Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(5): 5275–5291.
DOI:10.3934/math.2021312
Received: 06 January 2021
Accepted: 28 February 2021
Published: 12 March 2021

Research article

Initial boundary value problems for space-time fractional conformable differential equation

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Abstract: In this paper, the authors study a initial boundary value problems (IBVP) for space-time fractional conformable partial differential equation (PDE). Several inequalities of fractional conformable derivatives at extremum points are presented and proved. Based on these inequalities at extremum points, a new maximum principle for the space-time fractional conformable PDE is demonstrated. Moreover, the maximum principle is employed to prove a new comparison principle and estimation of solutions. Beside that, the uniqueness and continuous dependence of the solution of the space-time fractional conformable PDE are demonstrated.

Keywords: space-time fractional conformable differential equation; maximum principle; comparison theorem; uniqueness and continuous dependence
Mathematics Subject Classification: 26A33, 35K55

1. Introduction

Maximum principle, one of the most useful tools, is applied to study of complex dynamic systems without knowing explicit form of solutions [1–7]. In 2009, maximum principle for a fractional partial differential equation (PDE) was formulated in explicit form by Luchko [3]. In addition, he and his partners [1–5] proved the maximum principle for the generalized time-fractional and multi-terms time-fractional diffusion equations. By using the maximum principle, they also obtained the uniqueness and continuous dependence of solutions for the the generalized time-fractional and multi-terms time-fractional diffusion equations on the initial and boundary conditions. In 2019, Wang, Ren and Baleanu [8] applied maximum principle to investigating initial boundary value problems (IBVP) for Hadamard fractional differential equations involving fractional Laplace operator and got some existence and uniqueness results. In 2020, Mokhtar and Berikbol [9] proposed maximum principle for the space-time fractional diffusion and pseudo-parabolic equations with Caputo and Riemann-Liouville time-fractional derivatives. Based on the maximum principle, it is proved that the uniqueness and continuous dependence of the solution of IBVP for the nonlinear space-time fractional diffusion and pseudo-parabolic equations. It is provided that maximum principle for time-fractional diffusion equations with singular kernel fractional derivatives [10, 11], non-singular kernel fractional derivatives [6, 7, 12–14] or fractional Laplace operators [8, 15–19]. Extremum principles for fractional differential equations have huge potential application and attract the attention for more and more scholars [20–39].

Jarad et al. [40] introduced the fractional conformable derivatives in the sense of Caputo and Riemann-Liouville and stated their properties. To the best of our knowledge, the mathematical literature on the maximum principles and their applications for Caputo fractional conformable derivatives is rarely mentioned. Inspired by the above works, in this paper we investigate a IBVP for space-time Caputo fractional conformable PDE. First, we present several inequalities of Caputo fractional conformable derivatives at extremum points and give detailed proof of two inequalities. After that, by using these inequalities, a new maximum principle is established. The maximum principle is employed to show that estimation of solutions, comparison principle and the uniqueness and continuous dependence of solutions on the initial and boundary conditions.

The rest of this article is organized as follows: In Section 2, we introduce some definitions about Caputo fractional conformable derivatives and establish several extremum principles. In Section 3, these extremum principles are employed to derive maximum principle. Finally, the maximum principle is applied to obtain estimation of solutions, comparison principle and the properties of the solution of the space-time fractional conformable differential equations in Section 4.

2. Problem formulation and extreme principles

In this paper, we shall investigate the following space-time Caputo fractional conformable PDE

$$ {}^{C\beta}_{\;T}D^{\epsilon}_{t}z(x,t) - [{}^{C\gamma}_{\;a}D^{\epsilon}_{x}z(x,t) + {}^{C\gamma}D^{\;\epsilon}_{bx}z(x,t)] - a(x,t)z(x,t) = g(x,t), \quad (x,t) \in (a,b) \times (T, T_1]. \tag{2.1} $$

Here x and t are the space and time variables, $a(x,t) \in C^{1,1}([a,b]\times[T,T_1])$, and $0 < \epsilon, \beta < 1$, $1 < \gamma < 2$. ${}^{C\beta}_{\;T}D^{\epsilon}_{t}$ is the left Caputo fractional conformable derivative of order β. ${}^{C\gamma}_{\;a}D^{\epsilon}_{x}$ and ${}^{C\gamma}D^{\;\epsilon}_{bx}$ are the left and right Caputo fractional conformable derivatives of order γ. For $f \in C^{n}_{\epsilon,T}([T,T_1])$, the left Caputo fractional conformable derivative of order β is defined by

$$ {}^{C\beta}_{\;T}D^{\epsilon}_{t}f(t) = \frac{1}{\Gamma(n-\beta)} \int_{T}^{t} \left(\frac{(t-T)^{\epsilon} - (s-T)^{\epsilon}}{\epsilon} \right)^{n-\beta-1} \frac{{}^{n}_{T}T^{\epsilon} f(s)}{(s-T)^{1-\epsilon}} ds. \tag{2.2} $$

For $f \in C^{m}_{\epsilon,a}([a,b])$ ($f \in C^{m}_{\epsilon,b}([a,b])$), the left (right) Caputo fractional conformable derivatives of order γ can be written, respectively, as

$$ {}^{C\gamma}_{\;a}D^{\epsilon}_{x}f(t) = \frac{1}{\Gamma(m-\gamma)} \int_{a}^{x} \left(\frac{(x-a)^{\epsilon} - (s-a)^{\epsilon}}{\epsilon} \right)^{m-\gamma-1} \frac{{}^{m}_{a}T^{\epsilon} f(s)}{(s-a)^{1-\epsilon}} ds, \tag{2.3} $$

and

$$ {}^{C\gamma}D^{\;\epsilon}_{bx}f(t) = \frac{(-1)^{m}}{\Gamma(m-\gamma)} \int_{x}^{b} \left(\frac{(b-x)^{\epsilon} - (b-s)^{\epsilon}}{\epsilon} \right)^{m-\gamma-1} \frac{{}^{m}T^{\epsilon}_{b} f(s)}{(b-s)^{1-\epsilon}} ds, \tag{2.4} $$

with $n = [\beta]+1$, $m = [\gamma]+1$, ${}_aT^{\epsilon}f(t) = (t-a)^{1-\epsilon}f'(t)$, $T_b^{\epsilon}f(t) = (b-t)^{1-\epsilon}f'(t)$, ${}_a^nT^{\epsilon} = \underbrace{{}_aT^{\epsilon}\,{}_aT^{\epsilon}\cdots{}_aT^{\epsilon}}_{n\ times}$, ${}^mT_b^{\epsilon} = \underbrace{T_b^{\epsilon}T_b^{\epsilon}\cdots T_b^{\epsilon}}_{m\ times}$, $C^n_{\epsilon,T}[T,T_1] = \{f : [T,T_1] \to \mathbb{R}|\ {}^{n-1}_{T}T^{\epsilon}f \in I_{\epsilon}[T,T_1]\}$, $C^m_{\epsilon,a}[a,b] = \{f : [a,b] \to \mathbb{R}|\ {}^{m-1}_{a}T^{\epsilon}f \in I_{\epsilon}[a,b]\}$ and $C^m_{\epsilon,b}[a,b] = \{f : [a,b] \to \mathbb{R}|\ {}^{m-1}T_b^{\epsilon}f \in {}_{\epsilon}I[a,b]\}$ (where $I_{\epsilon}[T,T_1]$, $I_{\epsilon}[a,b]$ and ${}_{\epsilon}I[a,b]$ are defined in Definition 3.1 in [41]). The detailed information of Caputo fractional conformable derivative, see [40].

Denote

$$H(\bar{U}) = \{z(x,t)|\ z(x,t) \in C^{2,1}((a,b)\times(T,T_1)), z(x,t) \in C([a,b]\times[T,T_1])\}. \tag{2.5}$$

For our maximum principle, we make use of the following three Caputo fractional conformable extremum principles.

Lemma 2.1. *If $f \in C^2_{\epsilon,a}([a,b])$ reaches its maximum at a point $x_0 \in (a,b)$. Then the inequality*

$${}^{C\gamma}_{a}D^{\epsilon}_{x_0}f(x_0) \leq 0 \tag{2.6}$$

holds.

Proof. Let

$$g(x) = f(x_0) - f(x) \geq 0, \quad x \in [a,b]. \tag{2.7}$$

Concurrently, $g(x) \in C^2_{\epsilon,a}([a,b])$, $g(x_0) = 0$ and ${}^{C\gamma}_{a}D^{\epsilon}_{x}g(x) = -{}^{C\gamma}_{a}D^{\epsilon}_{x}f(x)$.

By calculation, we notice that

$$\begin{aligned}{}^{C\gamma}_{a}D^{\epsilon}_{x_0}g(x_0) &= \frac{1}{\Gamma(2-\gamma)}\int_a^{x_0}\left(\frac{(x_0-a)^{\epsilon}-(s-a)^{\epsilon}}{\epsilon}\right)^{1-\gamma}\left((s-a)^{1-\epsilon}g'(s)\right)' ds \\ &= \frac{1}{\Gamma(2-\gamma)}\left(\frac{(x_0-a)^{\epsilon}-(s-a)^{\epsilon}}{\epsilon}\right)^{1-\gamma}(s-a)^{1-\epsilon}g'(s)\Bigg|_a^{x_0} \\ &+ \frac{1-\gamma}{\Gamma(2-\gamma)}\int_a^{x_0}\left(\frac{(x_0-a)^{\epsilon}-(s-a)^{\epsilon}}{\epsilon}\right)^{-\gamma}g'(s)ds.\end{aligned} \tag{2.8}$$

Since

$$\begin{aligned}&\lim_{s\to x_0}\frac{1}{\Gamma(2-\gamma)}\left(\frac{(x_0-a)^{\epsilon}-(s-a)^{\epsilon}}{\epsilon}\right)^{1-\gamma}(s-a)^{1-\epsilon}g'(s) \\ &= \frac{1}{\Gamma(2-\gamma)}\lim_{s\to x_0}\frac{g''(s)(s-a)^{1-\epsilon}+g'(s)(1-\epsilon)(s-a)^{-\epsilon}}{(1-\gamma)\left(\frac{(x_0-a)^{\epsilon}-(s-a)^{\epsilon}}{\epsilon}\right)^{\gamma-2}(s-a)^{1-\epsilon}} \\ &= 0.\end{aligned} \tag{2.9}$$

Therefore, the formula (2.8) becomes

$$\begin{aligned}{}^{C\gamma}_{a}D^{\epsilon}_{x_0}g(x_0) &= \frac{1-\gamma}{\Gamma(2-\gamma)}\int_a^{x_0}\left(\frac{(x_0-a)^{\epsilon}-(s-a)^{\epsilon}}{\epsilon}\right)^{-\gamma}g'(s)ds \\ &= \frac{1-\gamma}{\Gamma(2-\gamma)}\left(\frac{(x_0-a)^{\epsilon}-(s-a)^{\epsilon}}{\epsilon}\right)^{-\gamma}g(s)\Bigg|_a^{x_0} \\ &- \frac{\gamma(1-\gamma)}{\Gamma(2-\gamma)}\int_a^{x_0}\left(\frac{(x_0-a)^{\epsilon}-(s-a)^{\epsilon}}{\epsilon}\right)^{-\gamma-1}(s-a)^{\epsilon-1}g(s)ds.\end{aligned} \tag{2.10}$$

Since

$$\begin{aligned}
&\lim_{s\to x_0}\frac{1-\gamma}{\Gamma(2-\gamma)}\left(\frac{(x_0-a)^\epsilon-(s-a)^\epsilon}{\epsilon}\right)^{-\gamma}g(s)\\
&=-\frac{1-\gamma}{\gamma\Gamma(2-\gamma)}\lim_{s\to x_0}\frac{g'(s)(s-a)^{1-\epsilon}}{\left(\frac{(x_0-a)^\epsilon-(s-a)^\epsilon}{\epsilon}\right)^{\gamma-1}}\\
&=0.
\end{aligned}\tag{2.11}$$

Therefore, the formula (2.10) becomes

$$\begin{aligned}
{}^{C\gamma}_{a}D^{\epsilon}_{x_0}g(x_0)=&\frac{\gamma-1}{\Gamma(2-\gamma)}\left(\frac{(x_0-a)^\epsilon}{\epsilon}\right)^{-\gamma}g(a)\\
&+\frac{\gamma(\gamma-1)}{\Gamma(2-\gamma)}\int_a^{x_0}\left(\frac{(x_0-a)^\epsilon-(s-a)^\epsilon}{\epsilon}\right)^{-\gamma-1}(s-a)^{\epsilon-1}g(s)ds\\
&\geq 0.
\end{aligned}\tag{2.12}$$

We can obtain ${}^{C\gamma}_{a}D^{\epsilon}_{x_0}f(x_0)\leq 0$.
The lemma is proved. □

Lemma 2.2. *If $f\in C^2_{\epsilon,b}([a,b])$ reaches its maximum at a point $x_0\in(a,b)$. Then the inequality*

$${}^{C\gamma}D^{\epsilon}_{bx_0}f(x_0)\leq 0\tag{2.13}$$

holds.

Proof. Let

$$g(x)=f(x_0)-f(x)\geq 0,\quad x\in[a,b].\tag{2.14}$$

Concurrently, $g(x)\in C^2_{\epsilon,b}([a,b])$, $g(x_0)=0$ and ${}^{C\gamma}D^{\epsilon}_{bx}g(x)=-{}^{C\gamma}D^{\epsilon}_{bx}f(x)$.
By calculation, we notice that

$$\begin{aligned}
{}^{C\gamma}D^{\epsilon}_{bx_0}g(x_0)=&\frac{1}{\Gamma(2-\gamma)}\int_{x_0}^{b}\left(\frac{(b-x_0)^\epsilon-(b-s)^\epsilon}{\epsilon}\right)^{1-\gamma}\left((b-s)^{1-\epsilon}g'(s)\right)'ds\\
=&\frac{1}{\Gamma(2-\gamma)}\left(\frac{(b-x_0)^\epsilon-(b-s)^\epsilon}{\epsilon}\right)^{1-\gamma}(b-s)^{1-\epsilon}g'(s)\Bigg|_{x_0}^{b}\\
&-\frac{1-\gamma}{\Gamma(2-\gamma)}\int_{x_0}^{b}\left(\frac{(b-x_0)^\epsilon-(b-s)^\epsilon}{\epsilon}\right)^{-\gamma}g'(s)ds.
\end{aligned}\tag{2.15}$$

Since

$$\begin{aligned}
&\lim_{s\to x_0}\frac{1}{\Gamma(2-\gamma)}\left(\frac{(b-x_0)^\epsilon-(b-s)^\epsilon}{\epsilon}\right)^{1-\gamma}(b-s)^{1-\epsilon}g'(s)\\
&=\frac{1}{\Gamma(2-\gamma)}\lim_{s\to x_0}\frac{g''(s)(b-s)^{1-\epsilon}-g'(s)(1-\epsilon)(b-s)^{-\epsilon}}{(\gamma-1)\left(\frac{(b-x_0)^\epsilon-(b-s)^\epsilon}{\epsilon}\right)^{\gamma-2}(b-s)^{\epsilon-1}}\\
&=0.
\end{aligned}\tag{2.16}$$

Therefore, the formula (2.15) becomes

$$\begin{aligned}{}^{C\gamma}D^{\epsilon}_{bx_0}g(x_0) &= -\frac{1-\gamma}{\Gamma(2-\gamma)}\int_{x_0}^{b}\left(\frac{(b-x_0)^{\epsilon}-(b-s)^{\epsilon}}{\epsilon}\right)^{-\gamma}g'(s)ds.\\ &= -\frac{1-\gamma}{\Gamma(2-\gamma)}\left(\frac{(b-x_0)^{\epsilon}-(b-s)^{\epsilon}}{\epsilon}\right)^{-\gamma}g(s)\Bigg|_{x_0}^{b}\\ &-\frac{\gamma(1-\gamma)}{\Gamma(2-\gamma)}\int_{x_0}^{b}\left(\frac{(b-x_0)^{\epsilon}-(b-s)^{\epsilon}}{\epsilon}\right)^{-\gamma-1}(b-s)^{\epsilon-1}g(s)ds.\end{aligned} \tag{2.17}$$

Since

$$\begin{aligned}&\lim_{s\to x_0}-\frac{1-\gamma}{\Gamma(2-\gamma)}\left(\frac{(b-x_0)^{\epsilon}-(b-s)^{\epsilon}}{\epsilon}\right)^{-\gamma}g(s)\\ &=-\frac{1-\gamma}{\gamma\Gamma(2-\gamma)}\lim_{s\to x_0}\frac{g'(s)(b-s)^{1-\epsilon}}{\left(\frac{(b-x_0)^{\epsilon}-(b-s)^{\epsilon}}{\epsilon}\right)^{\gamma-1}}\\ &=0.\end{aligned} \tag{2.18}$$

Therefore, the formula (2.17) becomes

$$\begin{aligned}{}^{C\gamma}D^{\epsilon}_{bx_0}g(x_0) &= \frac{\gamma-1}{\Gamma(2-\gamma)}\left(\frac{(b-x_0)^{\epsilon}}{\epsilon}\right)^{-\gamma}g(b)\\ &+\frac{\gamma(\gamma-1)}{\Gamma(2-\gamma)}\int_{x_0}^{b}\left(\frac{(b-x_0)^{\epsilon}-(b-s)^{\epsilon}}{\epsilon}\right)^{-\gamma-1}(b-s)^{\epsilon-1}g(s)ds\\ &\geq 0.\end{aligned} \tag{2.19}$$

We can obtain ${}^{C\gamma}D^{\epsilon}_{bx_0}f(x_0)\leq 0$.
The lemma is proved. □

Using the same method, it is easy to obtain the following lemmas.

Lemma 2.3. *If $f\in C^1_{\epsilon,T}([T,T_1])$ reaches its maximum at a point $t_0\in(T,T_1]$. Then the inequality*

$${}^{C\beta}_{T}D^{\epsilon}_{t_0}f(t_0)\geq 0 \tag{2.20}$$

holds.

Lemma 2.4. *If $f\in C^1_{\epsilon,T}([T,T_1])$ reaches its minimum at a point $t_0\in(T,T_1]$. Then the inequality*

$${}^{C\beta}_{T}D^{\epsilon}_{t_0}f(t_0)\leq 0 \tag{2.21}$$

holds.

Lemma 2.5. *If $f\in C^2_{\epsilon,a}([a,b])$ reaches its minimum at a point $x_0\in(a,b)$. Then the inequality*

$${}^{C\gamma}_{a}D^{\epsilon}_{x_0}f(x_0)\geq 0 \tag{2.22}$$

holds.

Lemma 2.6. *If $f \in C^2_{\epsilon,b}([a,b])$ reaches its minimum at a point $x_0 \in (a,b)$. Then the inequality*

$$^{C\gamma}D^{\epsilon}_{bx_0} f(x_0) \geq 0 \tag{2.23}$$

holds.

Example 2.1

If $f(x) = -(x - \frac{b+a}{2})^2$, Lemma 2.1 and 2.2 hold.

If $f(t) = -(t - \frac{T_1+T}{2})^2$, Lemma 2.3 holds.

If $f(t) = (t - \frac{T_1+T}{2})^2$, Lemma 2.4 holds.

If $f(x) = (x - \frac{b+a}{2})^2$, Lemma 2.5 and 2.6 hold.

3. Maximum principle

In this section, we shall consider the linear space-time Caputo fractional conformable PDE (2.1) on the initial-boundary conditions:

$$z(x,T) = \varphi(x), \quad x \in [a,b], \tag{3.1}$$

$$z(a,t) = \mu_1(t),\ z_x(b,t) + hz(b,t) = \mu_2(t) \quad t \in [T,T_1], \tag{3.2}$$

where h is a given positive constant, $U = (a,b) \times (T,T_1]$, $\bar{U} = [a,b] \times [T,T_1]$ and $S = ([a,b] \times \{T\} \bigcup \{a\} \times [a,b] \bigcup \{b\} \times [a,b])$.

Theorem 3.1. *Assume $g(x,t) \leq 0$, $\forall (x,t) \in U$. If $z \in H(\bar{U})$ satisfies the linear space-time Caputo fractional conformable PDE* (2.1)*,* (3.1) *and* (3.2)*, then*

$$z(x,t) \leq \max\{\max_{x\in[a,b]} \varphi(x), \max_{t\in[T,T_1]} \mu_1(t), \frac{1}{h}\max_{t\in[T,T_1]} \mu_2(t), 0\}, \quad \forall (x,t) \in \bar{U} \tag{3.3}$$

holds.

Proof. Arguing by contradiction, assume that there exists a point $(x_0,t_0) \in U$ satisfies

$$z(x_0,t_0) > \max\{\max_{x\in[a,b]} \varphi(x), \max_{t\in[T,T_1]} \mu_1(t), \frac{1}{h}\max_{t\in[T,T_1]} \mu_2(t), 0\} = M > 0.$$

Denote $\varepsilon = z(x_0,t_0) - M > 0$ and

$$w(x,t) = z(x,t) + \frac{\varepsilon}{2}\frac{T_1 - (t-T)}{T_1}, \quad (x,t) \in \bar{U}. \tag{3.4}$$

According to the definition of w, we have

$$\begin{aligned} &w(x,t) \leq z(x,t) + \frac{\varepsilon}{2}, \quad (x,t) \in \bar{U}, \\ &w(x_0,t_0) \geq z(x_0,t_0) = \varepsilon + M \geq \varepsilon + z(x,t) \geq \varepsilon + w(x,t) - \frac{\varepsilon}{2} \\ &\qquad \geq \frac{\varepsilon}{2} + w(x,t), \quad (x,t) \in S. \end{aligned}$$

The latter property implies that the maximum of w cannot be attained on S. Let $w(x_1,t_1) = \max\limits_{(x,t)\in\bar{U}} w(x,t)$, then

$$w(x_1,t_1) \geq w(x_0,t_0) \geq \varepsilon + M > \varepsilon.$$

By Lemma 2.1, 2.2 and 2.3, we know

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t}w(x,t)\Big|_{(x_1,t_1)} \geq 0, & 0<\epsilon<1, 0<\beta<1, \\ {}^{C\gamma}_{a}D^{\epsilon}_{x}w(x,t)\Big|_{(x_1,t_1)} \leq 0, & 0<\epsilon<1, 1<\gamma<2, \\ {}^{C\gamma}D^{\epsilon}_{bx}w(x,t)\Big|_{(x_1,t_1)} \leq 0, & 0<\epsilon<1, 1<\gamma<2. \end{cases} \tag{3.5}$$

By calculation, we can show

$${}^{C\beta}_{T}D^{\epsilon}_{t}\left(\frac{\varepsilon}{2}\frac{T_1-(t-T)}{T_1}\right) = -\frac{1}{\Gamma(1-\beta)}\frac{\varepsilon}{2T_1}\int_T^t\left(\frac{(t-T)^{\epsilon}-(s-T)^{\epsilon}}{\epsilon}\right)^{-\beta}ds. \tag{3.6}$$

Assume $u = (\frac{s-T}{t-T})^{\epsilon}$, substituting into the formula (3.6), we get

$$\begin{aligned} {}^{C\beta}_{T}D^{\epsilon}_{t}\left(\frac{\varepsilon}{2}\frac{T_1-(t-T)}{T_1}\right) &= -\frac{1}{\Gamma(1-\beta)}\frac{\varepsilon}{2T_1}\epsilon^{\beta-1}\int_0^1(t-T)^{1-\epsilon\beta}(1-u)^{-\beta}u^{1-\epsilon}du \\ &= -\epsilon^{\beta-1}(t-T)^{1-\epsilon\beta}\frac{\varepsilon}{2T_1}\frac{\Gamma(2-\epsilon)}{\Gamma(3-\epsilon-\beta)}. \end{aligned} \tag{3.7}$$

Applying (3.5) − (3.7), it holds

$$\begin{aligned} &{}^{C\beta}_{T}D^{\epsilon}_{t}z(x,t)\Big|_{(x_1,t_1)} - [{}^{C\gamma}_{a}D^{\epsilon}_{x}z(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}z(x,t)]\Big|_{(x_1,t_1)} - a(x_1,t_1)z(x_1,t_1) - g(x_1,t_1) \\ &= {}^{C\beta}_{T}D^{\epsilon}_{t}w(x,t)\Big|_{(x_1,t_1)} - {}^{C\beta}_{T}D^{\epsilon}_{t}\left(\frac{\varepsilon}{2}\frac{T_1-(t_1-T)}{T_1}\right) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}w(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}w(x,t)]\Big|_{(x_1,t_1)} \\ &\quad - a(x_1,t_1)\left(w(x_1,t_1) - \frac{\varepsilon}{2}\frac{T_1-(t_1-T)}{T_1}\right) - g(x_1,t_1) \\ &\geq \epsilon^{\beta-1}(t_1-T)^{1-\epsilon\beta}\frac{\varepsilon}{2T_1}\frac{\Gamma(2-\epsilon)}{\Gamma(3-\epsilon-\beta)} - a(x_1,t_1)\varepsilon\left(1-\frac{T_1-(t_1-T)}{2T_1}\right) \\ &> 0, \end{aligned} \tag{3.8}$$

which is in contradiction with (2.1).

This completes the proof of the theorem. □

Similarly, the following minimum principle can be obtained by substituting $-z$ for z in the Theorem 3.1.

Theorem 3.2. *Assume $g(x,t) \geq 0$, $\forall(x,t) \in U$. If $z \in H(\bar{U})$ satisfies the linear space-time Caputo fractional conformable PDE* (2.1)*,* (3.1) *and* (3.2)*, then*

$$z(x,t) \geq \min\{\min_{x\in[a,b]}\varphi(x), \min_{t\in[T,T_1]}\mu_1(t), \frac{1}{h}\min_{t\in[T,T_1]}\mu_2(t), 0\}, \quad \forall(x,t)\in\bar{U} \tag{3.9}$$

holds.

4. Applications of the maximum principle

Theorem 4.1. *If $z(x,t) \in H(\bar{U})$ is a solution of the Eq* (2.1) *on initial boundary conditions* (3.1) *and* (3.2)*, then the inequality*

$$\|z\|_{C(\bar{U})} \leq \max\{\max_{x\in[a,b]} \|\varphi(x)\|, \max_{t\in[T,T_1]} \|\mu_1(t)\|, \frac{1}{h}\max_{t\in[T,T_1]} \|\mu_2(t)\|\} + 2M\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(T_1-T)^{\epsilon\beta} \tag{4.1}$$

holds, where

$$M = \|g\|_{C(\bar{U})}. \tag{4.2}$$

Proof. Let

$$w(x,t) = z(x,t) - M\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(t-T)^{\epsilon\beta}, \quad (x,t) \in \bar{U}.$$

If $z(x,t)$ is a solution of the Eqs (2.1), (3.1) and (3.2), then $w(x,t)$ is a solution of the problem (2.1) with

$$\begin{aligned} g^*(x,t) &= g(x,t) - M\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}\, {}^{C\beta}_{\ T}D^{\epsilon}_{t}(t-T)^{\epsilon\beta} \\ &= g(x,t) - M, \\ \mu_1^*(t) &= \mu_1(t) - M\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(t-T)^{\epsilon\beta}, \\ \mu_2^*(t) &= \mu_2(t) - hM\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(t-T)^{\epsilon\beta}. \end{aligned}$$

$g^*(x,t)$, $\mu_1^*(t)$ and $\mu_2^*(t)$ replace $g(x,t)$, $\mu_1(t)$ and $\mu_2(t)$, respectively. Due to $g^*(x,t) \leq 0$, by using Theorem 3.1 (Maximum principle), we have

$$\begin{aligned} z(x,t) \leq \max\{&\max_{x\in[a,b]} \|\varphi(x)\|, \max_{t\in[T,T_1]} \|\mu_1(t)\| + M\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(T_1-T)^{\epsilon\beta}, \\ &\frac{1}{h}\max_{t\in[T,T_1]} \|\mu_2(t)\| + M\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(T_1-T)^{\epsilon\beta}\}. \end{aligned}$$

Therefore,

$$z(x,t) \leq \max\{\max_{x\in[a,b]} \|\varphi(x)\|, \max_{t\in[T,T_1]} \|\mu_1(t)\|, \frac{1}{h}\max_{t\in[T,T_1]} \|\mu_2(t)\|\} + 2M\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(T_1-T)^{\epsilon\beta}. \tag{4.3}$$

In a similar fashion, we get

$$z(x,t) \geq -\max\{\max_{x\in[a,b]} \|\varphi(x)\|, \max_{t\in[T,T_1]} \|\mu_1(t)\|, \frac{1}{h}\max_{t\in[T,T_1]} \|\mu_2(t)\|\} + 2M\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(T_1-T)^{\epsilon\beta}. \tag{4.4}$$

Combining (4.3) and (4.4), the theorem is proved. □

Theorem 4.2. *If $z(x,t)$ is a solution of the IBVP* (2.1), (3.1) *and* (3.2). *$z(x,t)$ continuously depends on the data given in the problem in the sense that if*

$$\|g-g^*\|_{C(\bar{U})} \le \varepsilon, \quad \|\varphi(x)-\varphi^*(x)\|_{C([a,b])} \le \varepsilon_0, \quad \|\mu_1(t)-\mu_1^*(t)\|_{C([T,T_1])} \le \varepsilon_1,$$
$$\|\mu_2(t)-\mu_2^*(t)\|_{C([T,T_1])} \le \varepsilon_2,$$

then, the estimate

$$\|z-z^*\|_{C(\bar{U})} \le max\{\varepsilon_0,\varepsilon_1,\varepsilon_2\} + 2\frac{\Gamma(2+\epsilon\beta-\epsilon-\beta)}{\beta\epsilon^{\beta}\Gamma(1+\epsilon\beta-\epsilon)}(T_1-T)^{\epsilon\beta}\varepsilon$$

for the corresponding classical solution $z(x,t)$ and $z^*(x,t)$ true.

The demonstrate process is similar to Theorem 4.1.

Theorem 4.3. *Assume $g(x,t) \le 0$, $a(x,t) \le 0$ for $(x,t) \in U$, $\varphi(x) \le 0$ for $x \in (a,b)$ and $\mu_1(t) \le 0$, $\mu_2(t) \le 0$ for $t \in (T,T_1]$. If $z \in H(\bar{U})$ is a solution of the IBVP* (2.1), (3.1) *and* (3.2)*, then*

$$z(x,t) \le 0, (x,t) \in \bar{U}$$

holds.

Theorem 4.4. *Assume $g(x,t) \ge 0$, $a(x,t) \ge 0$ for $(x,t) \in U$, $\varphi(x) \ge 0$ for $x \in (a,b)$ and $\mu_1(t) \ge 0$, $\mu_2(t) \ge 0$ for $t \in (T,T_1]$. If $z \in H(\bar{U})$ satisfy the IBVP* (2.1), (3.1) *and* (3.2)*, then*

$$z(x,t) \ge 0, (x,t) \in \bar{U}$$

holds.

The conclusion of Theorem 4.3 and Theorem 4.4 can be obtained by Theorem 3.1.

Remark 4.1. *Assume $g(x,t) = a(x,t) = 0$ for $(x,t) \in U$, $\varphi(x) = 0$ for $x \in (a,b)$ and $\mu_1(t) = \mu_2(t) \ge 0$ for $t \in (T,T_1]$. If $z \in H(\bar{U})$ satisfies the IBVP* (2.1), (3.1) *and* (3.2)*, then*

$$z(x,t) = 0, \forall (x,t) \in \bar{U},$$

holds.

Theorem 4.5. *Assume $\frac{\partial G}{\partial z} + a(x,t) \le 0$, $\forall (x,t) \in U$, then IBVP of the following nonlinear space-time fractional conformable PDE*

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t} z(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x} z(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z(x,t)] - a(x,t)z(x,t) = G(x,t,z), \ (x,t) \in U \\ z(x,T) = \varphi(x), \qquad x \in [a,b], \\ z(a,t) = \mu_1(t), \qquad t \in [T,T_1], \\ z_x(b,t) + hz(b,t) = \mu_2(t), \quad t \in [T,T_1], \end{cases} \tag{4.5}$$

has a unique solution on $H(\bar{U})$.

Proof. Suppose $z_1,\ z_2$ are two solutions of IBVP (4.5). Let

$$z(x,t) = z_1(x,t) - z_2(x,t),\ \forall (x,t) \in \bar{U},$$

satisfy the equation

$$\begin{cases} {}^{C\beta}_{\ T}D^{\epsilon}_{t} z(x,t) - [{}^{C\gamma}_{\ a}D^{\epsilon}_{x} z(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z(x,t)] - a(x,t)z(x,t) = G(x,t,z_1 - G(x,t,z_2), & (x,t) \in U, \\ z(x,T) = 0, & x \in [a,b], \\ z(a,t) = 0, & t \in [T,T_1], \\ z_x(b,t) + hz(b,t) = 0, & t \in [T,T_1]. \end{cases} \tag{4.6}$$

In view of

$$G(x,t,z_1) - G(x,t,z_2) = \frac{\partial G}{\partial z}(z^*)(z_1 - z_2), \tag{4.7}$$

where $z^* = (1-\lambda)z_1 + \lambda z_2,\ 0 < \lambda < 1.$

Using (4.6) and (4.7), we have that

$$\begin{cases} {}^{C\beta}_{\ T}D^{\epsilon}_{t} z(x,t) - [{}^{C\gamma}_{\ a}D^{\epsilon}_{x} z(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z(x,t)] = \left(\frac{\partial G}{\partial z}(z^*) + a(x,t)\right) z(x,t),\ (x,t) \in U \\ z(x,T) = 0, & x \in [a,b], \\ z(a,t) = 0, & t \in [T,T_1], \\ z_x(b,t) + hz(b,t) = 0, & t \in [T,T_1]. \end{cases} \tag{4.8}$$

Since $\frac{\partial G}{\partial z} + a(x,t) \le 0$, applying Theorem 4.3, we have

$$z(x,t) \le 0,\ (x,t) \in \bar{U}. \tag{4.9}$$

By the same way, using Theorem 4.4 to $-z(x,t)$ we have

$$z(x,t) \ge 0, (x,t) \in \bar{U}. \tag{4.10}$$

Combining (4.9) and (4.10), we can get

$$z(x,t) = 0,\ \forall (x,t) \in \bar{U}.$$

Thus, the theorem holds. □

Example 4.1

Consider the following space-time Caputo fractional conformable PDE:

$$\begin{cases} {}^{C\beta}_{\ T}D^{\epsilon}_{t} z(x,t) - [{}^{C\gamma}_{\ a}D^{\epsilon}_{x} z(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z(x,t)] - a(x,t)z(x,t) = G(x,t,z),\ (x,t) \in U \\ z(x,T) = \varphi(x), & x \in [a,b], \\ z(a,t) = \mu_1(t), & t \in [T,T_1], \\ z_x(b,t) + hz(b,t) = \mu_2(t), & t \in [T,T_1], \end{cases} \tag{4.11}$$

where $0 < \lambda < 1$, $\alpha, \beta \in (0,1)$, $\gamma \in (1,2)$.

If $G(x,t,z) = -x^2$, $a(x,t) = -x^2t^2$, $\varphi(x) = -x^2$, $\mu_1(t) = 0$ and $\mu_2(t) = 0$, Theorem 3.1, 4.1, 4.2 and 4.3 hold.

If $G(x,t,z) = x^2$, $a(x,t) = x^2t^2$, $\varphi(x) = x^2$, $\mu_1(t) = 0$ and $\mu_2(t) = 0$, Theorem 3.2, 4.1, 4.2 and 4.4 hold.

If $a(x,t) = -t^2$ and $G(x,t,z) = \dfrac{x^2}{z}$. We have $\dfrac{\partial G}{\partial u} + a(x,t) = -\dfrac{x^2}{z^2} - t^2 \le 0$, then Theorem 4.5 holds.

Theorem 4.6. *(Comparison Theorem) Assume $c(x,t) \ge 0$, $d(x,t) \ge 0$ and $d(x,t) \ge c(x,t)$ for $(x,t) \in U$. If $(z_1, z_2) \in H(\bar{U}) \times H(\bar{U})$ satisfies*

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t} z_1(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x} z_1(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z_1(x,t)] - c(x,t)z_2(x,t) - d(x,t)z_1(x,t) \ge 0, \\ \qquad\qquad\qquad\qquad (x,t) \in U, \\ {}^{C\beta}_{T}D^{\epsilon}_{t} z_2(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x} z_2(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z_2(x,t)] - c(x,t)z_1(x,t) - d(x,t)z_2(x,t) \ge 0, \\ \qquad\qquad\qquad\qquad (x,t) \in U, \\ z_1(x,T) \ge 0, \quad z_2(x,T) \ge 0, \quad x \in [a,b], \\ z_1(a,t) \ge 0, \quad z_2(a,t) \ge 0, \quad t \in [T,T_1], \\ (z_1)_x(b,t) + hz_1(b,t) \ge 0, \quad (z_2)_x(b,t) + hz_2(b,t) \ge 0, \quad t \in [T,T_1], \end{cases} \tag{4.12}$$

then

$$z_1(x,t) \ge 0, z_2(x,t) \ge 0, (x,t) \in \bar{U},$$

hold.

Proof. Denote $\xi(x,t) = z_1(x,t) + z_2(x,t)$, $\forall (x,t) \in \bar{U}$. Then, by (4.12), we have

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t} \xi(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x} \xi(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} \xi(x,t)] - c(x,t)\xi(x,t) - d(x,t)\xi(x,t) \ge 0, \quad (x,t) \in U, \\ \xi(x,T) \ge 0, \quad x \in [a.b], \\ \xi(a,t) \ge 0, \quad t \in [T,T_1], \\ \xi_x(b,t) + h\xi(b,t) \ge 0, \quad t \in [T,T_1]. \end{cases} \tag{4.13}$$

Thus, by (4.13) and Theorem 4.4, we obtain

$$\xi(x,t) \ge 0, \forall (x,t) \in \bar{U},$$

that is

$$z_1(x,t) + z_2(x,t) \ge 0, (x,t) \in \bar{U}. \tag{4.14}$$

Using (4.13) and (4.14), we have that

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t} z_1(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x} z_1(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z_1(x,t)] - (d(x,t) - c(x,t))z_1(x,t) \ge 0, \quad (x,t) \in U, \\ z_1(x,T) \ge 0, \quad x \in [a.b], \\ z_1(a,t) \ge 0, \quad t \in [T,T_1], \\ (z_1)_x(b,t) + hz_1(b,t) \ge 0, \quad t \in [T,T_1], \end{cases} \tag{4.15}$$

and

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t}z_2(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}z_2(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}z_2(x,t)] - (d(x,t) - c(x,t))z_2(x,t) \geq 0, \quad (x,t) \in U, \\ z_2(x,T) \geq 0, \qquad x \in [a.b], \\ z_2(a,t) \geq 0, \qquad t \in [T, T_1], \\ (z_2)_x(b,t) + hz_2(b,t) \geq 0, \quad t \in [T, T_1]. \end{cases} \tag{4.16}$$

Applying Theorem 4.4 to (4.15) and (4.16), we get

$$z_1(x,t) \geq 0, z_2(x,t) \geq 0, (x,t) \in \bar{U}.$$

Thus, the Theorem holds. □

Using the same way, the following Theorem holds.

Theorem 4.7. *Assume $c(x,t) \leq 0$, $d(x,t) \leq 0$ and $c(x,t) \geq d(x,t)$ for $(x,t) \in U$. If $(z_1, z_2) \in H(\bar{U}) \times H(\bar{U})$ satisfies*

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t}z_1(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}z_1(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}z_1(x,t)] - c(x,t)z_2(x,t) - d(x,t)z_1(x,t) \leq 0, \\ \qquad (x,t) \in U, \\ {}^{C\beta}_{T}D^{\epsilon}_{t}z_2(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}z_2(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}z_2(x,t)] - c(x,t)z_1(x,t) - d(x,t)z_2(x,t) \leq 0, \\ \qquad (x,t) \in U, \\ z_1(x,T) \leq 0, \qquad z_2(x,T) \leq 0, \qquad x \in [a,b], \\ z_1(a,t) \leq 0, \qquad z_2(a,t) \leq 0, \qquad t \in [T, T_1], \\ (z_1)_x(b,t) + hz_1(b,t) \leq 0, \quad (z_2)_x(b,t) + hz_2(b,t) \leq 0, \quad t \in [T, T_1], \end{cases} \tag{4.17}$$

then

$$z_1(x,t) \leq 0, z_2(x,t) \leq 0, (x,t) \in \bar{U},$$

hold.

Remark 4.2. *Assume $c(x,t) = d(x,t) = 0$ for $(x,t) \in U$, $\varphi(x) = \varphi^*(x) = 0$ for $x \in (a,b)$ and $\mu_1(t) = \mu_1^*(t) = \mu_2(t) = \mu_2^*(t) = 0$ for $t \in (T, T_1]$. If $(z_1, z_2) \in H(\bar{U}) \times H(\bar{U})$ satisfies*

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t}z_1(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}z_1(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}z_1(x,t)] - c(x,t)z_2(x,t) - d(x,t)z_1(x,t) = 0, \\ \qquad (x,t) \in U, \\ {}^{C\beta}_{a}D^{\epsilon}_{t}z_2(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}z_2(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}z_2(x,t)] - c(x,t)z_1(x,t) - d(x,t)z_2(x,t) = 0, \\ \qquad (x,t) \in U, \\ z_1(x,T) = \varphi(x), \qquad z_2(x,T) = \varphi^*(x), \qquad x \in [a,b], \\ z_1(a,t) = \mu_1(t), \qquad z_2(a,t) = \mu_1^*(t), \qquad t \in [T, T_1], \\ (z_1)_x(b,t) + hz_1(b,t) = \mu_2(t), \quad (z_2)_x(b,t) + hz_2(b,t) = \mu_2^*(t), \quad t \in [T, T_1], \end{cases} \tag{4.18}$$

then

$$z_1(x,t) = 0, z_2(x,t) = 0, \forall (x,t) \in \bar{U},$$

hold.

Next, we consider the following linear space-time fractional conformable PDE

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t}z_1(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}z_1(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}z_1(x,t)] - c(x,t)z_2(x,t) - d(x,t)z_1(x,t) = g_1(x,t), \\ \qquad (x,t) \in U, \\ {}^{C\beta}_{T}D^{\epsilon}_{t}z_2(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}z_2(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}z_2(x,t)] - c(x,t)z_1(x,t) - d(x,t)z_2(x,t) = g_2(x,t), \\ \qquad (x,t) \in U, \\ z_1(x,T) = \varphi(x), \quad z_2(x,T) = \varphi^*(x), \quad x \in [a,b], \\ z_1(a,t) = \mu_1(t), \quad z_2(a,t) = \mu_1^*(t), \quad t \in [T,T_1], \\ (z_1)_x(b,t) + hz_1(b,t) = \mu_2(t), \quad (z_2)_x(b,t) + hz_2(b,t) = \mu_2^*(t), \quad t \in [T,T_1]. \end{cases} \tag{4.19}$$

Theorem 4.8. *If* $(z_1, z_2) \in H(\bar{U}) \times H(\bar{U})$ *is a solution of the linear space-time fractional conformable PDE* (4.19)*, then*

$$\|z_1\| \le \frac{1}{2}(M_1 + M_2 + M_3 + M_4), \|z_2\| \le \frac{1}{2}(M_1 + M_2 + M_3 + M_4), \quad (x,t) \in \bar{U},$$

hold,where

$$M_1 = \max\{\max_{x\in[a,b]} \|\varphi(x) + \varphi^*(x)\|, \max_{t\in[T,T_1]} \|\mu_1(t) + \mu_1^*(t)\|, \frac{1}{h} \max_{t\in[T,T_1]} \|\mu_2(t) + \mu_2^*(t)\|\}$$

$$M_2 = \max\{\max_{x\in[a,b]} \|\varphi(x) - \varphi^*(x)\|, \max_{t\in[T,T_1]} \|\mu_1(t) - \mu_1^*(t)\|, \frac{1}{h} \max_{t\in[T,T_1]} \|\mu_2(t) - \mu_2^*(t)\|\}$$

$$M_3 = 2\|g_1 + g_2\|_{C(\bar{U})} \frac{\Gamma(2 + \epsilon\beta - \epsilon - \beta)}{\beta\epsilon^{\beta}\Gamma(1 + \epsilon\beta - \epsilon)}(T_1 - T)^{\epsilon\beta},$$

$$M_4 = 2\|g_1 - g_2\|_{C(\bar{U})} \frac{\Gamma(2 + \epsilon\beta - \epsilon - \beta)}{\beta\epsilon^{\beta}\Gamma(1 + \epsilon\beta - \epsilon)}(T_1 - T)^{\epsilon\beta}.$$

Proof. Let $\xi(x,t) = z_1(x,t) + z_2(x,t)$, $\eta(x,t) = z_1(x,t) - z_2(x,t)$, $\forall (x,t) \in \bar{U}$. Then, by (4.19), we have

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t}\xi(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}\xi(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}\xi(x,t)] - (c(x,t) + d(x,t))\xi(x,t) = g_1(x,t) + g_2(x,t), \\ \qquad (x,t) \in U, \\ {}^{C\beta}_{T}D^{\epsilon}_{t}\eta(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x}\eta(x,t) + {}^{C\gamma}D^{\epsilon}_{bx}\eta(x,t)] - (d(x,t) - c(x,t))\eta(x,t) = g_1(x,t) - g_2(x,t), \\ \qquad (x,t) \in U, \\ \xi(x,T) = \varphi(x) + \varphi^*(x), \eta(x,T) = \varphi(x) - \varphi^*(x) \quad x \in [a.b], \\ \xi(a,t) = \mu_1(t) + \mu_1^*(t), \eta(a,t) = \mu_1(t) - \mu_1^*(t) \quad t \in [T,T_1], \\ \xi_x(b,t) + h\xi(b,t) = \mu_2(t) + \mu_2^*(t), \eta_x(b,t) + h\eta(b,t) = \mu_2(t) - \mu_2^*(t) \quad t \in [T,T_1]. \end{cases} \tag{4.20}$$

Thus, by (4.20) and Theorem 4.1, we obtain

$$\|\xi\|_{C(\bar{U})} \le M_1 + M_3, \tag{4.21}$$

and

$$\|\eta\|_{C(\bar{U})} \le M_2 + M_4. \tag{4.22}$$

Using (4.21) and (4.22), we have that

$$\|z_1\|_{C(\bar{U})} \leq \frac{1}{2}(M_1 + M_2 + M_3 + M_4),$$

and

$$\|z_2\|_{C(\bar{U})} \leq \frac{1}{2}(M_1 + M_2 + M_3 + M_4).$$

Thus, the Theorem holds. □

Theorem 4.9. *Assume* $c(x,t) \leq 0$*,* $d(x,t) \leq 0$*,* $d(x,t) < c(x,t)$*,* $g_1(x,t) \leq 0$ *and* $g_2(x,t) \leq 0$ *for* $(x,t) \in U$*, then IBVP* (4.19) *has a unique solution on* $H(\bar{U}) \times H(\bar{U})$*.*

Proof. Suppose (z_{11}, z_{21}), (z_{12}, z_{22}) are two solutions of IBVP (4.19). Let

$$z_1(x,t) = z_{11}(x,t) - z_{12}(x,t), v(x,t) = z_{21}(x,t) - z_{22}(x,t),\ \forall (x,t) \in \bar{U},$$

satisfy the equation

$$\begin{cases} {}^{C\beta}_{T}D^{\epsilon}_{t} z_1(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x} z_1(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z_1(x,t)] - c(x,t)z_2(x,t) - d(x,t)z_1(x,t) = 0, \\ \qquad\qquad (x,t) \in U, \\ {}^{C\beta}_{T}D^{\epsilon}_{t} z_2(x,t) - [{}^{C\gamma}_{a}D^{\epsilon}_{x} z_2(x,t) + {}^{C\gamma}D^{\epsilon}_{bx} z_2(x,t)] - c(x,t)z_1(x,t) - d(x,t)z_2(x,t) = 0, \\ \qquad\qquad (x,t) \in U, \\ z_1(x,T) = 0, \quad z_2(x,T) = 0, \quad x \in [a,b], \\ z_1(a,t) = 0, \quad z_2(a,t) = 0, \quad t \in [T,T_1], \\ (z_1)_x(b,t) + hz_1(b,t) = 0, \quad (z_2)_x(b,t) + hz_2(b,t) = 0, \quad t \in [T,T_1]. \end{cases} \tag{4.23}$$

Applying Theorem 4.7, we get

$$z_1(x,t) \leq 0, z_2(x,t) \leq 0, (x,t) \in \bar{U}. \tag{4.24}$$

Similarly, employing Theorem 4.7 to $-z_1(x,t)$ and $-z_2(x,t)$ we get

$$z_1(x,t) \geq 0, z_2(x,t) \geq 0, (x,t) \in \bar{U}. \tag{4.25}$$

Combining (4.24) and (4.25), we have

$$z_1(x,t) = 0, z_2(x,t) = 0,\ \forall (x,t) \in \bar{U}.$$

Thus, the Theorem holds. □

Example 4.2

Consider the linear space-time Caputo fractional conformable PDE (4.19), if $g_1(x,t) = x^2t^2$, $g_2(x,t) = x^4$, $c(x,t) = \frac{t^2}{2}$, $d(x,t) = t^2$, $\varphi(x) = x^2$, $\mu_1(t) = 0$ and $\mu_2(t) = 0$, Theorem 4.6 and 4.8 hold.

If $g_1(x,t) = -x^2t^2$, $g_2(x,t) = -x^4$, $c(x,t) = -\frac{t^2}{2}$, $d(x,t) = -t^2$, $\varphi(x) = -x^2$, $\mu_1(t) = 0$ and $\mu_2(t) = 0$, Theorem 4.7, 4.8 and 4.9 hold.

5. Conclusion

In this paper, we have proved two extreme principles for the Caputo fractional conformable derivatives. Based on these extreme principles, a maximum principle for the space-time fractional conformable diffusion equation is established. Furthermore, the maximum principle is applied to show a new comparison principle, estimation of solutions and the uniqueness and continuous dependence of the solution for the IBVP to the space-time Caputo fractional conformable equations. Our results are new and contribute significantly to the literature on the topic.

Acknowledgments

We would like to express our gratitude to the anonymous reviewers and editors for their valuable comments and suggestions which led to the improvement of the original manuscript.

Conflict of interest

The authors declare no conflict of interest.

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