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Research article

Existence and global behavior of the solution to a parabolic equation with nonlocal diffusion

Fengfei Jin and Baoqiang Yan*

School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China

* **Correspondence:** Email: yanbqcn@aliyun.com.

Abstract: In this paper, we are concerned with the existence, uniqueness and long-time behavior of the solutions for a parabolic equation with nonlocal diffusion even if the reaction term is not Lipschitz-continuous at 0 and grows superlinearly or exponentially at $+\infty$. First, we give a special sub-supersolution pair for some parabolic equations with nonlocal diffusion and establish the method of sub-supersolution. Second, using the sub-supersolution method, we prove the existence, uniqueness and long-time behavior of positive solutions. Finally, some one-dimensional numerical experiments are presented.

Keywords: parabolic equation with nonlocal diffusion; sub-supersolution method, existence; uniqueness; long-time behavior
Mathematics Subject Classification: 35K57, 35B35

1. Introduction

In this paper, we consider the following parabolic problem with nonlocal nonlinearity:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\left(\displaystyle\int_\Omega |u(x,t)|^\gamma dx\right)\Delta u = f(u), & (x,t) \text{ in } \Omega\times(0,T), \\ u(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad x \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subseteq R^N$ $(N \geq 1)$ is a sufficiently regular domain, $\gamma \in [1,+\infty)$, $0 < T \leq +\infty$, $a \in C^1([0,+\infty),[0,+\infty))$, $u_0 \in C^{2+\alpha}(\overline{\Omega})$, $f \in C^1(R,R)$.

This type of problem was studied initially by Chipot and Lovat in [8], where they proposed the equation

$$\begin{cases} u_t - a\left(\int_\Omega u(z,t)dz\right)\Delta u = f(x,t), & \text{in } \Omega\times(0,T),\\ u(x,t)=0, \ (x,t) \text{ on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x), \ x \text{ in } \Omega \end{cases} \tag{1.2}$$

for modelling the density of a population, for example, of bacteria, subject to spreading and presented the existence and uniqueness of a weak solution to this equation. Another interesting result was obtained in [3] where Almeida, Antontsev and Duque considered the following problem

$$\begin{cases} u_t - \left(\int_\Omega u^2(z,t)dz\right)^\gamma \Delta u = f(x,t), & \text{in } \Omega\times(0,T),\\ u(x,t)=0, \ (x,t) \text{ on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x), \ x \text{ in } \Omega \end{cases} \tag{1.3}$$

and proved the existence, uniqueness and asymptotic behaviour of the weak solutions. Note that $f(x,t)$ is independent on u in problem (1.2) and if $a(t)=t^\gamma$, problem (1.1) can be changed into problem (1.3). Therefore, problem (1.1) is a generalization of problem (1.2) and (1.3).

For f depending on the state u such as $f(u)=ru(k-u)$ or $f(u)=ru/(k+u)$, Ackleh and Ke [1] studied the problem

$$\begin{cases} u_t - \dfrac{1}{a\left(\int_\Omega u(z,t)dz\right)}\Delta u = f(u), & \text{in } \Omega\times(0,T),\\ u(x,t)=0, & \text{on } \partial\Omega\times(0,T), \\ u(x,0)=u_0(x), & \text{in } \Omega, \end{cases} \tag{1.4}$$

proved the existence-uniqueness of a solution to this problem and gave some conditions for the extinction in finite time and the persistence respectively. If the coefficient $1/a$ is an unbounded function around the origin (e.g., $a(u(\cdot,t))=\int_\Omega u(x,t)dx$), then a diffusion of this type could model a population that is anxious to move quickly out of zones experiencing a sharp decrease in population densities. For example, consider a population attempting to leave a spatial region due to a sudden dangerous situation. The individuals in the population move randomly (due to lack of information) in an attempt to leave the area. In this case, diffusion out of the region will increase as population decreases due to a decrease in the interaction between individuals that hinders their movement out. One can imagine such an occurrence related to an epidemic. The asymptotic behaviour of the solutions as time tends to infinity was studied for nonlinear parabolic equations with two classes of nonlocal terms or a non-autonomous sublinear terms also (see [6,30]).

We point out that there is only one unknown function in (1.1). In fact, there are many types of species in some areas and then it is interesting to discuss nonlocal coupled systems. For examples, in [4], Duque et al. presented some results on the existence, uniqueness of weak and strong global in time solutions, polynomial and exponential decay and vanishing of the solutions in finite time. In [19], Raposo et al. discussed the existence, uniqueness and asymptotic behavior of the solutions for a nonlinear coupled system.

In this work, we present some conditions for reaction term f and diffusion coefficient a different from most previous papers: (1) reaction term f may grow superlinearly or exponentially at $+\infty$ or be lack of local Lipschiz continuity at 0; (2) diffusion coefficient a is unbounded and even grows exponentially at $+\infty$. This work is concerned with the proof of the existence, uniqueness and asymptotic behavior of the solutions and extend some results in previous literature (see [1, 3, 6, 8, 30]).

The paper is organized as follows. In Section 2, according to the proof in [1], we consider a generalized problem (2.1) of (1.1), transform (2.1) into (2.2) and show that the existence of solution to (2.1) is equivalent to the existence of solution to (2.2). In Section 3, we define the sub-supersolution pair for (3.1) which generalizes (2.2) and present the existence of solutions between the subsolution and supersolution. Section 4 is devoted to the proof of the existence, uniqueness and long-time behavior of solutions to (1.1) by using the method of sub-supersolution in Section 3. In Section 5, motivated by the idea in [1], we develop a finite difference scheme to approximate the solution of some reaction-diffusion equations. This scheme is then used to numerically study the long time behavior of the some models. Some ideas in our paper come from [5, 7, 9, 11–16, 20, 21, 23–29] also.

2. Equivalence of some generalized system to (1.1)

Now we generalize (1.1) to the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - a\left(\displaystyle\int_\Omega |u(x,t)|^\gamma dx\right)\Delta u = F(x,u), \quad (x,t) \text{ in } \Omega\times(0,T), \\ u(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad x \text{ in } \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subseteq R^N$ is a bounded domain with $\partial\Omega \in C^{2+\alpha}$, $u_0 \in C^{2+\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, $T \le +\infty$, $\gamma \in [1,+\infty)$.

In order to consider (2.1), we consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{1}{a\left(\displaystyle\int_\Omega |u(x,t)|^\gamma dx\right)} F(x,u), \quad (x,t) \text{ in } \Omega\times(0,T), \\ u(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \quad x \text{ in } \Omega, \end{cases} \tag{2.2}$$

where a satisfies

$$a \in C^1([0,+\infty), R), \quad \inf_{t\in[0,+\infty)} a(t) \ge a(0) \overset{def.}{=} a_0 > 0. \tag{2.3}$$

Theorem 2.1 *Suppose that $u_0 \in C^{2+\alpha}(\overline{\Omega})$, with $u_0 \neq 0$ and (2.3) holds. Then (2.1) has a solution $u(x,t)$ on $[0, \overline{T}_{\max})$ if and only if that (2.2) has a solution $v(x,t)$ on $[0, T_{\max})$, where the relation between $\overline{T}_{\max}$ and $T_{\max}$ is*

$$\overline{T}_{\max} = \int_0^{T_{\max}} \frac{1}{a(\int_\Omega |v(x,s)|^\gamma dx)} ds, \quad T_{\max} = \int_0^{\overline{T}_{\max}} a(\int_\Omega |u(x,s)|^\gamma dx) ds.$$

Proof. **Sufficiency.** Suppose that v is a solution of (2.2). Let $\tau(t)$ be a solution to the following ordinary differential equation

$$\begin{cases} \tau'(t) = a(\displaystyle\int_\Omega |v(x,\tau(t))|^\gamma dx), \quad t > 0, \\ \tau(0) = 0. \end{cases} \tag{2.4}$$

Separating variables and integrating in t we get the following equation:

$$\int_0^{\tau(t)} \frac{1}{a(\int_\Omega |v(x,s)|^\gamma dx)} ds = t, \quad t \in [0, \overline{T}_{\max}).$$

Set

$$G(\xi) = \int_0^{\xi} \frac{1}{a(\int_\Omega |v(x,s)|^\gamma dx)} ds, \quad \xi \in [0, T_{\max}).$$

It can be easily shown that G is a C^1 diffeomorphism from $[0, T_{\max})$ onto $[0, \overline{T}_{\max})$ and (2.3) implies that (2.4) has a unique solution given by $\tau(t) = G^{-1}(t)$ on $[0, \overline{T}_{\max})$.

Now let

$$u(x,t) = v(x, \tau(t)), \quad x \in \overline{\Omega} \times [0, \overline{T}_{\max}).$$

Then clearly u satisfies the following: $u(x,0) = v(x,\tau(0)) = u_0(x)$ and $u(x,t)$ is continuous for $t \geq 0$, continuously differentiable for $t > 0$. Furthermore, we have that

$$\begin{aligned}
\frac{u(x,t)}{\partial t} &= \frac{v(x,r)}{\partial r}|_{r=\tau(t)} \tau'(t) \\
&= [-\Delta v(x,\tau(t)) + \frac{1}{a(\int_\Omega |v(x,\tau(t))|^\gamma dx)} F(x, v(x,\tau(t)))] a(\int_\Omega |v(x,\tau(t))|^\gamma) dx \\
&= -a(\int_\Omega |v(x,\tau(t))|^\gamma dx) \Delta v(x,\tau(t)) + F(x, v(x,\tau(t))) \\
&= -a(\int_\Omega |u(x,t)|^\gamma dx) \Delta u(x,t) + F(x, u(x,t)), \quad (x,t) \in \Omega \times (0, \overline{T}_{\max}).
\end{aligned}$$

Hence, u is a local solution of Eq (2.1).

Necessity. Let $u(x,t)$ be a solution to (2.1) and let G be the solution to the differential equation

$$\begin{cases} G'(s) = \dfrac{1}{a(\int_\Omega |u(x,G(s))|^\gamma dx)}, \ t > 0, \\ G(0) = 0. \end{cases} \tag{2.5}$$

Separating variables and integrating in t we get the following equation:

$$\int_0^{G(t)} a(\int_\Omega |u(x,s)|^\gamma dx) ds = t, \quad t \in [0, T_{\max}).$$

Set

$$\tau(\xi) = \int_0^{\xi} a(\int_\Omega |u(x,s)|^\gamma dx) ds, \quad t \in [0, \overline{T}_{\max}).$$

It can be easily shown that τ is a C^1 diffeomorphism from $[0, \overline{T}_{\max})$ onto $[0, T_{\max})$ and (2.3) implies that (2.5) has a unique solution given by $G(t) = \tau^{-1}(t)$ on $[0, T_{\max})$.

Set

$$v(x,t) = u(x, G(t)), \quad (x,t) \in \overline{\Omega} \times [0, T_{\max}).$$

We have that $v(x,0) = u(x,0) = u_0(x)$, $v(x,t)$ is continuous for $t \geq 0$, continuously differentiable for $t > 0$. Moreover,

$$\begin{aligned}
\frac{v(x,t)}{\partial t} &= \frac{u(x,r)}{\partial r}|_{r=G(t)}G'(t) \\
&= [-a(\int_\Omega |u(x,G(t))|^\gamma dx)\Delta u(x,G(t)) + F(x,u(x,G(t))]\frac{1}{a(\int_\Omega |u(x,G(t))|^\gamma dx)} \\
&= (-\Delta v(x,t) + \frac{1}{a(\int_\Omega |v(x,t)|^\gamma dx)}F(x,v(x,t)), t > 0.
\end{aligned}$$

The proof is complete. □

Remark 2.1 Under the uniform Lipschitz continuity requirement on the functions F, condition (2.2) on a and $\gamma \geq 1$, Eq (2.2) has a unique mild solution (see [17, 22]).

Remark 2.2 The idea of our theorem comes from Theorem 2.1 in [1]. But in our proof, it is not necessary that F satisfies uniform Lipschitz continuity condition.

3. Sub-supersolution method

Let $k \geq 0$ be an integer, $C^k(\overline{\Omega}) = \{u : \overline{\Omega} \to \mathbb{R} | D^l u \in C(\overline{\Omega}),\ \forall\ |l| \leq k\}$ with the norm

$$||u||_{\overline{\Omega}}^{(k)} = \sum_{|s|\leq k} \max_{x\in\overline{\Omega}} |D^s u(x)|,$$

$C^{k+\alpha}(\overline{\Omega}) = \{u : \overline{\Omega} \to \mathbb{R} | u \in C^{(k)}(\overline{\Omega}), H_\alpha(D^s u) < +\infty\ \forall\ |s| = k\}$ with the norm

$$||u||_{\overline{\Omega}}^{(k+\alpha)} = \sum_{|s|\leq k} \max_{x\in\overline{\Omega}} |D^s u(x)| + \sum_{|s|=k} H_\alpha(D^s u),$$

where

$$H_\alpha(u) = \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha},$$

$C(\overline{Q}_T) = \{u : \overline{Q}_T \to \mathbb{R} | u \text{ is continuous on } \overline{Q}_T\}$ with the norm

$$||u||_{\overline{Q}_T} = \max_{(x,t)\in\overline{Q}_T} |u(x,t)|,$$

$C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T) = \{u : \overline{Q}_T \to \mathbb{R} | u \in C(\overline{Q}_T), H_{\alpha,\frac{\alpha}{2}}(u) < +\infty\}$ with the norm

$$||u||_{\overline{Q}_T}^{(\alpha)} = \max_{(x,t)\in\overline{Q}_T} |u(x,t)| + H_{\alpha,\frac{\alpha}{2}}(u),$$

and $C^{2k+\alpha,k+\frac{\alpha}{2}}(\overline{Q}_T) = \{u : \overline{Q}_T \to \mathbb{R} | u \in C^{2k,k}(\overline{Q}_T), H_{\alpha,\frac{\alpha}{2}}(D_t^r D_x^s u) < +\infty\ \ \forall 2r + |s| = 2k\}$ with the norm

$$||u||_{\overline{Q}_T}^{(2k+\alpha)} = \sum_{0\leq 2r+|s|\leq 2k} \max_{(x,t)\in\overline{Q}_T} |D_t^r D_x^s u(x,t)| + \sum_{2r+|s|=2k} H_{\alpha,\frac{\alpha}{2}}(D_t^r D_x^s u),$$

where

$$H_{\alpha,\frac{\alpha}{2}}(u) = \sup_{x,y\in\Omega, x\neq y, s,t\in[0,T], s\neq t} \frac{|u(x,s) - u(y,t)|}{|x-y|^\alpha + |s-t|^{\frac{\alpha}{2}}}.$$

For $p > 1$, let $L_p(\Omega) = \{u : \Omega \to \mathbb{R} | u$ is measurable on Ω and $\int_Q |u|^p dx < +\infty\}$ with norm

$$\|u\|_p = \left(\int_\Omega |u(x)^p dx\right)^{\frac{1}{p}},$$

$W_p^k(\Omega) = \{u : \Omega \to \mathbb{R} | u \in L_p(\Omega), D^l u \in L_p(\Omega), |l| \le k\}$ with the norm

$$\|u\|_{k,p} = \sum_{|l|\le 2} \left(\int_\Omega |D^l u|^p dx\right)^{\frac{1}{p}}.$$

Let $Q_T = \Omega \times (0, T]$ and $L_p(Q_T) = \{u : Q \to \mathbb{R} | u$ is measurable on Q_T and $\int_{Q_T} |u|^p dxdt < +\infty\}$ with the norm

$$\|u\|_{p,Q_T} = (\int_{Q_T} |u(x)^p dxdt)^{\frac{1}{p}},$$

$W_p^{2k,k}(Q_T) = \{u : Q_T \to \mathbb{R} | u \in L_p(Q_T), D_t^s D_x^l u \in L_p(Q_T), 2s + |l| \le 2k\}$ with the norm

$$\|u\|_{p,Q_T}^{(2k)} = \sum_{0\le 2s+|l|\le 2k} \left(\int_{Q_T} |D_t^s D_x^l u|^p dxdt\right)^{\frac{1}{p}}.$$

In this section, we generalize (2.2) to the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u(x,t) = \dfrac{1}{a(\int_\Omega |u(x,t)|^\gamma dx)} F(x,t,u(x,t)), \quad (x,t) \text{ in } \mathrm{Q_T}, \\ u(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), \quad x \text{ in } \Omega, \end{cases} \tag{3.1}$$

where $\Omega \subseteq R^N$ is a bounded domain with $\partial\Omega \in C^{2+\alpha}$, $u_0 \in C^{2+\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$, $T < +\infty$, $Q_T = \Omega \times (0,T]$, $\gamma \in (0,+\infty)$ and $a : [0,+\infty) \to (0,+\infty)$ is continuous with

$$\inf_{t\in[0,+\infty)} a(t) \ge a(0) \overset{def.}{=} a_0 > 0. \tag{3.2}$$

Definition 3.1 *The pair functions α and β with α, $\beta \in C(\overline{Q}_T) \cap C^2(Q_T)$ are subsolution and supersolution of (3.1) if $\alpha(x,t) \le \beta(x,t)$ for $(x,t) \in \overline{Q}_T$ and*

$$\begin{cases} \dfrac{\partial \alpha}{\partial t} - \Delta\alpha(x,t) \le \min\{\dfrac{1}{a_0} F(x,t,\alpha(x,t)), \dfrac{1}{b_0(t)} F(x,t,\alpha(x,t))\}, \quad (x,t) \text{ in } \mathrm{Q_T}, \\ \alpha(x,t) \le 0, \quad (x,t) \text{ on } \partial\Omega \times (0,T], \\ \alpha(x,0) \le u_0(x), \quad x \text{ in } \Omega, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial \beta}{\partial t} - \Delta\beta(x,t) \ge \max\{\dfrac{1}{a_0} F(x,t,\beta(x,t)), \dfrac{1}{b_0(t)} F(x,t,\beta(x,t))\}, \quad (x,t) \text{ in } \mathrm{Q_T}, \\ \beta(x,t) \ge 0, \quad (x,t) \text{ on } \partial\Omega \times (0,\mathrm{T}], \\ \beta(x,0) \ge u_0(x), \quad x \text{ in } \Omega, \end{cases}$$

where $a_0 = a(0)$ *and* $b_0(t) = \sup_{s\in[0,\int_\Omega \max\{|\alpha(x,t)|,|\beta(x,t)|\}^\gamma dx]} a(s)$.

For fixed $\lambda > 0$, we list the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \lambda u(x) = h(x,t), \quad (x,t) \text{ in } Q_T, \\ u(x,t) = 0, \quad x \text{ on } \partial\Omega, \ t \in (0,T], \\ u(x,0) = \phi(x), \end{cases} \tag{3.3}$$

where $\Omega \subseteq R^N$ is a smooth bounded domain and give the deformation of Agmon-Douglas-Nirenberg theorem for (3.3).

Lemma 3.1 (see [2], Agmon-Douglas-Nirenberg) *If* $p > 1$ *and* $h \in L^p(Q_T)$, $\phi \in W_p^2(\Omega)$, *then (3.3) has a unique solution* $u \in W_p^{2,1}(Q_T)$ *such that*

$$\|u\|_{p,Q_T}^{(2)} \le C_1(\|h\|_{p,Q_T} + \|\phi\|_{2,p}),$$

where C_1, C_2 *are independent from u, h.*

We define the unique solution $u = (\frac{\partial}{\partial t} - \Delta + \lambda)^{-1}h$ of (3.3) and obviously $(\frac{\partial}{\partial t} - \Delta + \lambda)^{-1}$ is a linear operator. In order to prove our theorem, we list the Embedding theorem.

Lemma 3.2 (See [10, 18])*Suppose* $Q_T \subseteq R^{N+1}$ *is a bounded domain with smooth boundary and* $2p > N + 2$. *Then there exists a* $C(N + 2, p, Q_T) > 0$ *such that*

$$|u|_{\overline{Q}_T}^{(\alpha)} \le C(N+2, p, Q_T)\|u\|_{p,Q_T}^{(2)}, \quad \forall\, u \in W_p^{2,1}(Q_T),$$

where $0 < \alpha < 1 - \frac{N+2}{2p}$.

Then we have the following main theorem.

Theorem 3.1 *Suppose that* $F : \overline{Q}_T \times R \to R$ *is a continuous function. Assume* α *and* β *are the subsolution and supersolution of (3.1) respectively. Then problem (3.1) has at least one solution* $u \in C^2(Q_T) \cap C(\overline{Q}_T)$ *such that, for all* $(x,t) \in \overline{Q}_T$,

$$\alpha(x,t) \le u(x,t) \le \beta(x,t).$$

Proof. For $u \in C(\overline{Q}_T)$, define

$$\chi(x,t,u(x,t)) = \alpha(x,t) + (u(x,t) - \alpha(x,t))^+ - (u(x,t) - \beta(x,t))^+.$$

We will study the modified problem ($\lambda > 0$)

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \Delta u(x,t) + \lambda u(x,t) = \dfrac{F(x,t,\chi(x,t,u(x,t)))}{a(\int_\Omega |\chi(x,t,u(x,t))|^\gamma dx)} + \lambda\chi(x,t,u(x,t)), (x,t) \in Q_T, \\ u(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), \quad x \text{ in } \Omega. \end{cases} \tag{3.4}$$

Step 1. Every solution $u(x,t)$ of (3.4) satisfies $\alpha(x,t) \le u(x,t) \le \beta(x,t)$, $x \in \overline{Q}_T$.

We prove that $\alpha(x,t) \le u(x,t)$ on $\overline{Q}_T$. Obviously, $|\chi(x,t,u(x,t))| \le \max\{|\alpha(x,t)|, |\beta(x,t)|\}$, which implies that

$$a_0 \le a(\int_\Omega |\chi(x,t,u(x,t))|^\gamma dx) \le b_0(t).$$

By contradiction, assume that $\alpha(x_0,t_0)-u(x_0,t_0)=\max_{x\in\bar{Q}_T}(\alpha(x,t)-u(x,t))=M>0$. Note that: $\alpha(x,0)-u(x,0)\le 0$ on Ω, $\alpha(x,t)-u(x,t)\le 0$, $(x,t)\in\partial\overline{\Omega}\times(0,T]$, there are two cases: (1) $(x_0,t_0)\in\Omega\times(0,T)$; (2) $x_0\in\Omega$, $t_0=T$.

For the case (1), we have $0\le-\Delta(\alpha(x_0,t_0)-u(x_0,t_0))$ and $\frac{\partial(\alpha(x_0,t)-u(x_0,t))}{\partial t}|_{t=t_0}=0$, which together with the definition of subsolution $\alpha(x,t)$ implies

$$\begin{aligned}
0 &\le -\Delta(\alpha(x_0,t_0)-u(x_0,t_0))\\
&\le \min\{\frac{1}{a_0}F(x,t,\alpha(x,t)),\frac{1}{b_0(t)}F(x,t,\alpha(x,t))\}-\frac{\partial\alpha(x_0,t)}{\partial t}|_{t=t_0}\\
&\quad -\frac{F(x_0,t_0,\chi(x_0,t_0,u(x_0,t_0))}{a(\int_\Omega|\chi(x,t,u(x,t))|^\gamma dx)}+\frac{\partial u(x_0,t)}{\partial t}|_{t=t_0}-\lambda\chi(x_0,t_0,u(x_0,t_0))+\lambda u(x_0,t_0)\\
&= \min\{\frac{1}{a_0}F(x,t,\alpha(x,t)),\frac{1}{b_0(t)}F(x,t,\alpha(x,t))\}-\frac{\partial\alpha(x_0,t)}{\partial t}|_{t=t_0}\\
&\quad -\frac{F(x_0,t_0,\alpha(x_0,t_0))}{a(\int_\Omega|\chi(x,t,u(x,t))|^\gamma dx)}+\frac{\partial u(x_0,t)}{\partial t}|_{t=t_0}-\lambda\chi(x_0,t_0,u(x_0,t_0))+\lambda u(x_0,t_0)\\
&\le -\lambda(\alpha(x_0,t_0)-u(x_0,t_0))\\
&< 0.
\end{aligned}$$

This is a contradiction.

For the case (2), we have $0\le-\Delta(\alpha(x_0,t_0)-u(x_0,t_0))$ and $\frac{\partial(\alpha(x_0,t)-u(x_0,t))}{\partial t}|_{t=t_0}\ge 0$, which together with the definition of subsolution $\alpha(x,t)$ implies that

$$\begin{aligned}
0 &\le -\Delta(\alpha(x_0,t_0)-u(x_0,t_0))\\
&\le \min\{\frac{1}{a_0}F(x,t,\alpha(x,t)),\frac{1}{b_0(t)}F(x,t,\alpha(x,t))\}-\frac{\partial\alpha(x_0,t)}{\partial t}|_{t=t_0}\\
&\quad -\frac{1}{a(\int_\Omega|\chi(x,t,u(x,t))|^\gamma dx)}F(x_0,\chi(x_0,t_0,u(x_0,t_0))\\
&\quad +\frac{\partial u(x_0,t)}{\partial t}|_{t=t_0}-\lambda\chi(x_0,t_0,u(x_0))+\lambda u(x_0,t_0)\\
&= \min\{\frac{1}{a_0}F(x,t,\alpha(x,t)),\frac{1}{b_0(t)}F(x,t,\alpha(x,t))\}-\frac{\partial\alpha(x_0,t)}{\partial t}|_{t=t_0}\\
&\quad -\frac{1}{a(\int_\Omega|\chi(x,t,u(x,t))|^\gamma dx)}F(x_0,\alpha(x_0,t_0))\\
&\quad +\frac{\partial u(x_0,t)}{\partial t}|_{t=t_0}-\lambda\chi(x_0,t_0,u(x_0))+\lambda u(x_0,t_0,t_0)\\
&\le -\lambda(\alpha(x_0,t_0)-u(x_0,t_0))\\
&< 0.
\end{aligned}$$

This is a contradiction also.

Consequently, $\alpha(x,t)\le u(x,t)$ for $(t,x)\in\overline{Q}_T$.

A similar argument shows that $\beta(x,t)\ge u(x,t)$ for $(t,x)\in\overline{Q}_T$ and we omit the proof.

Hence,

$$\alpha(x,t)\le u(x,t)\le\beta(x,t),\quad (t,x)\in\overline{Q}_T.$$

Step 2. Every solution of (3.4) is a solution of (3.1). From step 1, every solution of (3.4) is such that: $\alpha(x,t)\le u(x,t)\le\beta(x,t)$, which implies that

$$\chi(x,t,u(x,t))=u(x,t),\quad F(x,t,\chi(x,t,u(x,t))=F(x,t,u(x,t)),\quad (x,t)\in Q_T,$$

and $u(x,t)$ is a solution of (3.1).

Step 3. The problem (3.4) has at least one solution.

Choose $2p > N+2$ and $0 < \alpha < 1 - \frac{N+2}{2p}$ and define an operator

$$\overline{N} : C(\overline{Q}_T) \to C(\overline{Q}_T) \subseteq L^p(Q_T); u \to F(\cdot,\cdot,\chi(\cdot,\cdot,\cdot)).$$

Since F is continuous, $\overline{N} : C(\overline{Q}_T) \to C(\overline{Q}_T)$ is well defined, continuous and maps bounded sets to bounded sets. Since (3.2) is true, a is continuous and $\dfrac{1}{a(\int_\Omega |\chi(x,t,u(x,t))|^\gamma dx)} \le \dfrac{1}{a_0}$, the operator $\overline{N}_1 u = \frac{1}{a(\int_\Omega |\chi(x,t,u(x,t))|^\gamma dx)} \overline{N}u$ is continuous, and maps bounded sets of $C(\overline{Q}_T)$ into bounded sets of $C(\overline{Q}_T) \subseteq L_p(\overline{Q}_T)$.

Now, for $u \in C(\overline{Q}_T)$, we define an operator $\overline{A} : C(\overline{Q}_T) \to C(\overline{Q}_T)$ by

$$\overline{A}(u) = (\frac{\partial}{\partial t} - \Delta + \lambda)^{-1}(\overline{N}_1 u + \lambda\chi(\cdot,\cdot,u)).$$

Now we show that $\overline{A} : C(\overline{Q}_T) \to C(\overline{Q}_T)$ is completely continuous.

(1) By the construction of χ, we have, for every $u \in C(\overline{Q}_T)$,

$$\begin{aligned} &\left| \frac{F(x,t,\chi(x,t,u(x,t))}{a(\int_\Omega |\chi(x,t,u(x,t))|^\gamma dx)} + \lambda\chi(x,t,u(x)) \right| \\ \le\ & \frac{1}{a_0} \max_{(x,t)\in\overline{Q}_T, \alpha(x,t)\le u\le\beta(x,t)} |F(x,t,u)| + \lambda \max\{\|\alpha\|_{\overline{Q}_T}, \|\beta\|_{\overline{Q}_T}\}, \forall (x,t) \in \overline{Q}_T, \end{aligned}$$

which guarantees that there exists a $K > 0$ big enough such that $N_1 u + \lambda\chi(\cdot,\cdot,u) \in B_{L^p}(0,K)$ for all $u \in C(\overline{Q}_T)$, where

$$B_{L^p}(0,R) = \{u \in L_p(Q_T) | \|u\|_{p,Q_T} \le K\}.$$

By Lemma 3.1, we have

$$\|\overline{A}(u)\|^{(2)}_{p,Q_T} = \|(\frac{\partial}{\partial t} - \Delta + \lambda)^{-1}(\overline{N}_1 u + \lambda\chi(\cdot,\cdot,u))\|^{(2)}_{p,Q_T} \le C_1(K + \|u_0\|_{2,p}), \quad \forall u \in C(\overline{Q}_T), \tag{3.5}$$

which implies that $\overline{A}(C(\overline{Q}_T))$ is bounded in $W^{2,1}_p(Q_T)$. Now Lemma 3.2 guarantees that $\overline{A}(C(\overline{Q}_T))$ is bounded in $C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)$. Therefore, $\overline{A}(C(\overline{Q}_T))$ is relatively compact in $C(\overline{Q}_T)$.

(2) For $u_1, u_2 \in C(\overline{Q}_T)$, by Lemma 3.1, one has

$$\|\overline{A}(u_1) - \overline{A}(u_2)\|^{(2)}_{p,Q_T} \le C_1 \|\overline{N}_1 u_1 + \lambda\chi(\cdot,\cdot,u_1) - (\overline{N}_1 u_2 + \lambda\chi(\cdot,\cdot,u_2))\|_{p,Q_T},$$

which together the continuity of the operator $N_1 + \lambda\chi$ guarantees that $A : C(\overline{Q}_T) \to C(\overline{Q}_T)$ is continuous.

Consequently, $A : C(\overline{Q}_T) \to C(\overline{Q}_T)$ is completely continuous.

By (3.5) and Lemma 3.2, there exists a $K_1 > 0$ big enough such that

$$\overline{A}(C(\overline{Q}_T)) \subseteq B_C(0,K_1),$$

where $B_C(0,K_1) = \{u \in C(\overline{Q}_T) | \|u\| \le K_1\}$, which implies that

$$\overline{A}(B_C(0,K_1)) \subseteq B_C(0,K_1).$$

The Schauder fixed point theorem guarantees that there exists a $u \in B_C(0, K_1)$ such that

$$u = \overline{A}u,$$

i.e., u is a solution of (3.4).

Consequently, the step 1 and step 2 guarantees that u in the step 3 is a solution of (3.1).

The proof is complete. □

Corollary 3.1 *Assume that the conditions of Theorem 3.1 hold and F satisfies local lipschitz condition. Then (3.1) has a unique solution u with*

$$\alpha(x,t) \le u(x,t) \le \beta(x,t), \quad (x,t) \in \overline{\Omega} \times [0,T].$$

Proof. Assume that there is another solution u_1 with $u_1(x,t) \not\equiv u(x,t)$. Then there is a $t_1 > 0$ with $u_1(x,t_1) \neq u(x,t_1)$. Let $t_* = \inf\{t < t_1 | u_1(x,s) \neq u(x,s)$ for all $s \in [t,t_1)\}$. Since $u(x,0) = u_1(x,0) = u_0(x)$, we have $t_* \ge 0$, $u_1(x,t_*) = u(x,t_*)$ and $u_1(x,t) \neq u(x,t)$ for all $t \in (t_*, t_1]$. Since $\frac{1}{a(\int_\Omega |u(x,t)|^\gamma dx)} F(x,t,u)$ is locally Lipschitz continuous,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{1}{a(\int_\Omega |u(x,t)|^\gamma dx)} F(x,t,u), & t > t_*, \\ u = 0, \quad x \text{ on } \partial\Omega, \ t > t^*; \\ u(x,t_*) = u_1(x,t_*), \end{cases}$$

has a unique solution. This is a contradiction. The proof is complete. □

4. Results

In this section, using the method of sub-supersolution in above section, we consider the existence, uniqueness and long time behavior of the solution for (1.1).

Now we list some results for a parabolic equation (see [18]) which will be used later. Assume that $\overline{u}$ and $\underline{u}$ are the super-subsolutions to the following equation

$$\begin{cases} -\Delta u = f(x,u), & x \text{ in } \Omega, \\ u(x) = g(x), & x \text{ on } \partial\Omega. \end{cases}$$

If $\underline{u}(x) \le \phi(x) \le \overline{u}(x)$, $x \in \overline{\Omega}$, then $u_{\overline{u}}$ and $u_{\underline{u}}$ are the super-subsolutions to the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x,u), & x \text{ in } \Omega, \\ u(x,t) = g(x), & (x,t) \text{ on } \partial\Omega \times [0,+\infty), \\ u(x,0) = \phi(x), & x \in \Omega. \end{cases} \tag{4.1}$$

If $f \in C^1(\overline{\Omega} \times R)$, then (4.1) has a unique solution $u(x,t)$ with

$$\underline{u}(x) \le V(x,t) \le u(x,t) \le U(x,t) \le \overline{u}(x),$$

where $V(x,t) = u_{\underline{u}}(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x,u), & x \text{ in } \Omega, \\ u(x,t) = g(x), & (x,t) \text{ on } \partial\Omega \times [0,+\infty), \\ u(x,0) = \underline{u}(x), & x \in \Omega \end{cases}$$

and $U(x,t) = u_{\overline{u}}(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(x,u), & x \text{ in } \Omega, \\ u(x,t) = g(x), & (x,t) \text{ on } \partial\Omega \times [0,+\infty), \\ u(x,0) = \overline{u}(x), & x \in \Omega. \end{cases}$$

Lemma 4.1 (see, [27]) *$V(x,t)$ is nondecreasing on $[0,+\infty)$ and $U(x,t)$ is nonincreasing on $[0,+\infty)$.*

Lemma 4.2 (see, [18,27]) *Suppose u, $v \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ satisfying*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u - F(x,u) \geq \dfrac{\partial v}{\partial t} - \Delta v - F(x,v), & x \text{ in } \Omega, \ \mathrm{t} \in (0,\mathrm{T}] \\ u(x,t) \geq v(x,t), & (x,t) \text{ on } \mathrm{S_T} = \partial\Omega \times (0,\mathrm{t}], \\ u(x,0) \geq v(x,0), & x \text{ in } \Omega. \end{cases}$$

If $(x,t) \in \overline{Q}_T$, u, $v \in [m,M]$ and $\frac{\partial F}{\partial u} \in C(\overline{\Omega} \times [m,M])$, then

$$u(x,t) \geq v(x,t), \quad \forall (x,t) \in \overline{Q}_T.$$

Moreover, if $u(x,0) \not\equiv v(x,0)$, $x \in \Omega$, we have

$$u(x,t) > v(x,t), \quad \forall (x,t) \in Q_T.$$

Let Φ_1 be the eigenfunction corresponding to the principle eigenvalue λ_1 of

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \tag{4.2}$$

It is found that $\lambda_1 > 0$,

$$\Phi_1(x) > 0, \quad |\nabla\Phi_1(x)| > 0, \quad \forall x \in \partial\Omega. \tag{4.3}$$

According to Theorem 2.1, in order to study system (1.1), we only consider the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{1}{a\left(\displaystyle\int_\Omega |u(x,t)|^\gamma dx\right)} f(u), & (x,t) \text{ in } \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \text{ in } \Omega, \end{cases} \tag{4.4}$$

where a satisfies (2.3) and $u_0 \in C^{2+\alpha}(\overline{\Omega})$ with $u_0 \neq 0$.

Theorem 4.1 *Suppose $f \in C^1(R,R)$ with $f(0) = f(1) = 0$ and $f(u) > 0$ for all $u \in (0,1)$ and $f(u) < 0$ for $u > 1$. Then, for any $u_0(x) \geq\not\equiv 0$ with $u_0 \in C^{2+\alpha}(\overline{\Omega})$, (1.1) has a unique nonnegative solution $u \in C^2(Q_{+\infty}) \cap C(\overline{Q}_{+\infty})$.*

Proof. Since $f \in C^1(R, R)$, (4.4) has a unique local solution v.

Step 1. We show that for any $l > 0$, $z_0 > 0$, the ordinary differential equation

$$\begin{cases} \dfrac{dz}{dt} = \dfrac{1}{l} f(z), t > 0, \\ z(0) = z_0 \end{cases} \tag{4.5}$$

has a unique positive solution $z(t, z_0)$ with

$$\lim_{t \to +\infty} z(t, z_0) = 1. \tag{4.6}$$

In fact, since $f \in C^1$, (4.5) has a unique solution $z(t, z_0)$. Since $f(1) = 0$ and $f \in C^1$, $z(t) \equiv 1$ is the unique solution of $\frac{dz}{dt} = \frac{1}{l} f(z)$ across any $(t_0, 1)$. If $1 > z_0 > 0$, since $f(u) > 0$ for all $u \in (0, 1)$ and $f(u) < 0$ for all $u > 1$, then $z(t, z_0)$ is increasing and $z(t, z_0) < 1$, which implies that there is a $1 \geq c > 0$ such that

$$\lim_{t \to +\infty} z(t, z_0) = c.$$

Therefore, there is a $\{t_n\}$ with $\lim\limits_{n \to +\infty} t_n = +\infty$ such that

$$\lim_{n \to +\infty} z'(t_n, z_0) = 0.$$

By $z'(t_n, z_0) = f(z(t_n, z_0))$, one has

$$0 = \lim_{n \to +\infty} z'(t_n, z_0) = \lim_{n \to +\infty} f(z(t_n, z_0)) = f(c).$$

Hence, $c = 1$, i.e.

$$\lim_{t \to +\infty} z(t, z_0) = 1,$$

which guarantees that (4.6) is true. By a same proof, we get if $z_0 \geq 1$,

$$\lim_{t \to +\infty} z(t, z_0) = 1$$

also.

Step 2. We establish the sub-supersolution pair for (4.4).

Choose $M = \max\{2, \max\limits_{x \in \overline{\Omega}} u_0(x)\}$. (4.5) and (4.6) imply that

$$\begin{cases} \dfrac{dz}{dt} = \dfrac{1}{a(M^\gamma |\Omega|)} f(z), t > 0, \\ z(0) = M \end{cases} \tag{4.7}$$

has a unique positive nonincreasing solution $z(t, M)$ with

$$M \geq z(t, M) > 1, \forall t > 0 \text{ and } \lim_{t \to +\infty} z(t, M) = 1. \tag{4.8}$$

Let $\beta(x, t) = z(t, M)$ and $\alpha(x, t) = 0$. Set

$$b_0(t) = \sup_{s \in [0, \int_\Omega \max\{|\alpha(x,t)|, |\beta(x,t)|\}^\gamma dx]} a(s) = \sup_{s \in [0, \int_\Omega \max\{0, |z(t,M)|\}^\gamma dx]} a(s) \leq a(M^\gamma |\Omega|),$$

which together with $f(\beta(x,t)) < 0$ implies that

$$\frac{1}{a(M^\gamma|\Omega|)}f(\beta(x,t)) \geq \max\{\frac{1}{a_0}f(\beta(x,t)), \frac{1}{b_0(t)}f(\beta(x,t))\}. \tag{4.9}$$

By (4.7)–(4.9) and the definitions of $\alpha(x,t)$ and $\beta(x,t)$, we have

(1) $\alpha(x,t) < \beta(x,t)$, $\forall (x,t) \in \Omega \times (0,+\infty)$;

(2)

$$\begin{cases} \frac{\partial \alpha}{\partial t} - \Delta\alpha(x,t) = 0 = \min\{\frac{1}{a_0}f(\alpha(x,t)), \frac{1}{b_0(t)}f(\alpha(x,t))\}, (x,t) \text{ in } \Omega \times (0,+\infty), \\ \alpha(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ \alpha(x,0) \leq u_0(x), \quad x \text{ in } \Omega \end{cases}$$

and

$$\begin{cases} \frac{\partial \beta}{\partial t} - \Delta\beta(x,t) \geq \max\{\frac{1}{a_0}f(\beta(x,t)), \frac{1}{b_0(t)}f(\beta(x,t))\}, \quad (x,t) \text{ in } \Omega \times (0,+\infty), \\ \beta(x,t) = z(t,M) > 0, \quad (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ \beta(x,0) = M > u_0(x), \quad x \text{ in } \Omega, \end{cases}$$

which imply that α and β are subsolution and supersolution to (4.4).

Hence, Theorem 3.1 together with $f \in C^1$ implies that (4.4) has a unique positive solution v such that

$$0 = \alpha(x,t) \leq v(x,t) \leq \beta(x,t) = z(t,M), \quad \forall (x,t) \in \overline{\Omega} \times [0,+\infty).$$

Step 3. We show that (1.1) has a unique solution u, $(x,t) \in \overline{\Omega} \times [0,+\infty)$.

Since $0 \leq v(x,t) \leq M$, one has $a_0 \leq a(\int_\Omega |v(x,t)|^\gamma dx) \leq a(M^\gamma|\Omega|)$. Since $T_{\max} = +\infty$ in Theorem 2.1, one has

$$\overline{T}_{\max} = \int_0^{T_{\max}} \frac{1}{a(\int_\Omega |v(x,t)|^\gamma dx)}dt = +\infty.$$

Let

$$G(\xi) = \int_0^{\xi} \frac{1}{a(\int_\Omega |v(x,t)|^\gamma dx)}dt, \quad t \in [0,+\infty)$$

and $\tau(t) = G^{-1}(t)$, $t \in [0,+\infty)$. Then $u(x,t) = v(x,\tau(t))$ is a unique nonnegative solution to (1.1) on $[0,+\infty)$.

The proof is complete. □

Corollary 4.1 *Suppose the conditions of Theorem 4.1 hold. If there is a* $\varepsilon_0 > 0$ *with* $u_0(x) \geq \varepsilon_0\phi_1(x)$ *for all* $x \in \overline{\Omega}$ *and* $f'(0) > a(M^\gamma|\Omega|)\lambda_1$*(*$M = \max\{1, \max_{x\in\overline{\Omega}} u_0(x)\}$*), then the unique solution* $u \in C^2(Q_{+\infty}) \cap C(\overline{Q}_{+\infty})$ *of (1.1) satisfies*

$$\lim_{t\to+\infty} u(x,t) = 1.$$

Proof. Choose a $\varepsilon : 0 < \varepsilon < \min\{1, \varepsilon_0\}$ small enough such that

$$\frac{f(s)}{s} \geq \frac{\lambda_1 a(M^\gamma|\Omega|) + f'(0)}{2} > \lambda_1 a(M^\gamma|\Omega|), \quad \forall s \in (0,\varepsilon],$$

which implies

$$\frac{1}{a(M^\gamma|\Omega|)}f(s) > \lambda_1 s, \quad \forall s \in (0, \varepsilon].$$

Let $\alpha_1(x) = \varepsilon\phi_1(x)$. Then

$$\frac{1}{a(M^\gamma|\Omega|)}f(\alpha_1(x)) \geq \lambda_1\alpha_1(x), \quad \forall x \in \Omega.$$

Hence,

$$\begin{cases} -\Delta\alpha_1 = \lambda_1\alpha_1(x) \leq \dfrac{1}{a(M^\gamma|\Omega|)}f(\alpha_1(x)), & x \text{ in } \Omega, \\ \alpha_1(x) = 0, & x \text{ on } \partial\Omega. \end{cases}$$

Step 1. Problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{1}{a(M^\gamma|\Omega|)}f(u), & (x,t) \text{ in } \Omega \times (0,+\infty), \\ u = 0, & (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ u(x,0) = \varepsilon\phi_1(x), & x \text{ in } \Omega \end{cases} \tag{4.10}$$

has a unique solution $V(x,t)$ such that

$$\lim_{t\to+\infty} V(x,t) = 1.$$

Let $\alpha_1(x,t) = \alpha_1(x)$ and $\beta_1(x,t) = 1$, $\forall (x,t) \in \overline{\Omega} \times [0,+\infty)$. Then

(1) $\alpha_1(x) \leq \beta_1(x,t)$, $\forall (x,t) \in \overline{\Omega} \times [0,+\infty)$;

(2)

$$\begin{cases} \dfrac{\partial\alpha_1}{\partial t} - \Delta\alpha_1(x,t) \leq \dfrac{1}{a(M^\gamma|\Omega|)}f(\alpha_1(x,t)), & (x,t) \text{ in } \Omega \times (0,+\infty), \\ \alpha_1(x,t) = 0, & (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ \alpha_1(x,0) = \varepsilon\phi_1(x), & x \text{ in } \Omega \end{cases}$$

and

$$\begin{cases} \dfrac{\partial\beta_1}{\partial t} - \Delta\beta_1(x,t) = 0 = \dfrac{1}{a(M^\gamma|\Omega|)}f(\beta_1(x,t)), & (x,t) \text{ in } \Omega \times (0,+\infty), \\ \beta_1(x,t) = 1 > 0, & (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ \beta_1(x,0) = 1 > \varepsilon\phi_1(x), & x \text{ in } \Omega, \end{cases}$$

which imply that α_1 and β_1 are subsolution and supersolution to (4.10) also. Thus, (4.10) has a unique positive solution $V(x,t)$ such that

$$\alpha_1(x,t) \leq V(x,t) \leq 1, \quad \forall (x,t) \in \overline{\Omega} \times [0,+\infty).$$

Choose an arbitrary $x_0 \in \Omega$. Then there exists a $\overline{B}(x_0,\delta) = \{x \in \Omega \| x - x_0| \leq \delta\} \subseteq \Omega$ and $t_1 > 0$ such that

$$V(x,t_1) > 0, \quad \forall x \in \overline{B}(x_0,\delta).$$

Set $\delta_0 = \min_{x\in\overline{B}(x_0,\delta)} V(x,t_1)$. Lemma 4.1 implies that $V(x,t)$ in nondecreasing on $[0,+\infty)$, which guarantees that that $V(x,t) \geq \delta_0$ for all $(x,t) \in \partial B(x_0,\delta) \times [t_1,+\infty)$.

Let $z_0 = \delta_0$ and $z(t,\delta_0)$ is the unique solution of the ordinary differential equation

$$\begin{cases} \dfrac{dz}{dt} = \dfrac{1}{a(M^\gamma|\Omega|)}f(z), t > 0, \\ z(0) = \delta_0. \end{cases}$$

(4.5) and (4.6) guarantee that

$$\lim_{t\to+\infty} z(t,\delta_0) = 1.$$

Let $V_1(x,t) = V(x,t+t_1)$, $t \geq 0$. We have

$$\begin{cases} \dfrac{\partial V_1}{\partial t} - \Delta V_1 - \dfrac{1}{a(M^\gamma|\Omega|)} f(V_1) = 0 \\ = \dfrac{\partial z(t,\delta_0)}{\partial t} - \Delta z(t,\delta_0) - \dfrac{1}{a(M^\gamma|\Omega|)} f(z(t,\delta_0)), \quad (x,t) \text{ in } \overline{B}(x_0,\delta_0)\times(0,+\infty), \\ V_1(x,t) \geq z(0,\delta_0) = \delta_0, \quad (x,t) \text{ on } \partial B(x_0,\delta_0)\times(0,+\infty), \\ V_1(x,0) \geq z(0,\delta_0), \quad x \text{ in } \Omega. \end{cases}$$

Lemma 4.2 implies that

$$V_1(x,t) = V(x,t+t_1) \geq z(t,\delta_0), \quad t \geq 0,$$

which together with $V(x,t) \leq 1$ guarantees that

$$\lim_{t\to+\infty} V(x,t) = 1.$$

Step 2. The unique solution $v(x,t)$ of problem (4.4) satisfies that

$$\lim_{t\to+\infty} v(x,t) = 1.$$

Let $\alpha_2(x,t) = V(x,t)$ and $\beta_2(x,t) = \beta(x,t)$, $(x,t) \in \overline{\Omega}\times[0,+\infty)$, where $\beta(x,t)$ is the unique positive solution of (4.7). Let

$$b_2(t) = \sup_{s\in[0,\int_\Omega \max\{|\alpha_2(x,t)|,|\beta_2(x,t)|\}^\gamma dx]} a(s).$$

Then

(1) $\alpha_2(x,t) < \beta_2(x,t)$, $\forall (x,t) \in \Omega\times(0,+\infty)$;

(2)

$$\begin{cases} \dfrac{\partial \alpha_2}{\partial t} - \Delta\alpha_2(x,t) & = \dfrac{1}{a(M^\gamma|\Omega|)} f(\alpha_2(x,t)) \\ & \leq \min\{\dfrac{1}{a_0} f(\alpha_2(x,t)), \dfrac{1}{b_2(t)} f(\alpha_2(x,t))\}, \quad (x,t) \text{ in } \Omega\times(0,+\infty), \\ \alpha_2(x,t) = 0, & (x,t) \text{ on } \partial\Omega\times(0,+\infty), \\ \alpha_2(x,0) \leq u_0(x), & x \text{ in } \Omega \end{cases}$$

and

$$\begin{cases} \dfrac{\partial \beta_2}{\partial t} - \Delta\beta_2(x,t) & = \dfrac{1}{a(M^\gamma|\Omega|)} f(\beta_2(x,t)) \\ & \geq \max\{\dfrac{1}{a_0} f(\beta_2(x,t)), \dfrac{1}{b_0(t)} f(\beta_2(x,t))\}, \quad (x,t) \text{ in } \Omega\times(0,+\infty), \\ \beta_2(x,t) = z(t,M) > 0, & (x,t) \text{ on } \partial\Omega\times(0,+\infty), \\ \beta_2(x,0) = M > u_0(x), & x \text{ in } \Omega, \end{cases}$$

which imply that α and β are subsolution and supersolution to (4.4) also. The corollary 3.1 implies that

$$\alpha_2(x,t) \leq v(x,t) \leq \beta_2(x,t), \quad (x,t) \in \overline{\Omega}\times[0,+\infty).$$

And so

$$\lim_{t\to+\infty} v(x,t) = 1.$$

Step 3. We show that

$$\lim_{t\to+\infty} u(x,t) = 1.$$

According to the step 3 in the proof of Theorem 4.1, one knows that $\tau(t) = G^{-1}(t)$ exists on $[0,+\infty)$ and

$$\lim_{t\to+\infty} \tau(t) = +\infty.$$

Hence

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} v(x,\tau(t)) = 1.$$

The proof is complete. □

Theorem 4.2 *Suppose $f \in C^1(R,R)$ with $f(0) = 0$, $f'(0) < a_0\lambda_1$ and $f(u) \ge 0$ for $u \in [0,+\infty)$. Then, there exists a $\varepsilon > 0$ such that for all $0 \le u_0 \le \varepsilon\phi_1(x)$ with $u_0 \in C^{2+\alpha}(\overline{\Omega})$, (1.1) has a unique nonnegative solution $u \in C^2(Q_{+\infty}) \cap C(\overline{Q}_{+\infty})$ such that*

$$\lim_{t\to+\infty} u(x,t) = 0.$$

Proof. Since $f(0) = 0$ and $f'(0) < a_0\lambda_1$, there is a $\lambda_1 > r > 0$ and $\varepsilon > 0$ such that

$$\frac{f(\eta)}{\eta} \le a_0(\lambda_1 - r), \quad \forall 0 < \eta \le \varepsilon,$$

which guarantees that

$$\frac{1}{a_0} f(\eta) \le (\lambda_1 - r)\eta, \quad \forall 0 < \eta \le \varepsilon. \tag{4.11}$$

Let $0 \le u_0(x) \le \varepsilon\phi_1(x)$, $\alpha(x,t) = 0$, $\beta(x,t) = \varepsilon e^{-rt}\phi_1(x)$ and

$$b_0(t) = \sup_{s\in[0,\int_\Omega \max\{|\alpha(x,t)|,|\beta(x,t)|\}^\gamma dx]} a(s).$$

Then, from (4.11), one has

(1) $\alpha(x,t) < \beta(x,t)$, $\forall (x,t) \in \Omega \times (0,+\infty)$;

(2)

$$\begin{cases} \dfrac{\partial\alpha}{\partial t} - \Delta\alpha(x,t) = 0 = \min\{\dfrac{1}{a_0} f(\alpha(x,t)), \dfrac{1}{b_0(t)} f(\alpha(x,t))\}, & (x,t) \text{ in } \Omega\times(0,+\infty), \\ \alpha(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega\times(0,+\infty), \\ \alpha(x,0) = 0 \le u_0(x), \quad x \text{ in } \Omega \end{cases}$$

and

$$\begin{cases} \dfrac{\partial\beta}{\partial t} - \Delta\beta(x,t) = (\lambda_1 - r)\beta(x,t) \ge \max\{\dfrac{1}{a_0} f(\beta(x,t)), \dfrac{1}{b_0(t)} f(\beta(x,t))\}, & (x,t) \text{ in } \Omega\times(0,+\infty), \\ \beta(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega\times(0,+\infty), \\ \beta(x,0) \ge u_0(x), \quad x \text{ in } \Omega, \end{cases}$$

which imply that α and β are subsolution and supersolution to (4.5).

Now Theorem 3.1 guarantees that (4.5) has a unique solution $v(x,t)$ such that

$$\alpha(x,t) \le v(x,t) \le \beta(x,t), \quad \forall (x,t) \in \overline{\Omega} \times [0,+\infty).$$

It is easy to see that

$$\lim_{t\to+\infty} v(x,t) = 0.$$

Since $0 \le v(x,t) \le \varepsilon$, one has $a_0 \le a(\int_\Omega |v(x,t)|^\gamma dx) \le a(|\Omega|\varepsilon^\gamma)$. Since $T_{\max} = +\infty$ in Theorem 2.1, one has

$$\overline{T}_{\max} = \int_0^{T_{\max}} \frac{1}{a(\int_\Omega |v(x,t)|^\gamma dx)} dt = +\infty.$$

Let

$$G(\xi) = \int_0^{\xi} \frac{1}{a(\int_\Omega |v(x,t)|^\gamma dx)} dt, \quad t \in [0,+\infty)$$

and $\tau(t) = G^{-1}(t)$, $t \in [0,+\infty)$. Then $u(x,t) = v(x,\tau(t))$ is a unique nonnegative solution to (1.1) on $[0,+\infty)$ and

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} v(x,\tau(t)) = 0.$$

The proof is complete. □

Next we consider another the special case of (1.1) for reaction function $f = (\lambda u^s - u^p)$, $s \in (0,1)$ and $p > 1$. According to theorem 2.1, we only consider the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \dfrac{1}{a\left(\displaystyle\int_\Omega |u(x,t)|^\gamma dx\right)}(\lambda u^s - u^p), & (x,t) \text{ in } \Omega \times (0,+\infty), \\ u(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \quad x \text{ in } \Omega, \end{cases} \tag{4.12}$$

where $s \in (0,1)$ and $p > 1$.

Theorem 4.3 *Suppose there exists a $\varepsilon_0 > 0$ such that $u_0(x) \ge \varepsilon_0\phi_1(x)$. Then, for any $\lambda > 0$, (1.1) has at least one positive solution $u \in C^2(Q_{+\infty}) \cap C(\overline{Q}_{+\infty})$.*

Proof. For $\lambda > 0$, choose $M > 1 + \frac{1}{\lambda} + \max_{x\in\overline{\Omega}} u_0(x)$ such that

$$\lambda M^s - M^p < 0.$$

Since $s \in (0,1)$ and $p > 1$, we choose $\varepsilon_0 > \varepsilon > 0$ small enough such that

$$\lambda_1 < \frac{1}{a(M^\gamma|\Omega|)}[\lambda\varepsilon^{s-1}\phi_1^{s-1}(x) - \varepsilon^{p-1}\phi_1^{p-1}(x)], \quad x \in \Omega. \tag{4.13}$$

Let $\underline{u}(x,t) = \varepsilon\phi_1(x)$ and $\overline{u}(x,t) = M$ for all $(x,t) \in \overline{\Omega} \times [0,+\infty)$. Set

$$b_0(t) = \sup_{s\in[0,\int_\Omega \max\{|\underline{u}(x,t)|,|\overline{u}(x,t)|\}^\gamma dx]} a(s) = a(M^\gamma|\Omega|),$$

which together (4.13) implies that

$$\begin{cases} \max\{\frac{1}{a_0}[\lambda\overline{u}^s - \overline{u}^p], \frac{1}{b_0(t)}[\lambda\overline{u}^s - \overline{u}^p]\} < 0, \\ \min\{\frac{1}{a_0}[\lambda\underline{u}^s - \underline{u}^p], \frac{1}{b_0(t)}[\lambda\underline{u}^s - \underline{u}^p]\} > \lambda_1\varepsilon\phi_1(x). \end{cases} \tag{4.14}$$

From (4.14), one has

(1) $\underline{u}(x,t) < \overline{u}(x,t) = M$ for all $(x,t) \in \overline{\Omega} \times [0,+\infty)$;

(2)

$$\begin{cases} \frac{\partial\overline{u}(x,t)}{\partial t} - \Delta\overline{u}(x,t) = 0 > \max\{\frac{1}{a_0}[\lambda\overline{u}^s - \overline{u}^p], \frac{1}{b_0(t)}[\lambda\overline{u}^s - \overline{u}^p)]\}, \quad (x,t) \text{ in } \Omega \times (0,+\infty), \\ \overline{u}(x,t) > 0, \quad (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ \overline{u}(x,0) > \phi(x), \quad x \text{ in } \Omega \end{cases}$$

and

$$\begin{cases} \frac{\partial\underline{u}(x,t)}{\partial t} - \Delta\overline{u}(x,t) = \lambda_1\varepsilon\phi_1(x) \le \min\{\frac{1}{a_0}[\lambda\underline{u}^s - \underline{u}^p], \frac{1}{b_0(t)}[\lambda\underline{u}^s - \underline{u}^p]\}, \quad (x,t) \text{ in } \Omega \times (0,+\infty), \\ \underline{u}(x,t) = 0, \quad (x,t) \text{ on } \partial\Omega \times (0,+\infty), \\ \underline{u}(x,0) \le \phi(x), \quad x \text{ in } \Omega, \end{cases}$$

which imply that $\underline{u}(x,t)$ and $\overline{u}(x,t)$ are sub-super solutions to (4.12).

Theorem 3.1 guarantees that (4.12) has at least one positive solution $v(x,t)$ such that

$$\underline{u}(x,t) \le v(x,t) \le \overline{u}(x,t), \quad \forall (x,t) \in \Omega \times [0,+\infty).$$

Since $0 \le v(x,t) \le M$, one has $a_0 \le a(\int_\Omega |v(x,t)|^\gamma dx) \le a(M^\gamma|\Omega|)$. Since $T_{\max} = +\infty$ in Theorem 2.1, one has

$$\overline{T}_{\max} = \int_0^{T_{\max}} \frac{1}{a(\int_\Omega |v(x,t)|^\gamma dx)} dt = +\infty.$$

Let

$$G(\xi) = \int_0^\xi \frac{1}{a(\int_\Omega |v(x,t)|^\gamma dx)} dt, \quad t \in [0,+\infty)$$

and $\tau(t) = G^{-1}(t)$, $t \in [0,+\infty)$. Then $u(x,t) = v(x,\tau(t))$ is a nonnegative solution to (1.1) on $[0,+\infty)$. The proof is complete. □

Remark. The function f in the above theorem does not satisfy *Lipschitz* condition.

5. Some one-dimensional numerical experiments

In this section we consider the following case of Eq (1.1):

$$\begin{cases} u_t - a\left(\int_0^1 u^\gamma(x,t)dx\right)u_{xx} = \lambda f(u), \quad (x,t) \text{ in } (0,1) \times (0,+\infty), \\ u(0,t) = u(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad x \text{ in } [0,1]. \end{cases} \tag{5.1}$$

To numerically solve this equation the following implicit backward finite difference approximation was employed:

$$\begin{cases} \dfrac{u_j^{i+1}-u_j^i}{\Delta t} - a(\sum_{j=1}^{N}(u_j^i)^\gamma \Delta x)\dfrac{u_{j+1}^{i+1}-2u_j^{i+1}+u_{j-1}^{i+1}}{\Delta x^2} = \lambda f(u_j^i), & j = 1,\cdots,N-1, \\ & i = 1,\cdots,K, \\ u_0^{i+1} = u_1^{i+1} = 0, & i = 1,\cdots,K, \\ u_j^0 = u_0(x_j), & j = 0,\cdots,N, \end{cases} \tag{5.1$'$}$$

where $\Delta t = \overline{T}_{\max}/K = 0.0002$, $\Delta x = 1/N = 0.02$, $x_j = j\Delta x$, $j = 0, 1, \cdots, N$, and $t_i = i\Delta t$, $i = 0, 1, \cdots, M$. In (5.1)$'$, u_j^i denotes the difference approximations of $u(t_i, x_j)$. Using above scheme, we simulate the solution of (5.1) under different f, a and λ.

According to section 4, it is necessary to analyse the first eigenvalue and corresponding eigenfunction of

$$\begin{cases} -\phi'' = \lambda\phi, & x \in (0,1), \\ \phi(0) = \phi(1) = 0. \end{cases} \tag{5.2}$$

Obviously, $\lambda_1 = \pi^2$ and $\phi_1(x) = \sin(\pi x)$, $x \in [0, 1]$.

First, for Eq (5.1), let $a(t) = 1 + t$, $\gamma = 1$, $\lambda = 500$, $f(u) = u - u^3$. If the initial condition is given by $u_0(x) = 2\sin(\pi x)$, which satisfies that $\max_{x\in[0,1]} u_0(x) = 2 > 1$ and $\frac{1}{a(M^\gamma|\Omega|)} f'(0) = \frac{500}{1+4/\pi} > \pi^2$, in our numerical simulations, we present the result in Figure 1. If $u_0(x) = 0.5\sin(\pi x)$ which satisfies that $\max_{x\in[0,1]} u_0(x) = 0.5 < 1$ and $\frac{1}{a(M^\gamma|\Omega|)} f'(0) = \frac{500}{1+1/\pi} > \pi^2$, in our numerical simulations, we present the result in Figure 2. The simulations indicate Corollary 4.1 is in agreement with the numerical results presented in Figure 1. Note f is independent on u in problem (1.2) and (1.3) and $f(u) = u - u^3$ in this example, which illustrates that our results improve these ones in [3, 8].

Second, for Eq (5.1), let $a(t) = 1 + t$, $\gamma = 1$, $\lambda = 7$. If $f(u) = e^u - 1$ and the initial condition is given by $u_0(x) = 0.5\sin(\pi x)$, which satisfies that $\frac{1}{a_0} f'(0) = 7 < \pi^2$, in our numerical simulations, we present the result in Figure 3. If $f(u) = u + u^3$ and $u_0(x) = 0.2\sin(\pi x)$ which satisfies that $\frac{1}{a_0} f'(0) = 7 < \pi^2$, in our numerical simulations, we present the result in Figure 4.

The simulations indicate Theorem 4.2 is in agreement with the numerical results presented in Figure 2. In these examples, $f(u) = e^u - 1$ or $f(u) = u + u^3$, which are different from $f(u) = ru(k - u)$ or $f(u) = ru/(k + u)$ in [1].

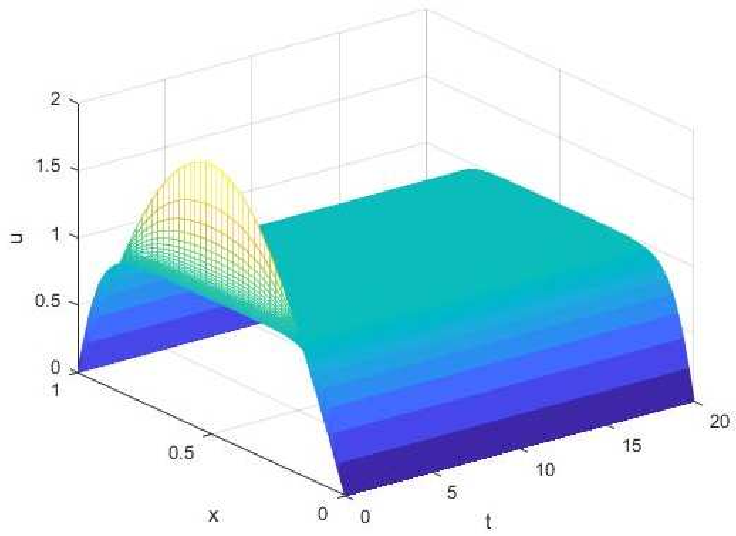

Figure 1. Solution to (5.1) with $a(t) = 1 + t$, $\gamma = 1$, $\lambda = 500$, $f(u) = u - u^3$, $u_0(x) = 2\sin(\pi x)$.

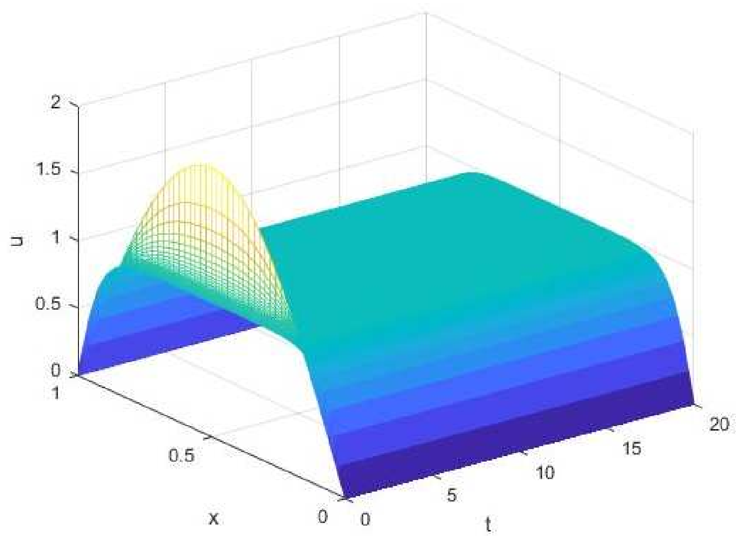

Figure 2. Solution to (5.1) with $a(t) = 1+t$, $\gamma = 1$, $\lambda = 500$, $f(u) = u-u^3$, $u_0(x) = 0.5\sin(\pi x)$.

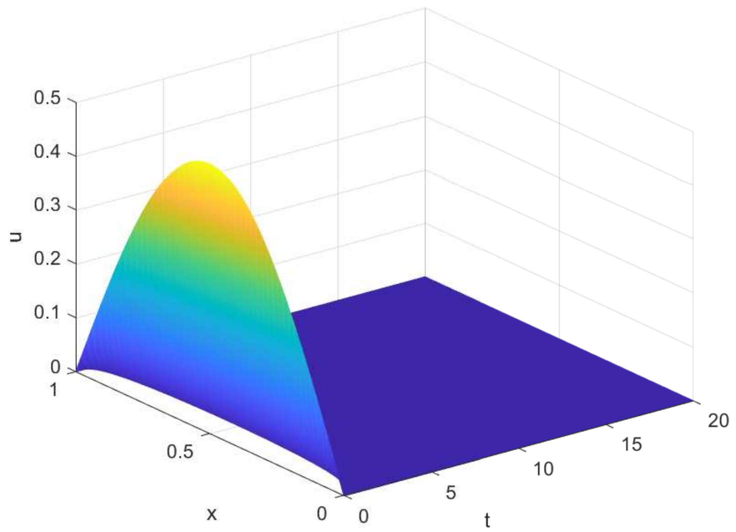

Figure 3. Solution to (5.1) with $a(t) = 1 + t$, $\gamma = 1$, $\lambda = 7$, $u_0(x) = 0.5\sin(\pi x)$ and $f(u) = e^u - 1$.

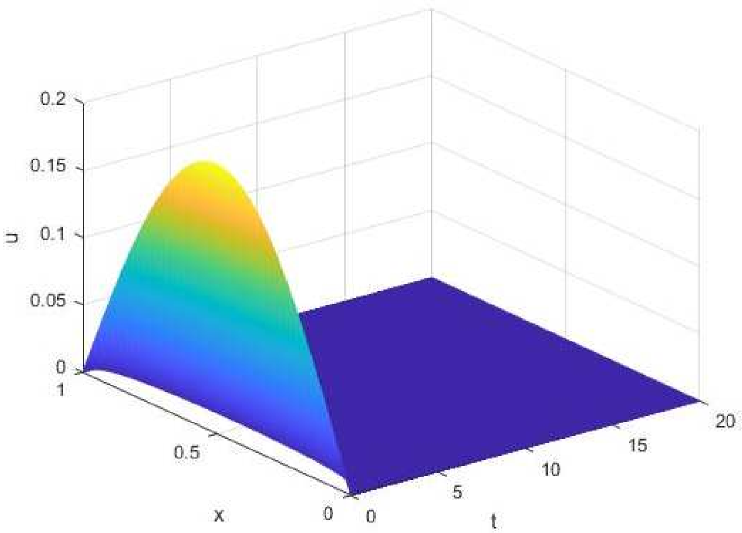

Figure 4. Solution to (5.1) with $a(t) = 1 + t$, $\gamma = 1$, $\lambda = 7$, $u_0(x) = 0.2\sin(\pi x)$ and $f(u) = u + u^3$.

Finally, for Eq (5.1), let $a(t) = 1 + t$, $\gamma = 1$, $s = \frac{1}{2}$, $p = 3$, $f(u) = u^s - u^p$ and $u_0(x) = 1.5\sin(\pi x)$. If $\lambda_1 = 50$, in our numerical simulations, we present the results in Figure 5. If $\lambda_1 = 100$, in our numerical simulations, we present the results in Figure 6. The simulations indicate Theorem 4.5 is in agreement with the numerical results presented in Figure 3.

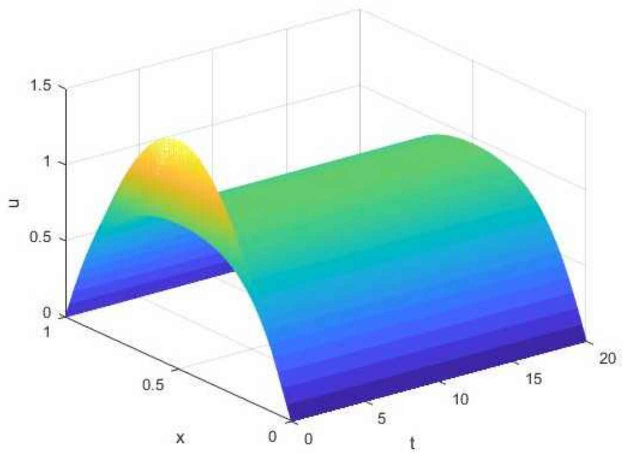

Figure 5. Solution to (5.1) for $a(t) = 1 + t$, $\gamma = 1$, $s = \frac{1}{2}$, $p = 3$, $f(u) = u^s - u^p$ and $u_0(x) = 1.5\sin(\pi x)$, $\lambda_1 = 50$.

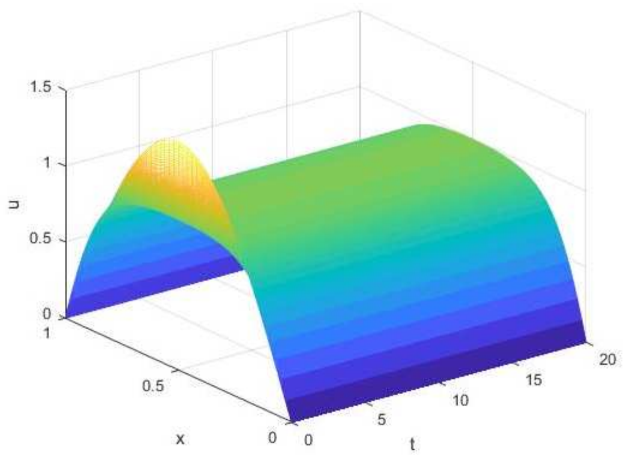

Figure 6. Solution to (5.1) for $a(t) = 1 + t$, $\gamma = 1$, $s = \frac{1}{2}$, $p = 3$, $f(u) = u^s - u^p$ and $u_0(x) = 1.5\sin(\pi x)$, $\lambda_2 = 100$.

6. Conclusions

This paper studies on a parabolic equation with nonlocal diffusion. Using the sub-supersolution method, we prove the existence, uniqueness and long-time behavior of positive solutions. Under the suitable cases for one-dimensional equations, we plotted 3D simulations of the the solutions. From these Figures 1–6, it may be observed that solutions to the studied nonlinear model show the estimated solution propagations.

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Conflict of interest

No potential conflict of interest was reported by the authors.

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