}))$ and f,g,F,G satisfy the following conditions:

(E1) There exists a number $M\geq 0$ such that for all $t\in\mathbb{R}$
$|f(t,0)|\leq M,\qquad \|g(t,0)Q^{\frac{1}{2}}\|_{L_2(U,\mathbb{H})}\leq M,$
$\int_{|x|_U<1}|F(t,0,x)|^2\nu(\mathrm{d}x)\leq M^2,\qquad \int_{|x|_U\geq 1}|G(t,0,x)|^2\nu(\mathrm{d}x)\leq M^2.$

(E1′) There exists a number $M\geq 0$ such that for some constant $p>2$ and all $t\in\mathbb{R}$
$|f(t,0)|\leq M,\qquad \|g(t,0)Q^{\frac{1}{2}}\|_{L_2(U,\mathbb{H})}\leq M,$
$\int_{|x|_U<1}|F(t,0,x)|^p\nu(\mathrm{d}x)\leq M^p,\qquad \int_{|x|_U\geq 1}|G(t,0,x)|^p\nu(\mathrm{d}x)\leq M^p.$

(E2) There exists a number $\mathcal{L}\geq 0$ such that for all $t\in\mathbb{R}$ and $y_1,y_2\in\mathbb{H}$
$|f(t,y_1)-f(t,y_2)|\leq\mathcal{L}|y_1-y_2|,\qquad \|(g(t,y_1)-g(t,y_2))Q^{\frac{1}{2}}\|_{L_2(U,\mathbb{H})}\leq\mathcal{L}|y_1-y_2|,$
$\int_{|x|_U<1}|F(t,y_1,x)-F(t,y_2,x)|^2\nu(\mathrm{d}x)\leq\mathcal{L}^2|y_1-y_2|^2,$
$\int_{|x|_U\geq 1}|G(t,y_1,x)-G(t,y_2,x)|^2\nu(\mathrm{d}x)\leq\mathcal{L}^2|y_1-y_2|^2.$

(E2′) There exists a number $\mathcal{L}\geq 0$ such that for some constant $p>2$ and $t\in\mathbb{R}$, $y_1,y_2\in\mathbb{H}$
$|f(t,y_1)-f(t,y_2)|\leq\mathcal{L}|y_1-y_2|,\qquad \|(g(t,y_1)-g(t,y_2))Q^{\frac{1}{2}}\|_{L_2(U,\mathbb{H})}\leq\mathcal{L}|y_1-y_2|,$
$\int_{|x|_U<1}|F(t,y_1,x)-F(t,y_2,x)|^p\nu(\mathrm{d}x)\leq\mathcal{L}^p|y_1-y_2|^p,$
$\int_{|x|_U\geq 1}|G(t,y_1,x)-G(t,y_2,x)|^p\nu(\mathrm{d}x)\leq\mathcal{L}^p|y_1-y_2|^p.$

(E3) f,g,F,G are continuous in t uniformly w.r.t. Y on each bounded subset $Q\subset\mathbb{H}$.

Remark 2.30. (i) If f,g,F,G satisfy (E1)–(E3), then $f\in BUC(\mathbb{R}\times\mathbb{H},\mathbb{H})$, $g\in BUC(\mathbb{R}\times\mathbb{H},L(U,\mathbb{H}))$, $F,G\in BUC(\mathbb{R}\times\mathbb{H},\mathcal{L}^2(\nu;\mathbb{H}))$ and $H(f,g,F,G)\subset BUC(\mathbb{R}\times\mathbb{H},\mathbb{H})\times BUC(\mathbb{R}\times\mathbb{H},L(U,\mathbb{H}))\times BUC(\mathbb{R}\times\mathbb{H},\mathcal{L}^2(\nu;\mathbb{H}))\times BUC(\mathbb{R}\times\mathbb{H},\mathcal{L}^2(\nu;\mathbb{H}))$.

(ii) If f,g,F,G satisfy (E1), (E2) (or (E1′), (E2′)) with the constants M and $\mathcal{L}$, then every quadruplet $(\tilde{f},\tilde{g},\tilde{F},\tilde{G})\in H(f,g,F,G):=\overline{\{(f^\tau,g^\tau,F^\tau,G^\tau):\tau\in\mathbb{R}\}}$, the hull of (f,g,F,G), also possesses the same properties with the same constants.

Theorem 2.31. *Consider Eq* (2.7)*. Suppose that $\mathcal{A} \in C(\mathbb{R}, L(\mathbb{H}))$; Eq* (2.4) *is uniformly asymptotically stable such that* (2.5) *holds; $f \in C(\mathbb{R} \times \mathbb{H}, \mathbb{H})$, $g \in C(\mathbb{R} \times \mathbb{H}, L(U, \mathbb{H}))$, $F, G \in C(\mathbb{R} \times \mathbb{H}, \mathcal{L}^2(\nu; \mathbb{H}))$. Suppose that W and N are the same as in Section 2.3; f, g, F, G satisfy the conditions (E1) and (E2). Then the following statements hold:*

(i) If $\mathcal{L} < \dfrac{\omega}{2K\sqrt{1 + 2\omega + 2b}}$, then Eq (2.7) *has a unique solution $\xi \in C(\mathbb{R}, B[0, r])$ defined by equality*

$$
\begin{aligned}
\xi(t) = &\int_{-\infty}^{t} T_{\mathcal{A}}(t, \tau) f(\tau, \xi(\tau))\mathrm{d}\tau + \int_{-\infty}^{t} T_{\mathcal{A}}(t, \tau) g(\tau, \xi(\tau))\mathrm{d}W(\tau) \\
&+ \int_{-\infty}^{t}\int_{|x|_U<1} T_{\mathcal{A}}(t, \tau) F(\tau, \xi(\tau-), x)\widetilde{N}(\mathrm{d}\tau, \mathrm{d}x) \\
&+ \int_{-\infty}^{t}\int_{|x|_U\geq 1} T_{\mathcal{A}}(t, \tau) G(\tau, \xi(\tau-), x) N(\mathrm{d}\tau, \mathrm{d}x),
\end{aligned}
$$

where $B[0, r] := \{Y \in \mathcal{L}^2(\mathbf{P}; \mathbb{H}) : \|Y\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \leq r\}$ and $r = \dfrac{2KM\sqrt{1 + 2\omega + 2b}}{\omega - 2K\mathcal{L}\sqrt{1 + 2\omega + 2b}}$.

(ii) If $\mathcal{L} < \left\{\dfrac{\omega}{2K\sqrt{2 + 8\omega + 4b}} \wedge \dfrac{\omega}{2K\sqrt{1 + 10\omega + 2b}}\right\}$ and f, g, F, G satisfy (E1′), (E2′) *and* (E3) *additionally, then we have $\mathfrak{M}^u_{(\mathcal{A},f,g,F,G)} \subseteq \tilde{\mathfrak{M}}^u_\xi$ and the solution ξ is strongly compatible in distribution (i.e. $\mathfrak{M}_{(\mathcal{A},f,g,F,G)} \subseteq \tilde{\mathfrak{M}}_\xi$).*

Proof. The proof is similar to Theorem 4.6 in [21]. □

Corollary 2.32. *Consider Eq* (2.7)*. Assume that the conditions of* Theorem 2.31 *hold.*

(i) If $\mathcal{A}, f, g, F, G$ are jointly stationary (respectively, τ-periodic, quasi-periodic with the spectrum of frequencies $\nu_1, \nu_2, \ldots, \nu_m$, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in $t \in \mathbb{R}$ uniformly w.r.t. $Y \in \mathbb{H}$ on every bounded subset, then so is the unique bounded solution ξ of Eq (2.7) *in distribution.*

(ii) If $\mathcal{A}, f, g, F, G$ are jointly pseudo-periodic (respectively, pseudo-recurrent) and jointly Lagrange stable in $t \in \mathbb{R}$ uniformly w.r.t. $Y \in \mathbb{H}$ on every bounded subset, then the unique bounded solution ξ of Eq (2.7) *is pseudo-periodic (respectively, pseudo-recurrent) in distribution.*

Proof. This statement follows from Theorems 2.16, 2.31 and Remark 2.18. □

3. Averaging principle for linear equations

Let ε_0 be some fixed positive number. We now study an averaging principle for the following equation

$$
\begin{aligned}
\mathrm{d}Y(t) = &\varepsilon(\mathcal{A}(t)Y(t) + f(t))\mathrm{d}t + \sqrt{\varepsilon} g(t)\mathrm{d}W(t) \\
&+ \sqrt{\varepsilon}\int_{|x|_U<1} F(t, x)\widetilde{N}(\mathrm{d}t, \mathrm{d}x) + \sqrt{\varepsilon}\int_{|x|_U\geq 1} G(t, x) N(\mathrm{d}t, \mathrm{d}x),
\end{aligned}
\tag{3.1}
$$

where $\mathcal{A} \in C(\mathbb{R}, L(\mathbb{H}))$, $f \in C(\mathbb{R}, \mathcal{L}^2(\mathbf{P}; \mathbb{H}))$, $g \in C(\mathbb{R}, \mathcal{L}^2(\mathbf{P}; L(U, \mathbb{H})))$, $F, G \in C(\mathbb{R}, \mathcal{L}^2(\mathbf{P}_\nu; \mathbb{H}))$, and $\varepsilon \in (0, \varepsilon_0]$ is a small parameter. Here W and N are the Lévy-Itô decomposition components of the two-sided Lévy process L as in Section 2.3.

Denote by Ψ the family of all decreasing, positive bounded functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim\limits_{t\to+\infty} \psi(t) = 0$. Let $\mathcal{A} \in L(\mathbb{H})$. Denote by $\sigma(\mathcal{A})$ the spectrum of $\mathcal{A}$. We respectively impose the following conditions on $\mathcal{A}, f, g, F, G$:

(A1) $\mathcal{A} \in C(\mathbb{R}, L(\mathbb{H}))$ and there exists $\bar{\mathcal{A}} \in L(\mathbb{H})$ such that

$$\lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} \mathcal{A}(s)\mathrm{d}s = \bar{\mathcal{A}}$$

uniformly w.r.t. $t \in \mathbb{R}$;

(A2) $f \in C(\mathbb{R}, \mathcal{L}^2(\mathbf{P}; \mathbb{H}))$ and there exist functions $\omega_1 \in \Psi$, $\bar{f} \in \mathcal{L}^2(\mathbf{P}; \mathbb{H})$ such that

$$\frac{1}{T} \left\| \int_t^{t+T} [f(s) - \bar{f}]\mathrm{d}s \right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \leq \omega_1(T) \tag{3.2}$$

for any $T > 0$ and $t \in \mathbb{R}$;

(A3) $g \in C(\mathbb{R}, \mathcal{L}^2(\mathbf{P}; L(U, \mathbb{H})))$ and there exist functions $\omega_2 \in \Psi$, $\bar{g} \in \mathcal{L}^2(\mathbf{P}; L(U, \mathbb{H}))$ such that

$$\frac{1}{T} \int_t^{t+T} \mathbb{E}\|(g(s) - \bar{g})Q^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})}\mathrm{d}s \leq \omega_2(T) \tag{3.3}$$

for any $T > 0$ and $t \in \mathbb{R}$;

(A4) $F \in C(\mathbb{R}, \mathcal{L}^2(\mathbf{P}_\nu; \mathbb{H}))$ and there exist functions $\omega_3 \in \Psi$, $\bar{F} \in \mathcal{L}^2(\mathbf{P}_\nu; \mathbb{H})$ such that

$$\frac{1}{T} \int_t^{t+T} \int_{|x|_U<1} \mathbb{E}|F(s, x) - \bar{F}(x)|^2 \nu(\mathrm{d}x)\mathrm{d}s \leq \omega_3(T) \tag{3.4}$$

for any $T > 0$ and $t \in \mathbb{R}$;

(A5) $G \in C(\mathbb{R}, \mathcal{L}^2(\mathbf{P}_\nu; \mathbb{H}))$ and there exist functions $\omega_4 \in \Psi$, $\bar{G} \in \mathcal{L}^2(\mathbf{P}_\nu; \mathbb{H})$ such that

$$\frac{1}{T} \int_t^{t+T} \int_{|x|_U\geq 1} \mathbb{E}|G(s, x) - \bar{G}(x)|^2 \nu(\mathrm{d}x)\mathrm{d}s \leq \omega_4(T) \tag{3.5}$$

for any $T > 0$ and $t \in \mathbb{R}$.

Theorem 3.1. ([36, Chapter IV]) *Suppose that* $\mathcal{A} \in C_b(\mathbb{R}, L(\mathfrak{B}))$ *and*

$$\lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} \mathcal{A}(s)\mathrm{d}s = \bar{\mathcal{A}}$$

uniformly w.r.t. $t \in \mathbb{R}$ *and the operator* $\bar{\mathcal{A}}$ *is Hurwitz, i.e.* $\mathcal{R}e\ \lambda < 0$ *for any* $\lambda \in \sigma(\bar{\mathcal{A}})$.

Then the following statements hold:

(i) There exists a positive constant $\alpha \leq \varepsilon_0$ *such that the equation*

$$\mathrm{d}y(t) = \mathcal{A}_\varepsilon(t)y(t)\mathrm{d}t,$$

where $\mathcal{A}_\varepsilon(t) := \mathcal{A}(\frac{t}{\varepsilon})$ *for any* $t \in \mathbb{R}$, *is uniformly asymptotically stable for any* $\varepsilon \in (0, \alpha]$.

(ii) There exists $\gamma_0 > 0$ such that

$$\lim_{\varepsilon\to 0}\sup_{(t\geq\tau;\ t,\tau\in\mathbb{R})} e^{\gamma_0(t-\tau)}\|T_{\mathcal{A}_\varepsilon}(t,\tau)-T_{\bar{\mathcal{A}}}(t,\tau)\|_{\mathfrak{B}}=0. \tag{3.6}$$

Remark 3.2. (i) Under the conditions of Theorem 3.1 there are positive constants α, K and ω such that

$$\|T_{\mathcal{A}_\varepsilon}(t,\tau)\|_{\mathfrak{B}}, \|T_{\bar{\mathcal{A}}}(t,\tau)\|_{\mathfrak{B}} \leq Ke^{-\omega(t-\tau)} \tag{3.7}$$

for any $t\geq\tau$ and $\varepsilon\in(0,\alpha]$.

(ii) By Eq (3.6) there exists a function $\mathcal{K}:(0,\alpha)\to\mathbb{R}_+$ such that $\mathcal{K}(\varepsilon)\to 0$ as $\varepsilon\to 0$ and

$$\|T_{\mathcal{A}_\varepsilon}(t,\tau)-T_{\bar{\mathcal{A}}}(t,\tau)\|_{\mathfrak{B}}\leq\mathcal{K}(\varepsilon)e^{-\gamma_0(t-\tau)}$$

for any $t\geq\tau$ $(t,\tau\in\mathbb{R})$.

Lemma 3.3. ([18]) *Let $f_\varepsilon\in C(\mathbb{R},\mathfrak{B})$ for $\varepsilon\in(0,\alpha]$ be functions satisfying the following conditions:*

(i) there exists a positive constant A such that $|f_\varepsilon(t)|_{\mathfrak{B}}\leq A$ for any $t\in\mathbb{R}$ and $\varepsilon\in(0,\alpha]$;

(ii) for any $l>0$

$$\lim_{\varepsilon\to 0}\sup_{|s|\leq l,\ t\in\mathbb{R}}\left|\int_t^{t+s} f_\varepsilon(\sigma)\mathrm{d}\sigma\right|_{\mathfrak{B}}=0.$$

Then for any $\omega>0$ we have

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^t e^{-\omega(t-\tau)}f_\varepsilon(\tau)\mathrm{d}\tau\right|_{\mathfrak{B}}=0.$$

Remark 3.4. ([18]) If the function $f\in C_b(\mathbb{R},\mathfrak{B})$ and $\bar{f}\in\mathfrak{B}$ are such that

$$\lim_{L\to+\infty}\frac{1}{L}\left|\int_t^{t+L}[f(s)-\bar{f}]\mathrm{d}s\right|=0$$

uniformly w.r.t. $t\in\mathbb{R}$, then the function $f_\varepsilon(\sigma):=f(\frac{\sigma}{\varepsilon})-\bar{f}$ satisfies the conditions of Lemma 3.3. Similarly, if the function g (respectively, F, G) in (A3) (respectively, (A4), (A5)) is $\mathcal{L}^2$-bounded, then the function $f_\varepsilon(\sigma):=\mathbb{E}\|(g(\frac{\sigma}{\varepsilon})-\bar{g})Q^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})}$ (respectively, $f_\varepsilon(\sigma):=\int_{|x|_U<1}\mathbb{E}|F(\frac{\sigma}{\varepsilon},x)-\bar{F}(x)|^2\nu(\mathrm{d}x)$, $f_\varepsilon(\sigma):=\int_{|x|_U\geq 1}\mathbb{E}|G(\frac{\sigma}{\varepsilon},x)-\bar{G}(x)|^2\nu(\mathrm{d}x)$) satisfies the conditions of Lemma 3.3.

Theorem 3.5. *Consider Eq (3.1). Suppose that $\mathcal{A}\in C_b(\mathbb{R},L(\mathbb{H}))$, $f\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$, $g\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};L(U,\mathbb{H})))$, $F,G\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H}))$ and conditions (A1)–(A5) are satisfied. Suppose further that $\bar{\mathcal{A}}$ in (A1) is Hurwitz such that (3.6) and (3.7) hold. Then for any $\varepsilon\in(0,\alpha]$, Eq (3.1) has a unique solution $\varphi_\varepsilon\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ and it is strongly compatible in distribution (i.e. $\mathfrak{M}_{(\mathcal{A},f,g,F,G)}\subseteq\tilde{\mathfrak{M}}_{\varphi_\varepsilon}$) and $\mathfrak{M}^u_{(\mathcal{A},f,g,F,G)}\subseteq\tilde{\mathfrak{M}}^u_{\varphi_\varepsilon}$. Besides we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\beta(\mathcal{L}(\varphi_\varepsilon(\frac{t}{\varepsilon})),\mathcal{L}(\bar{\phi}(t)))=0,$$

where $\mathcal{L}(X)$ denotes the law of random variable X and $\bar{\phi}$ is the unique stationary solution of

$$\mathrm{d}Y(t)=(\bar{\mathcal{A}}Y(t)+\bar{f})\mathrm{d}t+\bar{g}\mathrm{d}W(t)+\int_{|x|_U<1}\bar{F}(x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x)+\int_{|x|_U\geq 1}\bar{G}(x)N(\mathrm{d}t,\mathrm{d}x). \tag{3.8}$$

Proof. Consider the following equations

$$\begin{aligned} \mathrm{d}Y_\varepsilon(t) = (\mathcal{A}_\varepsilon(t)Y_\varepsilon(t) + f_\varepsilon(t))\mathrm{d}t + g_\varepsilon(t)\mathrm{d}W_\varepsilon(t) \\ + \int_{|x|_U<1} F_\varepsilon(t,x)\widetilde{N}_\varepsilon(\mathrm{d}t,\mathrm{d}x) + \int_{|x|_U\geq 1} G_\varepsilon(t,x)N_\varepsilon(\mathrm{d}t,\mathrm{d}x) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \mathrm{d}Y_\varepsilon(t) = (\mathcal{A}_\varepsilon(t)Y_\varepsilon(t) + f_\varepsilon(t))\mathrm{d}t + g_\varepsilon(t)\mathrm{d}W(t) \\ + \int_{|x|_U<1} F_\varepsilon(t,x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \int_{|x|_U\geq 1} G_\varepsilon(t,x)N(\mathrm{d}t,\mathrm{d}x), \end{aligned} \tag{3.10}$$

where $\mathcal{A}_\varepsilon(t) := \mathcal{A}(\frac{t}{\varepsilon})$, $f_\varepsilon(t) := f(\frac{t}{\varepsilon})$, $g_\varepsilon(t) := g(\frac{t}{\varepsilon})$, $F_\varepsilon(t,x) := F(\frac{t}{\varepsilon},x)$ and $G_\varepsilon(t,x) := G(\frac{t}{\varepsilon},x)$ for $t\in\mathbb{R}$ and $\varepsilon\in(0,\varepsilon_0]$. Here $W_\varepsilon(t) := \sqrt{\varepsilon}W(\frac{t}{\varepsilon})$, $N_\varepsilon(t,x) := \sqrt{\varepsilon}N(\frac{t}{\varepsilon},x)$ and $\widetilde{N}_\varepsilon(t,x) := \sqrt{\varepsilon}\widetilde{N}(\frac{t}{\varepsilon},x)$. By Theorem 2.29, Eq (3.9) has a unique bounded solution $\psi_\varepsilon\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ defined by equality

$$\begin{aligned} \psi_\varepsilon(t) = \int_{-\infty}^t T_{\mathcal{A}_\varepsilon}(t,\tau)f_\varepsilon(\tau)\mathrm{d}\tau + \int_{-\infty}^t T_{\mathcal{A}_\varepsilon}(t,\tau)g_\varepsilon(\tau)\mathrm{d}W_\varepsilon(\tau) \\ + \int_{-\infty}^t\int_{|x|_U<1} T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau,x)\widetilde{N}_\varepsilon(\mathrm{d}\tau,\mathrm{d}x) + \int_{-\infty}^t\int_{|x|_U\geq 1} T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x)N_\varepsilon(\mathrm{d}\tau,\mathrm{d}x) \end{aligned}$$

and the solution is strongly compatible in distribution (i.e. $\mathfrak{M}_{(\mathcal{A}_\varepsilon,f_\varepsilon,g_\varepsilon,F_\varepsilon,G_\varepsilon)}\subseteq\tilde{\mathfrak{M}}_{\psi_\varepsilon}$) and $\mathfrak{M}^u_{(\mathcal{A}_\varepsilon,f_\varepsilon,g_\varepsilon,F_\varepsilon,G_\varepsilon)}\subseteq\tilde{\mathfrak{M}}^u_{\psi_\varepsilon}$. Let $\varphi_\varepsilon(t) := \psi_\varepsilon(\varepsilon t)$, $t\in\mathbb{R}$. Then $\varphi_\varepsilon\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ is the unique bounded solution of Eq (3.1). If $\{t_n\}\in\mathfrak{M}_{(\mathcal{A},f,g,F,G)}$ (respectively, $\{t_n\}\in\mathfrak{M}^u_{(\mathcal{A},f,g,F,G)}$), then by Theorem 2.29 $\{\varepsilon t_n\}\in\mathfrak{M}_{(\mathcal{A}_\varepsilon,f_\varepsilon,g_\varepsilon,F_\varepsilon,G_\varepsilon)}\subseteq\tilde{\mathfrak{M}}_{\psi_\varepsilon}$ (respectively, $\{\varepsilon t_n\}\in\mathfrak{M}^u_{(\mathcal{A}_\varepsilon,f_\varepsilon,g_\varepsilon,F_\varepsilon,G_\varepsilon)}\subseteq\tilde{\mathfrak{M}}^u_{\psi_\varepsilon}$). Further we have $\{t_n\}\in\tilde{\mathfrak{M}}_{\varphi_\varepsilon}$ (respectively, $\{t_n\}\in\tilde{\mathfrak{M}}^u_{\varphi_\varepsilon}$).

We now prove the solution φ_ε of the original Eq (3.1) converges to the stationary solution $\bar{\phi}$ of the averaged Eq (3.8) uniformly in $t\in\mathbb{R}$ in distribution sense. By Theorem 2.29, Eq (3.10) has a unique bounded solution $\phi_\varepsilon\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ defined by equality

$$\begin{aligned} \phi_\varepsilon(t) = \int_{-\infty}^t T_{\mathcal{A}_\varepsilon}(t,\tau)f_\varepsilon(\tau)\mathrm{d}\tau + \int_{-\infty}^t T_{\mathcal{A}_\varepsilon}(t,\tau)g_\varepsilon(\tau)\mathrm{d}W(\tau) \\ + \int_{-\infty}^t\int_{|x|_U<1} T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau,x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x) \\ + \int_{-\infty}^t\int_{|x|_U\geq 1} T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x)N(\mathrm{d}\tau,\mathrm{d}x). \end{aligned} \tag{3.11}$$

By Theorem 2.29, Eq (3.8) has a unique bounded and stationary solution $\bar{\phi}$, which is given by the formula

$$\begin{aligned} \bar{\phi}(t) = \int_{-\infty}^t T_{\bar{\mathcal{A}}}(t,\tau)\bar{f}\mathrm{d}\tau + \int_{-\infty}^t T_{\bar{\mathcal{A}}}(t,\tau)\bar{g}\mathrm{d}W(\tau) \\ + \int_{-\infty}^t\int_{|x|_U<1} T_{\bar{\mathcal{A}}}(t,\tau)\bar{F}(x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x) + \int_{-\infty}^t\int_{|x|_U\geq 1} T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)N(\mathrm{d}\tau,\mathrm{d}x), \end{aligned} \tag{3.12}$$

where $T_{\bar{\mathcal{A}}}(t,\tau)=\exp\left\{\bar{\mathcal{A}}(t-\tau)\right\}$ for $t,\tau\in\mathbb{R}$. From (3.11) and (3.12), by the basic inequality $\left(\sum_{i=1}^{n}a_i\right)^2\le n\left(\sum_{i=1}^{n}a_i^2\right), n\in\mathbb{N}$, we have

$$
\begin{aligned}
&\mathbb{E}|\phi_\varepsilon(t)-\bar{\phi}(t)|^2 \\
&=\mathbb{E}\bigg|\int_{-\infty}^{t}T_{\mathcal{A}_\varepsilon}(t,\tau)f_\varepsilon(\tau)\mathrm{d}\tau+\int_{-\infty}^{t}T_{\mathcal{A}_\varepsilon}(t,\tau)g_\varepsilon(\tau)\mathrm{d}W(\tau) \\
&\quad+\int_{-\infty}^{t}\int_{|x|_U<1}T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau,x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)+\int_{-\infty}^{t}\int_{|x|_U\ge1}T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x)N(\mathrm{d}\tau,\mathrm{d}x) \\
&\quad-\int_{-\infty}^{t}T_{\bar{\mathcal{A}}}(t,\tau)\bar{f}\mathrm{d}\tau-\int_{-\infty}^{t}T_{\bar{\mathcal{A}}}(t,\tau)\bar{g}\mathrm{d}W(\tau) \\
&\quad-\int_{-\infty}^{t}\int_{|x|_U<1}T_{\bar{\mathcal{A}}}(t,\tau)\bar{F}(x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)-\int_{-\infty}^{t}\int_{|x|_U\ge1}T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)N(\mathrm{d}\tau,\mathrm{d}x)\bigg|^2 \\
&\le4\mathbb{E}\left|\int_{-\infty}^{t}[T_{\mathcal{A}_\varepsilon}(t,\tau)f_\varepsilon(\tau)-T_{\bar{\mathcal{A}}}(t,\tau)\bar{f}]\mathrm{d}\tau\right|^2+4\mathbb{E}\left|\int_{-\infty}^{t}[T_{\mathcal{A}_\varepsilon}(t,\tau)g_\varepsilon(\tau)-T_{\bar{\mathcal{A}}}(t,\tau)\bar{g}]\mathrm{d}W(\tau)\right|^2 \\
&\quad+4\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U<1}[T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau,x)-T_{\bar{\mathcal{A}}}(t,\tau)\bar{F}(x)]\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\right|^2 \\
&\quad+4\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U\ge1}[T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x)-T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)]N(\mathrm{d}\tau,\mathrm{d}x)\right|^2 \\
&=:I_1(t,\varepsilon)+I_2(t,\varepsilon)+I_3(t,\varepsilon)+I_4(t,\varepsilon).
\end{aligned}
\tag{3.13}
$$

Note that

$$
\begin{aligned}
I_1(t,\varepsilon)&=4\mathbb{E}\left|\int_{-\infty}^{t}[T_{\mathcal{A}_\varepsilon}(t,\tau)f_\varepsilon(\tau)-T_{\bar{\mathcal{A}}}(t,\tau)\bar{f}]\mathrm{d}\tau\right|^2 \\
&\le8\mathbb{E}\left|\int_{-\infty}^{t}T_{\mathcal{A}_\varepsilon}(t,\tau)(f_\varepsilon(\tau)-\bar{f})\mathrm{d}\tau\right|^2+8\mathbb{E}\left|\int_{-\infty}^{t}[T_{\mathcal{A}_\varepsilon}(t,\tau)\bar{f}-T_{\bar{\mathcal{A}}}(t,\tau)\bar{f}]\mathrm{d}\tau\right|^2 \\
&=:8I_{11}(t,\varepsilon)+8I_{12}(t,\varepsilon).
\end{aligned}
\tag{3.14}
$$

Since

$$\frac{\partial T_{\mathcal{A}}(t,\tau)}{\partial\tau}=-T_{\mathcal{A}}(t,\tau)\mathcal{A}(\tau),$$

we have

$$\left\|\frac{\partial T_{\mathcal{A}_\varepsilon}(t,t+s)}{\partial s}\right\|\le K\|\mathcal{A}\|_\infty e^{\omega s}$$

for any $t\in\mathbb{R}$ and $s<0$.

Similar to the proof of [18, Theorem 3.9], by making the change of variable $s:=\tau-t$, integrating by parts and letting l be an arbitrary positive number we have

$$\left\|\int_{-\infty}^{t}T_{\mathcal{A}_\varepsilon}(t,\tau)(f_\varepsilon(\tau)-\bar{f})\mathrm{d}\tau\right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \tag{3.15}$$

$$
\begin{aligned}
&= \left\| \int_{-\infty}^{0} T_{\mathcal{A}_\varepsilon}(t,t+s)(f_\varepsilon(t+s)-\bar{f})\mathrm{d}s \right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \\
&= \left\| \int_{-\infty}^{0} T_{\mathcal{A}_\varepsilon}(t,t+s)\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{t}^{t+s} [f_\varepsilon(\sigma)-\bar{f}]\mathrm{d}\sigma \right)\mathrm{d}s \right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \\
&= \left\| -\int_{-\infty}^{0} \frac{\partial T_{\mathcal{A}_\varepsilon}(t,t+s)}{\partial s}\left(\int_{t}^{t+s} [f_\varepsilon(\sigma)-\bar{f}]\mathrm{d}\sigma \right)\mathrm{d}s \right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \\
&= \left\| -\int_{-\infty}^{-l} \frac{\partial T_{\mathcal{A}_\varepsilon}(t,t+s)}{\partial s}\left(\int_{t}^{t+s} [f_\varepsilon(\sigma)-\bar{f}]\mathrm{d}\sigma \right)\mathrm{d}s \right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \\
&\quad + \left\| \int_{-l}^{0} \frac{\partial T_{\mathcal{A}_\varepsilon}(t,t+s)}{\partial s}\left(\int_{t}^{t+s} [f_\varepsilon(\sigma)-\bar{f}]\mathrm{d}\sigma \right)\mathrm{d}s \right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \\
&\leq K\|\mathcal{A}\|_\infty \left(2\|f\|_\infty e^{-\omega l}\left(\frac{l}{\omega}+\frac{1}{\omega^2}\right) + \frac{1}{\omega}(1-e^{-\omega l}) \sup_{|s|\leq l,\ t\in\mathbb{R}} \left\| \int_{t}^{t+s} [f_\varepsilon(\sigma)-\bar{f}]\mathrm{d}\sigma \right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \right).
\end{aligned}
$$

By Remark 3.4, passing to limit in (3.15) as $\varepsilon \to 0$ we obtain

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0} \sup_{t\in\mathbb{R}} \left\| -\int_{-\infty}^{0} \frac{\partial T_{\mathcal{A}_\varepsilon}(t,t+s)}{\partial s}\left(\int_{t}^{t+s} [f_\varepsilon(\sigma)-\bar{f}]\mathrm{d}\sigma \right)\mathrm{d}s \right\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})} \\
&\leq 2K\|\mathcal{A}\|_\infty \|f\|_\infty e^{-\omega l}\left(\frac{l}{\omega}+\frac{1}{\omega^2}\right).
\end{aligned}
$$

Since l is an arbitrary positive number, by letting $l \to \infty$ we get

$$
\lim_{\varepsilon\to 0} \sup_{t\in\mathbb{R}} I_{11}(t,\varepsilon) = 0.
$$

Note that by Cauchy-Schwarz inequality, we have

$$
I_{12}(t,\varepsilon) := \mathbb{E}\left| \int_{-\infty}^{t} [T_{\mathcal{A}_\varepsilon}(t,\tau) - T_{\bar{\mathcal{A}}}(t,\tau)]\bar{f}\mathrm{d}\tau \right|^2 \leq \left(\frac{\|\bar{f}\|_{\mathcal{L}^2(\mathbf{P};\mathbb{H})}\mathcal{K}(\varepsilon)}{\gamma_0} \right)^2 \to 0
$$

as $\varepsilon \to 0$. Consequently,

$$
\lim_{\varepsilon\to 0} \sup_{t\in\mathbb{R}} I_1(t,\varepsilon) = 0.
$$

As for $I_2(t,\varepsilon)$, by Itô's isometry property and Remark 3.2-(ii), we have

$$
\begin{aligned}
I_2(t,\varepsilon) &= 4\mathbb{E}\left| \int_{-\infty}^{t} [T_{\mathcal{A}_\varepsilon}(t,\tau)g_\varepsilon(\tau) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{g}]\mathrm{d}W(\tau) \right|^2 \\
&= 4\mathbb{E}\int_{-\infty}^{t} \|(T_{\mathcal{A}_\varepsilon}(t,\tau)g_\varepsilon(\tau) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{g})Q^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})}\mathrm{d}\tau \\
&\leq 8\mathbb{E}\int_{-\infty}^{t} \|T_{\mathcal{A}_\varepsilon}(t,\tau)(g_\varepsilon(\tau) - \bar{g})Q^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})}\mathrm{d}\tau \\
&\quad + 8\mathbb{E}\int_{-\infty}^{t} \|\left(T_{\mathcal{A}_\varepsilon}(t,\tau) - T_{\bar{\mathcal{A}}}(t,\tau)\right)\bar{g}Q^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})}\mathrm{d}\tau
\end{aligned} \tag{3.16}
$$

$$\leq 8K^2 \int_{-\infty}^{t} e^{-2\omega(t-\tau)} \mathbb{E}\|(g_\varepsilon(\tau) - \bar{g})Q^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})} \mathrm{d}\tau + \frac{4(\mathcal{K}(\varepsilon))^2}{\gamma_0} \|\bar{g}Q^{\frac{1}{2}}\|^2_{\mathcal{L}^2(\mathbf{P};L_2(U,\mathbb{H}))}.$$

Since by Lemma 3.3 the integral

$$\int_{-\infty}^{t} e^{-2\omega(t-\tau)} \mathbb{E}\|(g_\varepsilon(\tau) - \bar{g})Q^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})} \mathrm{d}\tau$$

goes to 0 as $\varepsilon \to 0$ uniformly w.r.t. $t \in \mathbb{R}$, taking to the limit in (3.16) we obtain

$$\lim_{\varepsilon \to 0} \sup_{t \in \mathbb{R}} I_2(t,\varepsilon) = 0.$$

Note that

$$\begin{aligned} I_3(t,\varepsilon) &= 4\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U<1} [T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau,x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{F}(x)]\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \\ &= 4\mathbb{E}\int_{-\infty}^{t}\int_{|x|_U<1} |T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau,x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{F}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau \\ &\leq 8\mathbb{E}\int_{-\infty}^{t}\int_{|x|_U<1} |T_{\mathcal{A}_\varepsilon}(t,\tau)(F_\varepsilon(\tau,x) - \bar{F}(x))|^2\nu(\mathrm{d}x)\mathrm{d}\tau \\ &\quad + 8\mathbb{E}\int_{-\infty}^{t}\int_{|x|_U<1} |(T_{\mathcal{A}_\varepsilon}(t,\tau) - T_{\bar{\mathcal{A}}}(t,\tau))\bar{F}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau \\ &\leq 8K^2\int_{-\infty}^{t}\int_{|x|_U<1} e^{-2\omega(t-\tau)}\mathbb{E}|F_\varepsilon(\tau,x) - \bar{F}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau + \frac{4(\mathcal{K}(\varepsilon))^2}{\gamma_0}\|\bar{F}\|^2_{\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H})}. \end{aligned} \tag{3.17}$$

According to Lemma 3.3 we have

$$\lim_{\varepsilon \to 0} \sup_{t \in \mathbb{R}} \int_{-\infty}^{t}\int_{|x|_U<1} e^{-2\omega(t-\tau)}\mathbb{E}|F_\varepsilon(\tau,x) - \bar{F}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau = 0. \tag{3.18}$$

Passing to the limit in (3.17) and considering (3.18), we get

$$\lim_{\varepsilon \to 0} \sup_{t \in \mathbb{R}} I_3(t,\varepsilon) = 0.$$

By properties of integrals for Poisson random measures, we obtain

$$\begin{aligned} I_4(t,\varepsilon) &= 4\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1} [T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)]N(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \\ &= 4\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1} [T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)]\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x) \\ &\quad + \int_{-\infty}^{t}\int_{|x|_U\geq 1} [T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)]\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2 \\ &\leq 8\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1} [T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)]\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&+ 8\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U\geq 1}[T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)]\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2\\
&=: I_{41}(t,\varepsilon) + I_{42}(t,\varepsilon).
\end{aligned}
$$

Using the similar arguments as (3.17), we get

$$
\begin{aligned}
I_{41}(t,\varepsilon) \leq{}& 16K^2\int_{-\infty}^{t}\int_{|x|_U\geq 1} e^{-2\omega(t-\tau)}\mathbb{E}|G_\varepsilon(\tau,x) - \bar{G}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau \\
&+ \frac{8(\mathcal{K}(\varepsilon))^2}{\gamma_0}\|\bar{G}\|^2_{\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H})}.
\end{aligned} \tag{3.20}
$$

By Cauchy-Schwarz inequality we have

$$
\begin{aligned}
I_{42}(t,\varepsilon) ={}& 8\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U\geq 1}[T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(x)]\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2 \\
\leq{}& 16\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U\geq 1} T_{\mathcal{A}_\varepsilon}(t,\tau)(G_\varepsilon(\tau,x) - \bar{G}(x))\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2 \\
&+ 16\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U\geq 1} (T_{\mathcal{A}_\varepsilon}(t,\tau) - T_{\bar{\mathcal{A}}}(t,\tau))\bar{G}(x)\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2 \\
\leq{}& 16\int_{-\infty}^{t}\int_{|x|_U\geq 1} K^2e^{-\omega(t-\tau)}\nu(\mathrm{d}x)\mathrm{d}\tau\cdot\int_{-\infty}^{t}\int_{|x|_U\geq 1} e^{-\omega(t-\tau)}\mathbb{E}|G_\varepsilon(\tau,x) - \bar{G}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau \\
&+ 16\int_{-\infty}^{t}\int_{|x|_U\geq 1} (\mathcal{K}(\varepsilon))^2e^{-\gamma_0(t-\tau)}\nu(\mathrm{d}x)\mathrm{d}\tau\cdot\int_{-\infty}^{t}\int_{|x|_U\geq 1} e^{-\gamma_0(t-\tau)}\mathbb{E}|\bar{G}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau \\
\leq{}& \frac{16K^2b}{\omega}\int_{-\infty}^{t}\int_{|x|_U\geq 1} e^{-\omega(t-\tau)}\mathbb{E}|G_\varepsilon(\tau,x) - \bar{G}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau + \frac{16(\mathcal{K}(\varepsilon))^2b}{\gamma_0^2}\|\bar{G}\|^2_{\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H})}.
\end{aligned} \tag{3.21}
$$

According to (3.19)–(3.21), we obtain

$$
\begin{aligned}
I_4(t,\varepsilon) \leq{}& 16K^2\left(1 + \frac{b}{\omega}\right)\int_{-\infty}^{t}\int_{|x|_U\geq 1} e^{-\omega(t-\tau)}\mathbb{E}|G_\varepsilon(\tau,x) - \bar{G}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau \\
&+ 8(\mathcal{K}(\varepsilon))^2\left(\frac{1}{\gamma_0} + \frac{2b}{\gamma_0^2}\right)\|\bar{G}\|^2_{\mathcal{L}^2(\mathbf{P}_\nu;\mathbb{H})}.
\end{aligned} \tag{3.22}
$$

By Lemma 3.3 we have

$$
\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\int_{|x|_U\geq 1} e^{-\omega(t-\tau)}\mathbb{E}|G_\varepsilon(\tau,x) - \bar{G}(x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau = 0,
$$

so taking limits in (3.22) we obtain

$$
\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}} I_4(t,\varepsilon) = 0.
$$

Consequently we have

$$
\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t) - \bar{\phi}(t)|^2 = 0. \tag{3.23}
$$

Since the $\mathcal{L}^2$ convergence implies convergence in probability, it follows from (3.23) that

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\beta(\mathcal{L}(\phi_\varepsilon(t)),\mathcal{L}(\bar{\phi}(t)))=0.$$

On the other hand taking into consideration that $\mathcal{L}(W)=\mathcal{L}(W_\varepsilon)$ and $\mathcal{L}(N)=\mathcal{L}(N_\varepsilon)$ with compensated Poisson random measure $\widetilde{N}$ we have $\mathcal{L}(\psi_\varepsilon(t))=\mathcal{L}(\phi_\varepsilon(t))$ for any $t\in\mathbb{R}$, and by the relationship between φ_ε and ψ_ε (i.e. $\varphi_\varepsilon(t):=\psi_\varepsilon(\varepsilon t)$, $t\in\mathbb{R}$) we get

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\beta(\mathcal{L}(\varphi_\varepsilon(\frac{t}{\varepsilon})),\mathcal{L}(\bar{\phi}(t)))=0.$$

The proof is complete. □

Corollary 3.6. *Under the assumptions of* Theorem 3.5*, it follows from Theorems 2.16 and 3.5 that*

(i) if $\mathcal{A},f,g,F,G$ are jointly stationary (respectively, τ-periodic, quasi-periodic with the spectrum of frequencies $\nu_1,\dots,\nu_k$, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable), then Eq (3.1) has a unique solution $\varphi_\varepsilon\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ which is stationary (respectively, τ-periodic, quasi-periodic with the spectrum of frequencies $\nu_1,\dots,\nu_k$, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in distribution;

(ii) if $\mathcal{A},f,g,F,G$ are Lagrange stable and jointly pseudo-periodic (respectively, pseudo-recurrent), then Eq (3.1) has a unique solution $\varphi_\varepsilon\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ which is pseudo-periodic (respectively, pseudo-recurrent) in distribution;

(iii)

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\beta(\mathcal{L}(\varphi_\varepsilon(\frac{t}{\varepsilon}),\mathcal{L}(\bar{\phi}(t)))=0.$$

4. Averaging principle for semilinear equations

Let ε_0 be some fixed positive number. Consider the stochastic differential equation driven by Lévy noise of type

$$\begin{aligned}dY(t)=\varepsilon\,(\mathcal{A}(t)Y(t)+f(t,Y(t)))\,dt+\sqrt{\varepsilon}g(t,Y(t))dW(t)\\ +\sqrt{\varepsilon}\int_{|x|_U<1}F(t,Y(t-),x)\widetilde{N}(dt,dx)+\sqrt{\varepsilon}\int_{|x|_U\geq 1}G(t,Y(t-),x)N(dt,dx),\end{aligned}\tag{4.1}$$

where $\mathcal{A}\in C(\mathbb{R},L(\mathbb{H}))$, $f\in C(\mathbb{R}\times\mathbb{H},\mathbb{H})$, $g\in C(\mathbb{R}\times\mathbb{H},L(U,\mathbb{H}))$, $F,G\in C(\mathbb{R}\times\mathbb{H},\mathcal{L}^2(\nu;\mathbb{H}))$; $\varepsilon\in(0,\varepsilon_0]$ is a small parameter; W as well as N is the Lévy-Itô decomposition components of the two-sided Lévy process L with assumptions stated in Section 2.3. Assume that conditions (E1)–(E2′) in Section 2.4 are satisfied. We also impose the following additional conditions on $\mathcal{A},f,g,F,G$:

(H1) there exists $\bar{\mathcal{A}}\in L(\mathbb{H})$ such that

$$\lim_{T\to+\infty}\frac{1}{T}\int_t^{t+T}\mathcal{A}(s)ds=\bar{\mathcal{A}}$$

uniformly w.r.t. $t\in\mathbb{R}$;

(H2) there exist functions $\omega_1 \in \Psi$ and $\bar{f} \in C(\mathbb{H}, \mathbb{H})$ such that

$$\frac{1}{T}\left|\int_t^{t+T} [f(s,y) - \bar{f}(y)]\mathrm{d}s\right| \le \omega_1(T)(1+|y|)$$

for any $T > 0$, $y \in \mathbb{H}$ and $t \in \mathbb{R}$;

(H3) there exist functions $\omega_2 \in \Psi$ and $\bar{g} \in C(\mathbb{H}, L(U,\mathbb{H}))$ such that

$$\frac{1}{T}\int_t^{t+T} \|(g(s,y) - \bar{g}(y))Q^{\frac{1}{2}}\|^2_{L_2(U,\mathbb{H})}\mathrm{d}s \le \omega_2(T)(1+|y|^2)$$

for any $T > 0$, $y \in \mathbb{H}$ and $t \in \mathbb{R}$;

(H4) there exist functions $\omega_3 \in \Psi$ and $\bar{F} \in C(\mathbb{H}, \mathcal{L}^2(\nu;\mathbb{H}))$ such that

$$\frac{1}{T}\int_t^{t+T}\int_{|x|_U<1} \left|F(s,y,x) - \bar{F}(y,x)\right|^2 \nu(\mathrm{d}x)\mathrm{d}s \le \omega_3(T)(1+|y|^2)$$

for any $T > 0$, $y \in \mathbb{H}$ and $t \in \mathbb{R}$;

(H5) there exist functions $\omega_4 \in \Psi$ and $\bar{G} \in C(\mathbb{H}, \mathcal{L}^2(\nu;\mathbb{H}))$ such that

$$\frac{1}{T}\int_t^{t+T}\int_{|x|_U\ge 1} \left|G(s,y,x) - \bar{G}(y,x)\right|^2 \nu(\mathrm{d}x)\mathrm{d}s \le \omega_4(T)(1+|y|^2)$$

for any $T > 0$, $y \in \mathbb{H}$ and $t \in \mathbb{R}$.

Remark 4.1. Under the conditions (E1), (E2) and (H2)–(H5) the functions $\bar{f}, \bar{g}, \bar{F}, \bar{G}$ also possess the properties (E1), (E2) with the same constants M and $\mathcal{L}$.

Lemma 4.2. *Suppose that $f \in C(\mathbb{R}\times\mathbb{H}, \mathbb{H})$, $g \in C(\mathbb{R}\times\mathbb{H}, L(U,\mathbb{H}))$, $F, G \in C(\mathbb{R}\times\mathbb{H}, \mathcal{L}^2(\nu;\mathbb{H}))$ and the conditions (E1) and (E2) hold. If φ is an $\mathcal{L}^2$-bounded solution of the equation*

$$\begin{aligned}\mathrm{d}Y(t) &= f(t,Y(t))\mathrm{d}t + g(t,Y(t))\mathrm{d}W(t)\\ &\quad + \int_{|x|_U<1} F(t,Y(t-),x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \int_{|x|_U\ge 1} G(t,Y(t-),x)N(\mathrm{d}t,\mathrm{d}x),\end{aligned}$$

then there exists a constant $C > 0$ depending only on $M, \mathcal{L}, \|\varphi\|_\infty, b$, such that

$$\mathbb{E}\,|\varphi(t+h) - \varphi(t)|^2 \le Ch^2 + Ch$$

and

$$\mathbb{E}\sup_{t\le s\le t+h} |\varphi(s)|^2 \le Ch^2 + Ch + C$$

for any $t \in \mathbb{R}$ and $h > 0$.

Proof. Note that

$$\varphi(t+h) = \varphi(t) + \int_t^{t+h} f(\tau,\varphi(\tau))\mathrm{d}\tau + \int_t^{t+h} g(\tau,\varphi(\tau))\mathrm{d}W(\tau)$$

$$+\int_t^{t+h}\int_{|x|_U<1}F(\tau,\varphi(\tau-),x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)+\int_t^{t+h}\int_{|x|_U\geq 1}G(\tau,\varphi(\tau-),x)N(\mathrm{d}\tau,\mathrm{d}x).$$

Since f satisfies (E1) and (E2), we have for any $\tau\in\mathbb{R}$

$$\mathbb{E}|f(\tau,\varphi(\tau))|^2\leq 2\mathbb{E}|f(\tau,\varphi(\tau))-f(\tau,0)|^2+2\mathbb{E}|f(\tau,0)|^2\leq 2\mathcal{L}^2\|\varphi\|_\infty^2+2M^2.$$

Using the same arguments as above we have the same estimates for functions g, F, G. By Cauchy-Schwartz inequality, Itô's isometry property and properties of integrals for Poisson random measures we have

$$\begin{aligned}
&\mathbb{E}|\varphi(t+h)-\varphi(t)|^2\\
&\leq 4\mathbb{E}\left|\int_t^{t+h}f(\tau,\varphi(\tau))\mathrm{d}\tau\right|^2+4\mathbb{E}\left|\int_t^{t+h}g(\tau,\varphi(\tau))\mathrm{d}W(\tau)\right|^2\\
&\quad+4\mathbb{E}\left|\int_t^{t+h}\int_{|x|_U<1}F(\tau,\varphi(\tau-),x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\right|^2+4\mathbb{E}\left|\int_t^{t+h}\int_{|x|_U\geq 1}G(\tau,\varphi(\tau-),x)N(\mathrm{d}\tau,\mathrm{d}x)\right|^2\\
&\leq 4h\int_t^{t+h}\mathbb{E}|f(\tau,\varphi(\tau))|^2\mathrm{d}\tau+4\int_t^{t+h}\mathbb{E}\|g(\tau,\varphi(\tau))Q^{\frac{1}{2}}\|_{L_2(U,\mathbb{H})}^2\mathrm{d}\tau\\
&\quad+4\int_t^{t+h}\int_{|x|_U<1}\mathbb{E}|F(\tau,\varphi(\tau-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau\\
&\quad+8\mathbb{E}\left|\int_t^{t+h}\int_{|x|_U\geq 1}G(\tau,\varphi(\tau-),x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\right|^2+8\mathbb{E}\left|\int_t^{t+h}\int_{|x|_U\geq 1}G(\tau,\varphi(\tau-),x)\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2\\
&\leq 8h\int_t^{t+h}(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau+8\int_t^{t+h}(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau\\
&\quad+8\int_t^{t+h}(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau+16\int_t^{t+h}(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau\\
&\quad+8\int_t^{t+h}\int_{|x|_U\geq 1}\mathbb{E}|G(\tau,\varphi(\tau-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau\cdot\int_t^{t+h}\int_{|x|_U\geq 1}1\nu(\mathrm{d}x)\mathrm{d}\tau\\
&\leq 8h\int_t^{t+h}(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau+32\int_t^{t+h}(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau+16bh\int_t^{t+h}(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau\\
&\leq Ch^2+Ch.
\end{aligned}$$

Note that the BDG inequality for stochastic integrals with $\widetilde{N}$ is very different from the BDG inequality with Brownian stochastic integrals. Here we need the following Kunita's first inequality ([31, Theorem 2.11])

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[t_0,t]}\left|\int_{t_0}^s\int_Z F(\tau,x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\right|^p\\
&\leq d_p\left\{\mathbb{E}\left(\int_{t_0}^t\int_Z|F(\tau,x)|^2m(\mathrm{d}x)\mathrm{d}\tau\right)^{\frac{p}{2}}+\mathbb{E}\int_{t_0}^t\int_Z|F(\tau,x)|^pm(\mathrm{d}x)\mathrm{d}\tau\right\},
\end{aligned}\tag{4.2}$$

where $(Z,\mathcal{Z},m)$ is a measurable space, $F:\Omega\times[t_0,t]\times Z\to\mathbb{H}$ is a predictable process, $p\geq 2$, and d_p is continuous w.r.t. p.

Employing the BDG inequality (see [32, Theorem 4.36]), Kunita's first inequality, Cauchy-Schwartz inequality, Itô's isometry property and properties of integrals for Poisson random measures, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{t\le s\le t+h}|\varphi(s)|^2\\
&\le 5\mathbb{E}|\varphi(t)|^2+5\mathbb{E}\sup_{t\le s\le t+h}\left|\int_t^s f(\tau,\varphi(\tau))\mathrm{d}\tau\right|^2+5\mathbb{E}\sup_{t\le s\le t+h}\left|\int_t^s g(\tau,\varphi(\tau))\mathrm{d}W(\tau)\right|^2\\
&\quad+5\mathbb{E}\sup_{t\le s\le t+h}\left|\int_t^s\int_{|x|_U<1}F(\tau,\varphi(\tau-),x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\right|^2\\
&\quad+10\mathbb{E}\sup_{t\le s\le t+h}\left|\int_t^s\int_{|x|_U\ge1}G(\tau,\varphi(\tau-),x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\right|^2\\
&\quad+10\mathbb{E}\sup_{t\le s\le t+h}\left|\int_t^s\int_{|x|_U\ge1}G(\tau,\varphi(\tau-),x)\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2\\
&\le 5\|\varphi\|_\infty^2+5\mathbb{E}\sup_{t\le s\le t+h}\left|\int_t^s(M+\mathcal{L}|\varphi(\tau)|)\mathrm{d}\tau\right|^2+5C\mathbb{E}\int_t^{t+h}\|g(\tau,\varphi(\tau))Q^{\frac12}\|_{L_2(U,\mathbb{H})}^2\mathrm{d}\tau\\
&\quad+5C\mathbb{E}\int_t^{t+h}\int_{|x|_U<1}|F(\tau,\varphi(\tau-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau\\
&\quad+10C\mathbb{E}\int_t^{t+h}\int_{|x|_U\ge1}|G(\tau,\varphi(\tau-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau\\
&\quad+10\mathbb{E}\sup_{t\le s\le t+h}\int_t^s\int_{|x|_U\ge1}|G(\tau,\varphi(\tau-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau\cdot\int_t^{t+h}\int_{|x|_U\ge1}1\nu(\mathrm{d}x)\mathrm{d}\tau\\
&\le 5\|\varphi\|_\infty^2+5h\int_t^{t+h}2(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau+5C\int_t^{t+h}2(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau\\
&\quad+5C\int_t^{t+h}2(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau+10C\int_t^{t+h}2(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau\\
&\quad+10bh\int_t^{t+h}2(M^2+\mathcal{L}^2\|\varphi\|_\infty^2)\mathrm{d}\tau\\
&\le Ch^2+Ch+C,
\end{aligned}
$$

with C denoting some positive constants and each C maybe different. □

Theorem 4.3. *Consider Eq* (4.1). *Suppose that the operator $\mathcal{A}$ is bounded; the functions $\mathcal{A}, f, g, F, G$ satisfy the conditions (E1), (E2), (H1)–(H5), and the operator $\bar{\mathcal{A}}$ in (H1) is Hurwitz, i.e. $\mathcal{R}e\,\lambda<0$ for any $\lambda\in\sigma(\bar{\mathcal{A}})$. Then there exists a positive constant $\varepsilon_1\le\alpha$ such that for any $\varepsilon\in(0,\varepsilon_1]$*

(i) if $\mathcal{L}<\dfrac{\omega}{2K\sqrt{1+2\omega+2b}}$, Eq (4.1) *has a unique solution $\varphi_\varepsilon\in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ and $\|\varphi_\varepsilon\|_\infty\le r$, where*

$$r:=\frac{2KM\sqrt{1+2\omega+2b}}{\omega-2K\mathcal{L}\sqrt{1+2\omega+2b}};$$

(ii) if $\mathcal{L} < \left\{\dfrac{\omega}{2K\sqrt{2+8\omega+4b}} \wedge \dfrac{\omega}{2K\sqrt{1+10\omega+2b}}\right\}$ *and* f, g, F, G *satisfy* (E1′), (E2′) *and* (E3) *additionally, then we have* $\mathfrak{M}^u_{(\mathcal{A},f,g,F,G)} \subseteq \tilde{\mathfrak{M}}^u_{\varphi_\varepsilon}$ *and the solution* φ_ε *is strongly compatible in distribution (i.e.* $\mathfrak{M}_{(\mathcal{A},f,g,F,G)} \subseteq \tilde{\mathfrak{M}}_{\varphi_\varepsilon}$*);*

(iii) if $\mathcal{L} < \dfrac{\omega}{2K\sqrt{3(1+2\omega+2b)}}$*, we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}} \beta(\mathcal{L}(\varphi_\varepsilon(\frac{t}{\varepsilon})), \mathcal{L}(\bar{\phi}(t))) = 0,$$

where $\bar{\phi}$ *is the unique stationary solution of the averaged equation*

$$\begin{aligned} dY(t) &= (\bar{\mathcal{A}}Y(t) + \bar{f}(Y(t)))dt + \bar{g}(Y(t))dW(t) \\ &\quad + \int_{|x|_U<1} \bar{F}(Y(t-), x)\widetilde{N}(dt, dx) + \int_{|x|_U\geq 1} \bar{G}(Y(t-), x)N(dt, dx). \end{aligned}$$

Proof. The first and second statements follow from Theorem 2.31.

We now consider the following equations

$$\begin{aligned} dY(t) &= (\mathcal{A}_\varepsilon(t)Y(t) + f_\varepsilon(t, Y(t)))dt + g_\varepsilon(t, Y(t))dW(t) \\ &\quad + \int_{|x|_U<1} F_\varepsilon(t, Y(t-), x)\widetilde{N}(dt, dx) + \int_{|x|_U\geq 1} G_\varepsilon(t, Y(t-), x)N(dt, dx), \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} dY(t) &= (\mathcal{A}_\varepsilon(t)Y(t) + f_\varepsilon(t, Y(t)))dt + g_\varepsilon(t, Y(t))dW_\varepsilon(t) \\ &\quad + \int_{|x|_U<1} F_\varepsilon(t, Y(t-), x)\widetilde{N}_\varepsilon(dt, dx) + \int_{|x|_U\geq 1} G_\varepsilon(t, Y(t-), x)N_\varepsilon(dt, dx), \end{aligned}$$

where $\mathcal{A}_\varepsilon(t) := \mathcal{A}(\frac{t}{\varepsilon})$, $f_\varepsilon(t,y) := f(\frac{t}{\varepsilon}, y)$, $g_\varepsilon(t,y) := g(\frac{t}{\varepsilon}, y)$, $F_\varepsilon(t,y,x) := F(\frac{t}{\varepsilon}, y, x)$ and $G_\varepsilon(t,y,x) := G(\frac{t}{\varepsilon}, y, x)$ for $t \in \mathbb{R}$, $y \in \mathbb{H}$ and $\varepsilon \in (0, \varepsilon_0]$. Here as before $W_\varepsilon(t) := \sqrt{\varepsilon}W(\frac{t}{\varepsilon})$, $\widetilde{N}_\varepsilon(t,x) := \sqrt{\varepsilon}\widetilde{N}(\frac{t}{\varepsilon}, x)$ and $N_\varepsilon(t,x) := \sqrt{\varepsilon}N(\frac{t}{\varepsilon}, x)$. Since $(f_\varepsilon, g_\varepsilon, F_\varepsilon, G_\varepsilon)$ satisfy conditions (E1) and (E2) with the same constants as (f, g, F, G), according to Theorem 2.31 Eq (4.3) has a unique solution ϕ_ε from $C_b(\mathbb{R}, \mathcal{L}^2(\mathbf{P}; \mathbb{H}))$ with $\phi_\varepsilon \in C(\mathbb{R}, B[0, r])$, where

$$r := \frac{2KM\sqrt{1+2\omega+2b}}{\omega - 2K\mathcal{L}\sqrt{1+2\omega+2b}}.$$

Note that

$$\begin{aligned} &\mathbb{E}|\phi_\varepsilon(t) - \bar{\phi}(t)|^2 \\ &= \mathbb{E}\Big| \int_{-\infty}^t T_{\mathcal{A}_\varepsilon}(t,\tau)f_\varepsilon(\tau, \phi_\varepsilon(\tau))d\tau + \int_{-\infty}^t T_{\mathcal{A}_\varepsilon}(t,\tau)g_\varepsilon(\tau, \phi_\varepsilon(\tau))dW(\tau) \\ &\quad + \int_{-\infty}^t \int_{|x|_U<1} T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau, \phi_\varepsilon(\tau-), x)\widetilde{N}(d\tau, dx) \\ &\quad + \int_{-\infty}^t \int_{|x|_U\geq 1} T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau, \phi_\varepsilon(\tau-), x)N(d\tau, dx) \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&\quad - \int_{-\infty}^{t} T_{\bar{\mathcal{A}}}(t,\tau)\bar{f}(\bar{\phi}(\tau))\mathrm{d}\tau - \int_{-\infty}^{t} T_{\bar{\mathcal{A}}}(t,\tau)\bar{g}(\bar{\phi}(\tau))\mathrm{d}W(\tau) \\
&\quad - \int_{-\infty}^{t}\int_{|x|_U<1} T_{\bar{\mathcal{A}}}(t,\tau)\bar{F}(\bar{\phi}(\tau-),x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x) \\
&\quad - \int_{-\infty}^{t}\int_{|x|_U\geq 1} T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(\bar{\phi}(\tau-),x)N(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \\
&\leq 4\mathbb{E}\Big|\int_{-\infty}^{t}(T_{\mathcal{A}_\varepsilon}(t,\tau)f_\varepsilon(\tau,\phi_\varepsilon(\tau)) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{f}(\bar{\phi}(\tau)))\mathrm{d}\tau\Big|^2 \\
&\quad + 4\mathbb{E}\Big|\int_{-\infty}^{t}(T_{\mathcal{A}_\varepsilon}(t,\tau)g_\varepsilon(\tau,\phi_\varepsilon(\tau)) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{g}(\bar{\phi}(\tau)))\mathrm{d}W(\tau)\Big|^2 \\
&\quad + 4\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U<1}(T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau,\phi_\varepsilon(\tau-),x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{F}(\bar{\phi}(\tau-),x))\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \\
&\quad + 4\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1}(T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,\phi_\varepsilon(\tau-),x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(\bar{\phi}(\tau-),x))N(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \\
&=: 4(I_1(t,\varepsilon) + I_2(t,\varepsilon) + I_3(t,\varepsilon) + I_4(t,\varepsilon)).
\end{aligned}
$$

Similar to the proof of [18, Theorem 4.3] with minor modifications, we have

$$
I_1(t,\varepsilon) + I_2(t,\varepsilon) \leq 3K^2\mathcal{L}^2\left(\frac{1}{2\omega} + \frac{1}{\omega^2}\right)\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t) - \bar{\phi}(t)|^2 + A(\varepsilon) + B(\varepsilon), \tag{4.5}
$$

for any $t \in \mathbb{R}$ and $\varepsilon \in (0,\alpha)$. Here $A, B : (0,\alpha) \to \mathbb{R}_+$ are functions such that $A(\varepsilon), B(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Next we estimate $I_3(t,\varepsilon)$ and $I_4(t,\varepsilon)$. By properties of integrals for Poisson random measures, we have

$$
\begin{aligned}
I_3(t,\varepsilon) &:= \mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U<1}(T_{\mathcal{A}_\varepsilon}(t,\tau)F_\varepsilon(\tau,\phi_\varepsilon(\tau-),x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{F}(\bar{\phi}(\tau-),x))\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \qquad (4.6)\\
&\leq 3\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U<1} T_{\mathcal{A}_\varepsilon}(t,\tau)(F_\varepsilon(\tau,\phi_\varepsilon(\tau-),x) - F_\varepsilon(\tau,\bar{\phi}(\tau-),x))\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \\
&\quad + 3\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U<1}(T_{\mathcal{A}_\varepsilon}(t,\tau) - T_{\bar{\mathcal{A}}}(t,\tau))F_\varepsilon(\tau,\bar{\phi}(\tau-),x)\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \\
&\quad + 3\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U<1} T_{\bar{\mathcal{A}}}(t,\tau)(F_\varepsilon(\tau,\bar{\phi}(\tau-),x) - \bar{F}(\bar{\phi}(\tau-),x))\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\Big|^2 \\
&\leq 3K^2\mathcal{L}^2\int_{-\infty}^{t} e^{-2\omega(t-\tau)}\mathbb{E}|\phi_\varepsilon(\tau) - \bar{\phi}(\tau)|^2\mathrm{d}\tau \\
&\quad + 6(\mathcal{K}(\varepsilon))^2\int_{-\infty}^{t} e^{-2\gamma_0(t-\tau)}(M^2 + \mathcal{L}^2\|\bar{\phi}\|_\infty^2)\mathrm{d}\tau \\
&\quad + 3K^2\int_{-\infty}^{t}\int_{|x|_U<1} e^{-2\omega(t-\tau)}\mathbb{E}|F_\varepsilon(\tau,\bar{\phi}(\tau-),x) - \bar{F}(\bar{\phi}(\tau-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau \\
&\leq \frac{3K^2\mathcal{L}^2}{2\omega}\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t) - \bar{\phi}(t)|^2 + \frac{3(\mathcal{K}(\varepsilon))^2}{\gamma_0}(M^2 + \mathcal{L}^2\|\bar{\phi}\|_\infty^2)
\end{aligned}
$$

$$+ 3K^2 \int_{-\infty}^{t} \int_{|x|_U<1} e^{-2\omega(t-\tau)} \mathbb{E}|F_\varepsilon(\tau, \bar{\phi}(\tau-), x) - \bar{F}(\bar{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau.$$

To prove

$$\lim_{\varepsilon\to 0} \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \int_{|x|_U<1} e^{-2\omega(t-\tau)} \mathbb{E}|F_\varepsilon(\tau, \bar{\phi}(\tau-), x) - \bar{F}(\bar{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau = 0,$$

by Lemma 3.3 we only need to illustrate

$$\lim_{\varepsilon\to 0} \sup_{|s|\le l, t\in\mathbb{R}} \left| \int_{t}^{t+s} \int_{|x|_U<1} \mathbb{E}|F_\varepsilon(\tau, \bar{\phi}(\tau-), x) - \bar{F}(\bar{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \right| = 0.$$

Divide $[0, l]$ into intervals of size δ, where $\delta > 0$ is a fixed constant depending only on ε. Denote an adapted process $\hat{\phi}$ such that $\hat{\phi}(\sigma) = \bar{\phi}(t + k\delta)$ for any $\sigma \in [t + k\delta, t + (k + 1)\delta)$. We may assume $s > 0$ without loss of generality. Then by Lemma 4.2 we have

$$\begin{aligned}
&\int_{t}^{t+s} \int_{|x|_U<1} \mathbb{E}|F_\varepsilon(\tau, \bar{\phi}(\tau-), x) - \bar{F}(\bar{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&= \int_{t}^{t+s} \int_{|x|_U<1} \mathbb{E}|F_\varepsilon(\tau, \bar{\phi}(\tau-), x) - F_\varepsilon(\tau, \hat{\phi}(\tau-), x) + F_\varepsilon(\tau, \hat{\phi}(\tau-), x) - \bar{F}(\hat{\phi}(\tau-), x) \\
&\qquad + \bar{F}(\hat{\phi}(\tau-), x) - \bar{F}(\bar{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&\le 3 \int_{t}^{t+s} \int_{|x|_U<1} \mathbb{E}|F_\varepsilon(\tau, \bar{\phi}(\tau-), x) - F_\varepsilon(\tau, \hat{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&\quad + 3 \int_{t}^{t+s} \int_{|x|_U<1} \mathbb{E}|F_\varepsilon(\tau, \hat{\phi}(\tau-), x) - \bar{F}(\hat{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&\quad + 3 \int_{t}^{t+s} \int_{|x|_U<1} \mathbb{E}|\bar{F}(\hat{\phi}(\tau-), x) - \bar{F}(\bar{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&\le 6\mathcal{L}^2 l(C\delta^2 + C\delta) + 3 \int_{t}^{t+s} \int_{|x|_U<1} \mathbb{E}|F_\varepsilon(\tau, \hat{\phi}(\tau-), x) - \bar{F}(\hat{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&=: 6\mathcal{L}^2 l(C\delta^2 + C\delta) + 3J_1.
\end{aligned}$$

Denote $s(\delta) := \left[\frac{|s|}{\delta}\right]$. For J_1, we have

$$\begin{aligned}
J_1 &:= \mathbb{E} \int_{t}^{t+s} \int_{|x|_U<1} |F_\varepsilon(\tau, \hat{\phi}(\tau-), x) - \bar{F}(\hat{\phi}(\tau-), x)|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&\le \mathbb{E} \sum_{k=0}^{s(\delta)-1} \int_{t+k\delta}^{t+(k+1)\delta} \int_{|x|_U<1} \left|F_\varepsilon(\tau, \bar{\phi}((t + k\delta)-), x) - \bar{F}(\bar{\phi}((t + k\delta)-), x)\right|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&\quad + \mathbb{E} \int_{t+s(\delta)\cdot\delta}^{t+s} \int_{|x|_U<1} \left|F_\varepsilon(\tau, \bar{\phi}((t + s(\delta) \cdot \delta)-), x) - \bar{F}(\bar{\phi}((t + s(\delta) \cdot \delta)-), x)\right|^2 \nu(\mathrm{d}x)\mathrm{d}\tau \\
&=: J_1^1 + J_1^2.
\end{aligned}$$

Then

$$J_1^1 := \mathbb{E} \sum_{k=0}^{s(\delta)-1} \int_{t+k\delta}^{t+(k+1)\delta} \int_{|x|_U<1} \left|F_\varepsilon(\tau, \bar{\phi}((t + k\delta)-), x) - \bar{F}(\bar{\phi}((t + k\delta)-), x)\right|^2 \nu(\mathrm{d}x)\mathrm{d}\tau$$

$$\leq \left[\frac{l}{\delta}\right] \max_{0\leq k\leq s(\delta)-1} \mathbb{E}\int_{t+k\delta}^{t+(k+1)\delta}\int_{|x|_U<1}\left|F_\varepsilon(\tau,\bar{\phi}((t+k\delta)-),x)-\bar{F}(\bar{\phi}((t+k\delta)-),x)\right|^2\nu(\mathrm{d}x)\mathrm{d}\tau$$
$$= \left[\frac{l}{\delta}\right] \max_{0\leq k\leq s(\delta)-1} \mathbb{E}\int_{\frac{t+k\delta}{\varepsilon}}^{\frac{t+(k+1)\delta}{\varepsilon}}\int_{|x|_U<1}\left|F(\tau,\bar{\phi}((t+k\delta)-),x)-\bar{F}(\bar{\phi}((t+k\delta)-),x)\right|^2\varepsilon\nu(\mathrm{d}x)\mathrm{d}\tau$$
$$\leq l\omega_3\left(\frac{\delta}{\varepsilon}\right)(1+\|\bar{\phi}\|_\infty^2)$$

and

$$J_1^2 := \mathbb{E}\int_{t+s(\delta)\cdot\delta}^{t+s}\int_{|x|_U<1}\left|F_\varepsilon(\tau,\bar{\phi}((t+s(\delta)\cdot\delta)-),x)-\bar{F}(\bar{\phi}((t+s(\delta)\cdot\delta)-),x)\right|^2\nu(\mathrm{d}x)\mathrm{d}\tau$$
$$\leq 8(M^2+\mathcal{L}^2\|\bar{\phi}\|_\infty^2)\delta.$$

Hence

$$\sup_{|s|\leq l,t\in\mathbb{R}}\left|\int_t^{t+s}\int_{|x|_U<1}\mathbb{E}|F_\varepsilon(\tau,\bar{\phi}(\tau-),x)-\bar{F}(\bar{\phi}(\tau-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau\right| \tag{4.7}$$
$$\leq 6\mathcal{L}^2 l(C\delta^2+C\delta)+3l\omega_3\left(\frac{\delta}{\varepsilon}\right)(1+\|\bar{\phi}\|_\infty^2)+24(M^2+\mathcal{L}^2\|\bar{\phi}\|_\infty^2)\delta.$$

Taking $\delta = \sqrt{\varepsilon}$ and letting $\varepsilon \to 0$ in (4.7), we have

$$\lim_{\varepsilon\to 0}\sup_{|s|\leq l,t\in\mathbb{R}}\left|\int_t^{t+s}\int_{|x|_U<1}\mathbb{E}|F_\varepsilon(\tau,\bar{\phi}(\tau-),x)-\bar{F}(\bar{\phi}(\tau-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau\right| = 0. \tag{4.8}$$

From (4.6) and (4.8) it follows that

$$I_3(t,\varepsilon) \leq \frac{3K^2\mathcal{L}^2}{2\omega}\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t)-\bar{\phi}(t)|^2 + C(\varepsilon), \tag{4.9}$$

where $C : (0,\alpha) \to \mathbb{R}_+$ is a function so that $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Note that

$$I_4(t,\varepsilon) := \mathbb{E}\left|\int_{-\infty}^t\int_{|x|_U\geq 1}(T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,\phi_\varepsilon(\tau-),x)-T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(\bar{\phi}(\tau-),x))N(\mathrm{d}\tau,\mathrm{d}x)\right|^2 \tag{4.10}$$
$$\leq 2\mathbb{E}\left|\int_{-\infty}^t\int_{|x|_U\geq 1}(T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,\phi_\varepsilon(\tau-),x)-T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(\bar{\phi}(\tau-),x))\widetilde{N}(\mathrm{d}\tau,\mathrm{d}x)\right|^2$$
$$+ 2\mathbb{E}\left|\int_{-\infty}^t\int_{|x|_U\geq 1}(T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,\phi_\varepsilon(\tau-),x)-T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(\bar{\phi}(\tau-),x))\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2$$
$$=: 2(I_{41}(t,\varepsilon)+I_{42}(t,\varepsilon)).$$

The similar arguments as $I_3(t,\varepsilon)$ yield that

$$I_{41}(t,\varepsilon) \leq \frac{3K^2\mathcal{L}^2}{2\omega}\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t)-\bar{\phi}(t)|^2 + D_1(\varepsilon), \tag{4.11}$$

where $D_1(\varepsilon)$ is some constant and $D_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. For $I_{42}(t,\varepsilon)$, by Cauchy-Schwartz inequality we have

$$\begin{aligned} I_{42}(t,\varepsilon) :=& \mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1}(T_{\mathcal{A}_\varepsilon}(t,\tau)G_\varepsilon(\tau,\phi_\varepsilon(\tau-),x) - T_{\bar{\mathcal{A}}}(t,\tau)\bar{G}(\bar{\phi}(\tau-),x))\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2 \\ \leq& 3\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1}(T_{\mathcal{A}_\varepsilon}(t,\tau)(G_\varepsilon(\tau,\phi_\varepsilon(\tau-),x) - G_\varepsilon(\tau,\bar{\phi}(\tau-),x))\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2 \\ &+ 3\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1}(T_{\mathcal{A}_\varepsilon}(t,\tau) - T_{\bar{\mathcal{A}}}(t,\tau))G_\varepsilon(\tau,\bar{\phi}(\tau-),x)\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2 \\ &+ 3\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1}T_{\bar{\mathcal{A}}}(t,\tau)[G_\varepsilon(\tau,\bar{\phi}(\tau-),x) - \bar{G}(\bar{\phi}(\tau-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2 \\ \leq& \frac{3K^2\mathcal{L}^2 b}{\omega^2}\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t)-\bar{\phi}(t)|^2 + \frac{6(\mathcal{K}(\varepsilon))^2 b}{\gamma_0^2}(M^2+\mathcal{L}^2\|\bar{\phi}\|_\infty^2) \\ &+ 3\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1}T_{\bar{\mathcal{A}}}(t,\tau)[G_\varepsilon(\tau,\bar{\phi}(\tau-),x) - \bar{G}(\bar{\phi}(\tau-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2. \end{aligned} \tag{4.12}$$

We now show that

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1}T_{\bar{\mathcal{A}}}(t,\tau)[G_\varepsilon(\tau,\bar{\phi}(\tau-),x) - \bar{G}(\bar{\phi}(\tau-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2 = 0.$$

To this end, making the change of variable $s = \tau - t$ and integrating by parts, we obtain for any $l \geq 0$

$$\begin{aligned} &\mathbb{E}\Big|\int_{-\infty}^{t}\int_{|x|_U\geq 1}T_{\bar{\mathcal{A}}}(t,\tau)[G_\varepsilon(\tau,\bar{\phi}(\tau-),x) - \bar{G}(\bar{\phi}(\tau-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2 \\ &= \mathbb{E}\Big|\int_{-\infty}^{0}\int_{|x|_U\geq 1}T_{\bar{\mathcal{A}}}(t,t+s)[G_\varepsilon(t+s,\bar{\phi}((t+s-),x) - \bar{G}(\bar{\phi}((t+s-),x)]\nu(\mathrm{d}x)\mathrm{d}s\Big|^2 \\ &= \mathbb{E}\Big|\int_{-\infty}^{0}T_{\bar{\mathcal{A}}}(t,t+s)\frac{\mathrm{d}}{\mathrm{d}s}\Big(\int_{t}^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x) - \bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big)\mathrm{d}s\Big|^2 \\ &\leq 2\mathbb{E}\Big|-\int_{-\infty}^{0}\frac{\partial T_{\bar{\mathcal{A}}}(t,t+s)}{\partial s}\Big(\int_{t}^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x) - \bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big)\mathrm{d}s\Big|^2 \\ &\leq 4\mathbb{E}\Big|-\int_{-\infty}^{-l}\frac{\partial T_{\bar{\mathcal{A}}}(t,t+s)}{\partial s}\Big(\int_{t}^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x) - \bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big)\mathrm{d}s\Big|^2 \\ &\quad + 4\mathbb{E}\Big|-\int_{-l}^{0}\frac{\partial T_{\bar{\mathcal{A}}}(t,t+s)}{\partial s}\Big(\int_{t}^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x) - \bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big)\mathrm{d}s\Big|^2 \\ &\leq 4\mathbb{E}\Big(-\int_{-\infty}^{-l}\Big|\int_{t}^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x) - \bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big|K\|\bar{\mathcal{A}}\|e^{\omega s}\mathrm{d}s\Big)^2 \\ &\quad + 4\mathbb{E}\sup_{-l\leq s\leq 0}\Big|\int_{t}^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x) - \bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big|^2\cdot\Big|\int_{-l}^{0}K\|\bar{\mathcal{A}}\|e^{\omega s}\mathrm{d}s\Big|^2 \\ &=: J_2 + J_3. \end{aligned} \tag{4.13}$$

For J_2, by using Cauchy-Schwartz inequality we have

$$
\begin{aligned}
J_2 :=& 4\mathbb{E}\Big(-\int_{-\infty}^{-l}\Big|\int_t^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x)-\bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big|K\|\bar{\mathcal{A}}\|e^{\omega s}\mathrm{d}s\Big)^2 \qquad (4.14)\\
\leq& 4K^2\|\bar{\mathcal{A}}\|^2\int_{-\infty}^{-l}e^{\omega s}\mathrm{d}s\cdot\int_{-\infty}^{-l}\mathbb{E}\Big|\int_t^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x)-\bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big|^2e^{\omega s}\mathrm{d}s\\
\leq& \frac{4K^2\|\bar{\mathcal{A}}\|^2}{\omega}e^{-\omega l}\int_{-\infty}^{-l}\Big(\int_t^{t+s}\int_{|x|_U\geq 1}\mathbb{E}|G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x)-\bar{G}(\bar{\phi}(\sigma-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\sigma\Big)\\
&\cdot\Big(\int_t^{t+s}\int_{|x|_U\geq 1}1\nu(\mathrm{d}x)\mathrm{d}\sigma\Big)e^{\omega s}\mathrm{d}s\\
\leq& \frac{4K^2\|\bar{\mathcal{A}}\|^2}{\omega}e^{-\omega l}\int_{-\infty}^{-l}\Big(\int_t^{t+s}8(M^2+\mathcal{L}^2\|\bar{\phi}\|_\infty^2)\mathrm{d}\sigma\Big)bse^{\omega s}\mathrm{d}s\\
\leq& \frac{32K^2\|\bar{\mathcal{A}}\|^2b}{\omega}(M^2+\mathcal{L}^2\|\bar{\phi}\|_\infty^2)\left(\frac{l^2}{\omega}+\frac{2l}{\omega^2}+\frac{2}{\omega^3}\right)e^{-2\omega l}.
\end{aligned}
$$

Denote an adapted process $\tilde{\phi}$ such that $\tilde{\phi}(\sigma)=\bar{\phi}(t-k\delta)$ for any $\sigma\in(t-(k+1)\delta,t-k\delta]$. By Lemma 4.2 and Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{-l\leq s\leq 0}\Big|\int_t^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x)-\bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big|^2 \qquad (4.15)\\
&=\mathbb{E}\sup_{-l\leq s\leq 0}\Big|\int_t^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\bar{\phi}(\sigma-),x)-G_\varepsilon(\sigma,\tilde{\phi}(\sigma-),x)\\
&\quad+G_\varepsilon(\sigma,\tilde{\phi}(\sigma-),x)-\bar{G}(\tilde{\phi}(\sigma-),x)+\bar{G}(\tilde{\phi}(\sigma-),x)-\bar{G}(\bar{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big|^2\\
&\leq 6\mathbb{E}\sup_{-l\leq s\leq 0}\Big|\int_t^{t+s}\mathcal{L}|\bar{\phi}(\sigma)-\tilde{\phi}(\sigma)|\mathrm{d}\sigma\Big|^2\\
&\quad+3\mathbb{E}\sup_{-l\leq s\leq 0}\Big|\int_t^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\tilde{\phi}(\sigma-),x)-\bar{G}(\tilde{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big|^2\\
&\leq 6\mathcal{L}^2l^2(C\delta^2+C\delta)+3\mathbb{E}\sup_{-l\leq s\leq 0}\Big|\int_t^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\sigma,\tilde{\phi}(\sigma-),x)-\bar{G}(\tilde{\phi}(\sigma-),x)]\nu(\mathrm{d}x)\mathrm{d}\sigma\Big|^2\\
&=:6\mathcal{L}^2l^2(C\delta^2+C\delta)+i_1.
\end{aligned}
$$

For i_1, recalling that $s(\delta):=\left[\frac{|s|}{\delta}\right]$ by Lemma 4.2 we have

$$
\begin{aligned}
i_1 :=& 3\mathbb{E}\sup_{-l\leq s\leq 0}\Big|\int_t^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\tau,\tilde{\phi}(\tau-),x)-\bar{G}(\tilde{\phi}(\tau-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2 \qquad (4.16)\\
=& 3\mathbb{E}\sup_{-l\leq s\leq 0}\Big|\sum_{k=0}^{s(\delta)-1}\int_{t-k\delta}^{t-(k+1)\delta}\int_{|x|_U\geq 1}[G_\varepsilon(\tau,\bar{\phi}((t-k\delta)-),x)-\bar{G}(\bar{\phi}((t-k\delta)-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\\
&+\int_{t-s(\delta)\cdot\delta}^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\tau,\bar{\phi}((t-s(\delta)\cdot\delta)-),x)-\bar{G}(\bar{\phi}((t-s(\delta)\cdot\delta)-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\Big|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq 6\left[\frac{l}{\delta}\right]\mathbb{E}\sup_{-l\leq s\leq 0}\sum_{k=0}^{s(\delta)-1}\left|\int_{t-k\delta}^{t-(k+1)\delta}\int_{|x|_U\geq 1}[G_\varepsilon(\tau,\bar{\phi}((t-k\delta)-),x)-\bar{G}(\bar{\phi}((t-k\delta)-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2\\
&\quad+6\mathbb{E}\sup_{-l\leq s\leq 0}\left|\int_{t-s(\delta)\cdot\delta}^{t+s}\int_{|x|_U\geq 1}[G_\varepsilon(\tau,\bar{\phi}((t-s(\delta)\cdot\delta)-),x)-\bar{G}(\bar{\phi}((t-s(\delta)\cdot\delta)-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2\\
&\leq \frac{6l^2}{\delta^2}\mathbb{E}\sup_{-l\leq s\leq 0}\max_{0\leq k\leq s(\delta)-1}\left|\int_{\frac{t-k\delta}{\varepsilon}}^{\frac{t-(k+1)\delta}{\varepsilon}}\int_{|x|_U\geq 1}[G(\tau,\bar{\phi}((t-k\delta)-),x)-\bar{G}(\bar{\phi}((t-k\delta)-),x)]\varepsilon\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2\\
&\quad+6\mathbb{E}\sup_{-l\leq s\leq 0}b\delta\int_{t+s}^{t-s(\delta)\cdot\delta}\int_{|x|_U\geq 1}|G_\varepsilon(\tau,\bar{\phi}((t-s(\delta)\cdot\delta)-),x)-\bar{G}(\bar{\phi}((t-s(\delta)\cdot\delta)-),x)|^2\nu(\mathrm{d}x)\mathrm{d}\tau\\
&\leq \frac{6l^2}{\delta^2}\mathbb{E}\sup_{-l\leq s\leq 0}\max_{0\leq k\leq s(\delta)-1}\int_{\frac{t-k\delta}{\varepsilon}}^{\frac{t-(k+1)\delta}{\varepsilon}}\int_{|x|_U\geq 1}|G(\tau,\bar{\phi}((t-k\delta)-),x)-\bar{G}(\bar{\phi}((t-k\delta)-),x)|^2\varepsilon\nu(\mathrm{d}x)\mathrm{d}\tau\\
&\qquad\cdot\int_{\frac{t-k\delta}{\varepsilon}}^{\frac{t-(k+1)\delta}{\varepsilon}}\int_{|x|_U\geq 1}\varepsilon\nu(\mathrm{d}x)\mathrm{d}\tau+6b\delta\mathbb{E}\sup_{-l\leq s\leq 0}\int_{t+s}^{t-s(\delta)\cdot\delta}8(M^2+\mathcal{L}^2|\bar{\phi}(t-s(\delta)\cdot\delta)|^2)\mathrm{d}\tau\\
&\leq 6l^2b\mathbb{E}\sup_{-l\leq s\leq 0}\max_{0\leq k\leq s(\delta)-1}\omega_4\left(\frac{\delta}{\varepsilon}\right)(1+|\bar{\phi}(t+k\delta)|^2)+6b\delta\int_{t-l}^{t}8(M^2+\mathcal{L}^2\mathbb{E}\sup_{t-l\leq\tau\leq t}|\bar{\phi}(\tau)|^2)\mathrm{d}\tau\\
&\leq 6l^2b(Cl^2+Cl+C+1)\omega_4\left(\frac{\delta}{\varepsilon}\right)+48lb\delta(M^2+\mathcal{L}^2(Cl^2+Cl+C))
\end{aligned}
$$

Therefore, by (4.15) and (4.16) we get

$$
\begin{aligned}
J_3 \leq \Big[&6\mathcal{L}^2l^2(C\delta^2+C\delta)+6l^2b(Cl^2+Cl+C+1)\omega_4\left(\frac{\delta}{\varepsilon}\right)\\
&+48lb\delta(M^2+\mathcal{L}^2(Cl^2+Cl+C))\Big]\cdot\frac{4K^2\|\bar{\mathcal{A}}\|^2}{\omega^2}\left(1-e^{-\omega l}\right)^2.
\end{aligned}\tag{4.17}
$$

Hence, according to (4.13), (4.14) and (4.17) we have

$$
\begin{aligned}
&\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U\geq 1}T_{\bar{\mathcal{A}}}(t,\tau)[G_\varepsilon(\tau,\bar{\phi}(\tau-),x)-\bar{G}(\bar{\phi}(\tau-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2\\
&\leq\frac{32K^2\|\bar{\mathcal{A}}\|^2b}{\omega}(M^2+\mathcal{L}^2\|\bar{\phi}\|_\infty^2)\left(\frac{l^2}{\omega}+\frac{2l}{\omega^2}+\frac{2}{\omega^3}\right)e^{-2\omega l}+\frac{4K^2\|\bar{\mathcal{A}}\|^2}{\omega^2}\left(1-e^{-\omega l}\right)^2\\
&\quad\cdot\left[6\mathcal{L}^2l^2(C\delta^2+C\delta)+6l^2b(Cl^2+Cl+C+1)\omega_4\left(\frac{\delta}{\varepsilon}\right)+48lb\delta(M^2+\mathcal{L}^2(Cl^2+Cl+C))\right].
\end{aligned}\tag{4.18}
$$

Taking $\delta=\sqrt{\varepsilon}$ and letting $\varepsilon\to 0$ in (4.18), we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U\geq 1}T_{\bar{\mathcal{A}}}(t,\tau)[G_\varepsilon(\tau,\bar{\phi}(\tau-),x)-\bar{G}(\bar{\phi}(\tau-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2\\
&\leq\frac{32K^2\|\bar{\mathcal{A}}\|^2b}{\omega}(M^2+\mathcal{L}^2\|\bar{\phi}\|_\infty^2)\left(\frac{l^2}{\omega}+\frac{2l}{\omega^2}+\frac{2}{\omega^3}\right)e^{-2\omega l}.
\end{aligned}
$$

Since l is arbitrary, by letting $l\to 0$ we have

$$
\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\mathbb{E}\left|\int_{-\infty}^{t}\int_{|x|_U\geq 1}T_{\bar{\mathcal{A}}}(t,\tau)[G_\varepsilon(\tau,\bar{\phi}(\tau-),x)-\bar{G}(\bar{\phi}(\tau-),x)]\nu(\mathrm{d}x)\mathrm{d}\tau\right|^2=0.\tag{4.19}
$$

From (4.12) and (4.19) it follows that

$$I_{42}(t,\varepsilon) \leq \frac{3K^2\mathcal{L}^2 b}{\omega^2}\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t)-\bar{\phi}(t)|^2 + D_2(\varepsilon), \tag{4.20}$$

where $D_2(\varepsilon)$ is some positive constant such that $D_2(\varepsilon)\to 0$ as $\varepsilon\to 0$. So by (4.10), (4.11) and (4.20) we have

$$I_4(t,\varepsilon) \leq \left(\frac{3K^2\mathcal{L}^2}{\omega} + \frac{6K^2\mathcal{L}^2 b}{\omega^2}\right)\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t)-\bar{\phi}(t)|^2 + D_1(\varepsilon) + D_2(\varepsilon). \tag{4.21}$$

Combing (4.4), (4.5), (4.9) and (4.21), we have

$$\left(1 - 12K^2\mathcal{L}^2\left(\frac{1}{\omega^2} + \frac{2}{\omega} + \frac{2b}{\omega^2}\right)\right)\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t)-\bar{\phi}(t)|^2 \leq 4(A(\varepsilon)+B(\varepsilon)+C(\varepsilon)+D_1(\varepsilon)+D_2(\varepsilon)).$$

By the assumption $\mathcal{L} < \dfrac{\omega}{2K\sqrt{3(1+2\omega+2b)}}$ the coefficient is positive, so

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\mathbb{E}|\phi_\varepsilon(t)-\bar{\phi}(t)|^2 = 0.$$

Since $\mathcal{L}^2$-convergence implies convergence in distribution,

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\beta(\mathcal{L}(\phi_\varepsilon(t)),\mathcal{L}(\bar{\phi}(t))) = 0.$$

We note that $\mathcal{L}(\varphi_\varepsilon(\frac{t}{\varepsilon})) = \mathcal{L}(\phi_\varepsilon(t))$. Then we have

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\beta(\mathcal{L}(\varphi_\varepsilon(\frac{t}{\varepsilon})),\mathcal{L}(\bar{\phi}(t))) = 0.$$

The proof is complete. □

Corollary 4.4. *Assume that the conditions of* Theorem 4.3 *hold.*

(i) If $\mathcal{A}, f, g, F, G$ are jointly stationary (respectively, τ-periodic, quasi-periodic with the spectrum of frequencies $\nu_1,\dots,\nu_k$, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable), then Eq (4.1) has a unique solution $\varphi_\varepsilon \in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ which is stationary (respectively, τ-periodic, quasi-periodic with the spectrum of frequencies $\nu_1,\dots,\nu_k$, almost periodic, almost automorphic, Birkhoff recurrent, Lagrange stable, Levitan almost periodic, almost recurrent, Poisson stable) in distribution.

(ii) If $\mathcal{A}, f, g, F, G$ are Lagrange stable and jointly pseudo-periodic (respectively, pseudo-recurrent), then Eq (4.1) has a unique solution $\varphi_\varepsilon \in C_b(\mathbb{R},\mathcal{L}^2(\mathbf{P};\mathbb{H}))$ which is pseudo-periodic (respectively, pseudo-recurrent) in distribution.

(iii)

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}}\beta(\mathcal{L}(\varphi_\varepsilon(\frac{t}{\varepsilon}),\mathcal{L}(\bar{\phi}(t))) = 0.$$

Proof. This statement follows from Theorems 2.16, 4.3 and Remark 2.18. □

5. Applications

In this section, we illustrate our theoretical results by two examples.

Example 5.1. Consider the following stochastic ordinary differential equation driven by a two-sided Lévy noise:

$$\begin{aligned} \mathrm{d}y = {}& \varepsilon(-3y + \frac{1}{4}y\sin^2 t)\mathrm{d}t + \frac{1}{4}\sqrt{\varepsilon}\mathrm{d}W \\ & + \sqrt{\varepsilon}\int_{|x|<1}\frac{1}{5}y\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \sqrt{\varepsilon}\int_{|x|\geq 1}\frac{1}{8}y(\sin t + \cos\sqrt{3}t)N(\mathrm{d}t,\mathrm{d}x), \end{aligned} \tag{5.1}$$

where ε is a small positive parameter; W is a one-dimensional two-sided Brownian motion and N is a Poisson random measure in $\mathbb{R}$, which is independent of W. Let

$$\bar{f}(y) = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \frac{1}{4}y\sin^2 t\mathrm{d}t = \frac{1}{8}y,$$

$$\bar{G}(y,x) = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \frac{1}{8}y(\sin t + \cos\sqrt{3}t)\mathrm{d}t = 0,$$

and define the corresponding averaged stochastic differential equation

$$\mathrm{d}\bar{y} = (-3\bar{y} + \frac{1}{8}\bar{y})\mathrm{d}t + \frac{1}{4}\mathrm{d}W + \int_{|x|<1}\frac{1}{5}\bar{y}\widetilde{N}(\mathrm{d}t,\mathrm{d}x). \tag{5.2}$$

It is clear that $\mathcal{A}$ generates a dissipative semigroup on $\mathbb{R}$ with $K = 1$, $\omega = 3$. Since the functions $\mathcal{A}, f, g, F, G$ are respectively stationary, periodic, stationary, stationary and quasi-periodic in t, uniformly with respect to y on any bounded subset of $\mathbb{R}$, it follows that functions $\mathcal{A}, f, g, F, G$ are jointly quasi-periodic. Conditions (E1) and (E1′) always hold for any constant M, and the Lipschitz constants of functions f, g, F, G in Conditions (E2) and (E2′) can be chosen as $\frac{1}{4}$ if

$$\int_{|x|<1}\left(\frac{1}{5}\right)^p \nu(\mathrm{d}x) \leq \left(\frac{1}{4}\right)^p \quad \text{and} \quad \int_{|x|\geq 1}\left(\frac{1}{4}\right)^p \nu(\mathrm{d}x) \leq \left(\frac{1}{4}\right)^p$$

for $p = 2$ and some constant $p > 2$, i.e.

$$\nu(-1,1) < \frac{25}{16} \quad \text{and} \quad b \leq 1.$$

As for Condition (E3), it naturally holds for stochastic ordinary differential equation. Conditions (H1)–(H5) are also satisfied for functions $\mathcal{A}, f, g, F, G$.

Since f, g, F, G satisfy (E1) and (E2), by Theorem 4.3-(i) Eq (5.1) has a unique $\mathcal{L}^2$-bounded solution provided $\nu(-1,1) < \frac{25}{16}$, $b \leq 1$. The restrictions in Theorem 4.3-(ii) and (iii) respectively become

$$\frac{1}{4} < \left\{\frac{3}{2\sqrt{2+24+4b}} \wedge \frac{3}{2\sqrt{1+30+2b}}\right\} \quad \text{and} \quad \frac{1}{4} < \frac{3}{2\sqrt{3(1+6+2b)}},$$

i.e. $b < \frac{5}{2}$. According to Corollary 4.4, the unique $\mathcal{L}^2$-bounded solution of Eq (5.1) is quasi-periodic in distribution and it uniformly converges to the unique stationary solution of the averaged Eq (5.2) on $\mathbb{R}$ in distribution sense.

Example 5.2. Consider the following equations:

$$\begin{aligned}\mathrm{d}y_i &= \varepsilon[-a_i y_i + f_i(t,y)]\mathrm{d}t + \sqrt{\varepsilon} g_i(t,y)\mathrm{d}W \\ &\quad + \sqrt{\varepsilon}\int_{|x|<1} F_i(t,y,x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \sqrt{\varepsilon}\int_{|x|\geq 1} G_i(t,y,x)N(\mathrm{d}t,\mathrm{d}x), (i=1,2,...,n,...)\end{aligned} \tag{5.3}$$

where $0 < \omega \leq a_i \leq \kappa$ for $i = 1,2,...,$ and κ and ω are constants; $f_i, g_i \in C(\mathbb{R}\times l^2, l^2)$, $F_i, G_i \in C(\mathbb{R}\times l^2, \mathcal{L}^2(\nu; l^2))$, $l^2 := \{y : y = (y_1, y_2, ..., y_n, ...), \sum_{i=1}^{\infty} y_i^2 < \infty\}$; W is a one-dimensional two-sided Brownian motion and N is a Poisson random measure independent of W. (5.3) can also be written in the following form:

$$\begin{aligned}\mathrm{d}Y &= \varepsilon[\mathcal{A}Y + f(t,Y)]\mathrm{d}t + \sqrt{\varepsilon} g(t,Y)\mathrm{d}W \\ &\quad + \sqrt{\varepsilon}\int_{|x|<1} F(t,Y,x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \sqrt{\varepsilon}\int_{|x|\geq 1} G(t,Y,x)N(\mathrm{d}t,\mathrm{d}x),\end{aligned} \tag{5.4}$$

where $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \\ \vdots \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} -a_1 & & & \\ & \ddots & & \\ & & -a_n & \\ & & & \ddots \end{pmatrix}$, $f(t,Y) = \begin{pmatrix} f_1(t,Y) \\ \vdots \\ f_n(t,Y) \\ \vdots \end{pmatrix}$, $g(t,Y) = \begin{pmatrix} g_1(t,Y) \\ \vdots \\ g_n(t,Y) \\ \vdots \end{pmatrix}$,

$$F(t,Y,x) = \begin{pmatrix} F_1(t,Y,x) \\ \vdots \\ F_n(t,Y,x) \\ \vdots \end{pmatrix}, G(t,Y,x) = \begin{pmatrix} G_1(t,Y,x) \\ \vdots \\ G_n(t,Y,x) \\ \vdots \end{pmatrix}.$$

According to (H1)–(H5), we define the corresponding averaged equations:

$$\begin{aligned}\mathrm{d}\bar{y}_i &= [-a_i\bar{y}_i + \bar{f}_i(\bar{y})]\mathrm{d}t + \bar{g}_i(\bar{y})\mathrm{d}W \\ &\quad + \int_{|x|<1} \bar{F}_i(\bar{y},x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \int_{|x|\geq 1} \bar{G}_i(\bar{y},x)N(\mathrm{d}t,\mathrm{d}x), \qquad (i=1,2,...).\end{aligned} \tag{5.5}$$

Eq (5.5) can also be written in the following form:

$$\begin{aligned}\mathrm{d}\bar{Y} &= [\mathcal{A}\bar{Y} + \bar{f}(\bar{Y})]\mathrm{d}t + \bar{g}(\bar{Y})\mathrm{d}W \\ &\quad + \int_{|x|<1} \bar{F}(\bar{Y},x)\widetilde{N}(\mathrm{d}t,\mathrm{d}x) + \int_{|x|\geq 1} \bar{G}(\bar{Y},x)N(\mathrm{d}t,\mathrm{d}x),\end{aligned} \tag{5.6}$$

where $\bar{Y} = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \\ \vdots \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} -a_1 & & & \\ & \ddots & & \\ & & -a_n & \\ & & & \ddots \end{pmatrix}$, $\bar{f}(\bar{Y}) = \begin{pmatrix} \bar{f}_1(\bar{Y}) \\ \vdots \\ \bar{f}_n(\bar{Y}) \\ \vdots \end{pmatrix}$, $\bar{g}(\bar{Y}) = \begin{pmatrix} \bar{g}_1(\bar{Y}) \\ \vdots \\ \bar{g}_n(\bar{Y}) \\ \vdots \end{pmatrix}$,

$$\bar{F}(\bar{Y},x) = \begin{pmatrix} \bar{F}_1(\bar{Y},x) \\ \vdots \\ \bar{F}_n(\bar{Y},x) \\ \vdots \end{pmatrix}, \bar{G}(\bar{Y},x) = \begin{pmatrix} \bar{G}_1(\bar{Y},x) \\ \vdots \\ \bar{G}_n(\bar{Y},x) \\ \vdots \end{pmatrix}.$$

Assume that the Lipschitz condition, growth condition and all the conditions of Theorem 4.3 are satisfied for f_i, g_i, F_i, G_i, (i=1,2,...). Thus Theorem 4.3 and Corollary 4.4 hold. If every f_i, g_i, F_i, G_i, (i=1,2,...) are periodic or quasi-periodic in t uniformly with respect to y on any bounded subset of l^2, then f_i, g_i, F_i, G_i are jointly almost periodic. Basis on Theorem 4.3 and Corollary 4.4, Eq (5.4) has a unique almost periodic solution and it converges to the stationary solution of the averaged Eq (5.6) in distribution sense on the whole real axis.

Now we give a concrete example with infinite dimension:

$$\begin{aligned} \mathrm{d}y_i &= \varepsilon[-4y_i + \frac{1}{4}(y_{i-1} + y_i + y_{i+1})\cos\sqrt{2}it]\mathrm{d}t \\ &+ \sqrt{\varepsilon}[\frac{1}{5}(y_{i-1} + y_i + y_{i+1})\sin^2 it]\mathrm{d}W \\ &+ \sqrt{\varepsilon}\int_{|x|<1}[\frac{1}{5}y_{i-1}\sin it + \frac{1}{5}y_i \sin\sqrt{3}it + \frac{1}{5}y_{i+1}\cos\sqrt{5}it]\widetilde{N}(\mathrm{d}t, \mathrm{d}x), \quad (i = 1, 2, ...) \end{aligned} \tag{5.7}$$

and the averaged equations

$$\mathrm{d}\bar{y}_i = -4\bar{y}_i\mathrm{d}t + [\frac{1}{10}(\bar{y}_{i-1} + \bar{y}_i + \bar{y}_{i+1})]\mathrm{d}W, \quad (i = 1, 2, ...). \tag{5.8}$$

It is immediate to check that the drift, diffusion, and jump coefficients of Eqs (5.7) and (5.8) satisfy all the conditions of Theorem 4.3 and Corollary 4.4. So the almost periodic solution of Eq (5.7) converges to the stationary solution of Eq (5.8) on $\mathbb{R}$ in distribution.

6. Conclusions

In this paper we study an averaging principle on infinite time intervals for semilinear stochastic ordinary differential equations with Lévy noise. We establish the existence of Poisson stable (including periodic, quasi-periodic, almost periodic, almost automorphic etc) solutions of the equations by a unified framework. We show that the Poisson stable solution of the original equation converges to the stationary solution of the averaged equation uniformly on the whole real axis in distribution sense, as the time scale goes to zero. Moreover, we illustrate our theoretical results with some examples.

In this paper we only consider stochastic ordinary differential equations case. The case of the averaging on $\mathbb{R}$ for stochastic partial differential equations is more complicated, but it is both interesting and important. It deserves a separate paper.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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