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Research article

Chen-Ricci inequality for biwarped product submanifolds in complex space forms

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Abstract: The main objective of this paper is to achieve the Chen-Ricci inequality for biwarped product submanifolds isometrically immersed in a complex space form in the expressions of the squared norm of mean curvature vector and warping functions.The equality cases are likewise discussed. In particular, we also derive Chen-Ricci inequality for CR-warped product submanifolds and point wise semi slant warped product submanifolds.

Keywords: Ricci curvature; biwarped product submanifolds; complex space form; CR-warped product submanifolds; semi slant warped product submanifolds
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1. Introduction

The accoplishment of warped product manifolds came into existent after the study of Bishop and O'Neill [1] on the manifolds of negative curvature. Examining the fact that a Riemannian product of manifolds can not have negative curvature, they constructed the model of warped product manifolds for the class of manifolds of negative (or non positive) curvature which is defined as follows:

Let (U_1, g_1) and (U_2, g_2) be two Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively and ψ be a positive differentiable function on U_1. If $\xi : U_1 \times U_2 \rightarrow U_1$ and $\eta : U_1 \times U_2 \rightarrow U_2$ are the projection maps given by $\xi(p, q) = p$ and $\eta(p, q) = q$ for every $(p, q) \in U_1 \times U_2$, then the *warped product manifold* is the product manifold $U_1 \times U_2$ equipped with the Riemannian structure such that

$$g(V_1, V_2) = g_1(\xi_* V_1, \xi_* V_2) + (\xi \circ \pi)^2 g_2(\eta_* V_1, \eta_* V_2),$$

for all $V_1, V_2 \in TU$. The function ψ is called the *warping function* of the warped product manifold. If the warping function is constant, then the warped product is trivial i.e., simply Riemannian product. On the basis of the fact that warped product manifolds admit a number of applications in Physics and theory of relativity [2], this has been a topic of extensive research. Warped products provide many fundamental solutions to Einstein field equations [2]. The concept of modelling of space-time near black holes adopts the idea of warped product manifolds [3]. Schwartzschild space-time is an example of warped product $U \times_r K^2$, where the base $U = R \times R^+$ is a half plane $r > 0$ and the fibre K^2 is the unit sphere. Under certain conditions, the Schwartzchild space-time becomes the black hole. A cosmological model to represent the universe as a space-time known as Robertson-Walker model is a warped product [4].

In [1] authors have studied some fundamental features of warped product manifolds. An extrinsic study on warped product submanifolds of the kaehler manifolds was performed by B. Y. Chen ([5,6]). Since then, many geometers have explored warped product manifolds in different settings like almost complex and almost contact manifolds and various existence results have been investigated (see the survey article [7]).

In 1999, Chen [8] discovered a relationship between Ricci curvature and squared mean curvature vector for an arbitrary Riemannian manifold. On the line of Chen a series of articles have been appeared to formulate the relationship between Ricci curvature and squared mean curvature in the setting of some important structures on Riemannian manifolds (see [9–14]). Recently, Mustafa et al. [15] proved a relationship between Ricci curvature and squared mean curvature for warped product submanifolds of a semi-slant submanifold of Kenmotsu space forms.

In this paper, our aim is to obtain a relationship between Ricci curvature and squared mean curvature for biwarped product submanifolds in the setting of complex space forms.

2. Preliminaries

Let $\bar{U}$ be an almost Hermitian manifold with an almost complex structure J and a Hermitian metric g, i.e., $J^2 = -I$ and $g(JV_1, JV_2) = g(V_1, V_2)$, for all vector fields V_1, V_2 on $\bar{U}$. If J is parallel with respect to the Levi-Civita connection $\bar{D}$ on $\bar{U}$, that mean

$$(\bar{D}_{V_1} J)V_2 = 0, \tag{2.1}$$

for all $V_1, V_2 \in T\bar{U}$, then $(\bar{U}, J, g, \bar{D})$ is called a *Kaehler manifold*. A Kaehler manifold $\bar{U}$ is called a *complex space form* if it has constant holomorphic sectional curvature denoted by $\bar{U}(c)$. The curvature tensor of the complex space form $\bar{U}(c)$ is given by

$$\begin{aligned}\bar{R}(V_1, V_2, V_2, V_4) =& \frac{c}{4}[g(V_2, V_3)g(V_1, V_4) - g(V_1, V_3)g(V_2, V_4) + g(V_1, JV_3)g(JV_2, V_4) \\ &- g(V_2, JV_3)g(JV_1, V_4) + 2g(V_1, JV_2)g(JV_3, V_4)],\end{aligned} \tag{2.2}$$

for any $V_1, V_2, V_3, V_4 \in T\bar{U}$.

Let U be an $n-$dimensional Riemannian manifold isometrically immersed in a $m-$dimensional Riemannian manifold $\bar{U}$. Then the Gauss and Weingarten formulas are $\bar{D}_{V_1} V_2 = D_{V_1} V_2 + h(V_1, V_2)$ and $\bar{D}_{V_1}\xi = -A_\xi V_1 + D^\perp_{V_1}\xi$ respectively, for all $V_1, V_2 \in TU$ and $\xi \in T^\perp U$. Where D is the induced Levi-civita connection on U, ξ is a vector field normal to U, h is the second fundamental form of U,

$D^{\perp}$ is the normal connection in the normal bundle $T^{\perp}U$ and A_{ξ} is the shape operator of the second fundamental form. The second fundamental form h and the shape operator are associated by the following formula

$$g(h(V_1, V_2), \xi) = g(A_{\xi}V_1, V_2). \tag{2.3}$$

The equation of Gauss is given by

$$R(V_1, V_2, V_3, V_4) = \bar{R}(V_1, V_2, V_3, V_4) + g(h(V_1, V_4), h(V_2, V_3)) - g(h(V_1, V_3), h(V_2, V_4)), \tag{2.4}$$

for all $V_1, V_2, V_3, V_4 \in TU$. Where, $\bar{R}$ and R are the curvature tensors of $\bar{U}$ and U respectively.

For any $V \in TU$ and $N \in T^{\perp}U$, JV_1 and JN can be decomposed as follows

$$JV_1 = PV_1 + FV_1 \tag{2.5}$$

and

$$JN = tN + fN, \tag{2.6}$$

where PV_1 (resp. tN) is the tangential and FV_1 (resp. fN) is the normal component of JV_1 (resp. JN).

For any orthonormal basis $\{e_1, e_2, \ldots, e_k\}$ of the tangent space T_xU, the mean curvature vector $H(x)$ and its squared norm are defined as follows

$$H(x) = \frac{1}{n}\sum_{i=1}^{k} h(e_i, e_i), \quad ||H||^2 = \frac{1}{k^2}\sum_{i,j=1}^{k} g(h(e_i, e_i), h(e_j, e_j)), \tag{2.7}$$

where k is the dimension of U. If $h = 0$ then the submanifold is said to be totally geodesic and minimal if $H = 0$. If $h(V_1, V_2) = g(V_1, V_2)H$ for all $V_1, V_2 \in TU$, then U is called totally umbilical.

The scalar curvature of $\bar{U}$ is denoted by $\bar{\tau}(\bar{U})$ and is defined as

$$\bar{\tau}(\bar{U}) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}, \tag{2.8}$$

where $\bar{\kappa}_{pq} = \bar{\kappa}(e_p \wedge e_q)$ and m is the dimension of the Riemannian manifold $\bar{M}$. Throughout this study, we shall use the equivalent version of the above equation, which is given by

$$2\bar{\tau}(\bar{U}) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}. \tag{2.9}$$

In a similar way, the scalar curvature $\bar{\tau}(L_x)$ of a $L-$plane is given by

$$\bar{\tau}(L_x) = \sum_{1 \le p < q \le m} \bar{\kappa}_{pq}. \tag{2.10}$$

Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of the tangent space T_xU and if e_r belongs to the orthonormal basis $\{e_{k+1}, \ldots e_m\}$ of the normal space $T^{\perp}U$, then we have

$$h^r_{pq} = g(h(e_p, e_q), e_r) \tag{2.11}$$

and

$$||h||^2 = \sum_{p,q=1}^{n} g(h(e_p, e_q), h(e_p, e_q)). \tag{2.12}$$

Let κ_{pq} and $\bar{\kappa}_{pq}$ be the sectional curvatures of the plane sections spanned by e_p and e_q at x in the submanifold U^k and in the Riemannian space form $\bar{U}^m(c)$, respectively. Thus by Gauss equation, we have

$$\kappa_{pq} = \bar{\kappa}_{pq} + \sum_{r=k+1}^{m} (h^r_{pp}h^r_{qq} - (h^r_{pq})^2). \tag{2.13}$$

The global tensor field for orthonormal frame of vector field $\{e_1, \ldots, e_k\}$ on U^k is defined as

$$\bar{T}(V_1, V_2) = \sum_{i=1}^{k} \{g(\bar{R}(e_i, V_1)V_2, e_i)\}, \tag{2.14}$$

for all $V_1, V_2 \in T_xU^k$. The above tensor is called the Ricci tensor. If we fix a distinct vector e_u from $\{e_1, \ldots, e_k\}$ on U^k, which is governed by χ. Then the Ricci curvature is defined by

$$R(\chi) = \sum_{\substack{p=1 \\ p\neq u}}^{k} \kappa(e_p \wedge e_u). \tag{2.15}$$

For a smooth function ψ on a Riemannian manifold U with Riemannian metric g, the gradient of ψ is denoted by $\nabla\psi$ and is defined as

$$g(\nabla\psi, U_1) = U_1\psi, \tag{2.16}$$

for all $U_1 \in TU$.

Let the dimension of U is k and $\{e_1, e_2, \ldots, e_k\}$ be a basis of TU. Then as a result of (2.16), we get

$$\|\nabla\psi\|^2 = \sum_{i=1}^{k} (e_i(\psi))^2. \tag{2.17}$$

The Laplacian of ψ is defined by

$$\Delta\psi = \sum_{i=1}^{k} \{(\nabla_{e_i}e_i)\psi - e_ie_i\psi\}. \tag{2.18}$$

3. Biwarped product submanifolds of a Kaehler manifold

B. Y. Chen and F. Dillen [16] generalize the definition of warped product submanifold to multiply warped product manifolds as follows.

Let $\{U_i\}$, $i = 1, 2, \ldots, k$ be Riemannian manifolds with respective Riemannian metrics $\{g_i\}_{i=1,2,\ldots,k}$ and $\{\psi\}_{i=2,3,\ldots,k}$ are positive valued functions on U_1. Then the product manifold $U = U_1 \times U_2 \times \cdots \times U_k$ endowed with the Riemannian metric g given by

$$g = h_1^*(g_1) + \sum_{i=2}^{k} (\psi_i \circ h_1)^2 h_i^*(g_i)$$

is called multiply warped product manifold and denoted by $U = U_1 \times_{f_2} U_2 \times \cdots \times_{f_k} U_k$ where $h_i (i = 1, 2, \ldots, k)$ are the projection maps of U onto U_i respectively. The functions f_i are known as the warping functions [16]. If the warping functions are constants, the warped product is simply Riemannian product of manifolds. As a paricular case of multiply warped product manifolds, we can define biwarped product manifolds for $i = 3$. For $i = 2$, multiply warped product manifold reduces to single warped product manifold. Consider the biwarped product manifold $U = U_0 \times_{f_1} U_1 \times_{f_2} U_2$ with the Levi-civita connection of U_i for $i = 0, 1, 2$. Now, we have the following result for biwarped product submanifold.

Lemma 3.1. [17] *Let $U = U_0 \times_{f_1} U_1 \times_{f_2} U_2$ be a biwarped product manifold. Then we have*

$$D_{V_1} V_2 = D_{V_2} V_1 = V_1(lnf_i)V_2 \tag{3.1}$$

for $V_1 \in TU_0$ and $V_2 \in TU_i$, for $i = 1, 2$.

Recently, H. M. Tastan [18] studied biwarped submanifolds in the Kaehler manifolds and this was followed by M. A. Khan and K. Khan [19]. Basically, M. A. Khan and K. Khan explored biwarped product submnaifolds of the type $U = U_T \times_{f_1} U_\perp \times_{f_2} U_\theta$ in the setting of complex space forms. Where U_T, $U_\perp$ and U_θ are the invarianat, totally real and pointwise slant submanifolds respectively. Throughout this study we consider $k-$dimensional biwarped product submanifold $U^k = U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ of a complex space form, where k_1, k_2, k_3 are the dimensions of the invariant, totally real and pointwise slant submanifolds. If $U_\theta^{k_3} = \{0\}$ then the biwarped product submanifold becomes the CR-warped product submanifold. Similarly, if $U_\perp^{k_2} = \{0\}$ then the biwarped product submanifold reduced to pointwise semi-slant warped product submanifold.

For a biwarped product submanifold $U^k = U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ of a Riemannian manifold from Eq (3.5) of [16] one can conclude the following result

$$\frac{\Delta f_2}{f_2} + \frac{\Delta f_3}{f_3} = \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} \kappa(e_i, e_j) + \sum_{i=1}^{k_1}\sum_{k=1}^{k_3} \kappa(e_i, e_k). \tag{3.2}$$

Now, we have the following initial result.

Lemma 3.2. *Let $U^k = U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ be a biwarped product submanifold isometrically immersed in a Kaehler manifold $\bar{U}$. Then*

(i) $g(h(V_1, V_2), FV_3) = 0$,
(ii) $g(h(V_1, V_2), JV_4) = 0$,
(iii) $g(h(JV_1, JV_1), N) = -g(h(V_1, V_1), N)$,

for any $V_1, V_2 \in TU_T^{k_1}$, $V_4 \in TU_\perp^{k_2}$, $V_3 \in TU_\theta^{k_3}$ and N belongs to invariant subbundle of $T^\perp U$.

Proof. By using Gauss and Weingarten formulae in Eq (2.1), we have

$$D_{V_1} PV_3 + h(V_1, PV_3) - A_{FV_3} V_1 + D^\perp_{V_1} FV_3 + JD_{V_1} V_3 + D_{V_3} JV_1 + Jh(V_1, V_3) = 0,$$

taking inner product with V_2 and using 3.1, we get the required result. In a similar way, we can prove the part (ii).
To prove (iii), for any $V_1 \in TU_T$ we have

$$\bar{D}_{V_1} JV_1 = J\bar{D}_{V_1} V_1,$$

using Gauss formula and (2.1), we get

$$D_{V_1}JV_1 + h(JV_1, V_1) = JD_{V_1}V_1 + Jh(V_1, V_1),$$

taking inner product with JN, above equation yields

$$g(h(JV_1, V_1), JN) = g(h(V_1, V_1), N), \tag{3.3}$$

interchanging V_1 by JV_1 the above equation gives

$$g(h(JV_1, V_1), JN) = -g(h(JV_1, JV_1), N). \tag{3.4}$$

From (3.3) and (3.4), we get the required result. □

Definition 3.1. The warped product $U_1 \times_{f_2} U_2 \times_{f_3} U_3$ isometrically immersed in a Riemannian manifold $\bar{U}$ is called U_i totally geodesic if the partial second fundamental form h_i vanishes identically. It is called U_i-minimal if the partial mean curvature vector H^i becomes zero for $i = 1, 2, 3$.

Assumme that the distributions corresponding to the submanifolds $U_T^{k_1}$, $U_\perp^{k_2}$ and $U_\theta^{k_3}$ are S, $S^\perp$ and S^θ respectively. From the Lemma 3.2 it is evident that the isometric immersion $U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ into a Kaehler manifold is $D-$ minimal. The $S-$ minimality property provides us a useful relationship between the biwarped product submanifold $U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ and the equation of Gauss.

Let $\{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{k_1} = Je_p, e_{k_1+1}, \ldots, e_{k_2}, e_{k_2+1} = e^1, \ldots, e_{k_2+q} = e^q, e_{k_2+q+1} = e^{q+1} = \sec\theta Pe^1, \ldots e_{(k_3=2q)} = e^{k_3} = \sec\theta Pe^q\}$ be a local orthonormal frame of vector fields on the biwarped product submanifold $U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ such that the set $\{e_1, \ldots, e_p, e_{p+1} = Je_1, \ldots, e_{k_1} = Je_p\}$ is tangent to $U_T^{k_1}$, the set $\{e_{k_1+1}, \ldots, e_{k_2}\}$ is tangent to $U_\perp^{k_2}$ and the set $\{e_{k_2+1}, \ldots, e_{k_2+q}, \ldots e^{k_3}\}$ is tangent to $U_\theta^{k_3}$. Moreover, $\{e_{k+1} = Je_{k_1+1}, \ldots e_{k+k_2} = Je_{k_2}, e_{k+k_2+1} = \csc\theta Fe^1, \ldots, e_{k+k_3} = \csc\theta Fe^q, e_{k+k_2+k_3+1} = \bar{e}^1, \ldots, e_m = \bar{e}^k\}$ is a basis for the normal bundle $T^\perp U$, such that the sets $\{e_{k+1} = Je_{k_1+1}, \ldots e_{k+k_2} = Je_{k_2}\}$ is tangent to $JS^\perp$, $\{e_{k+1} = \csc\theta Fe^1, \ldots, e_{k+k_2} = \csc\theta Fe^q\}$ is tangent to FS^θ and $\{\bar{e}^1, \ldots, \bar{e}^l\}$ is tangent to the complementary invariant subbundle μ with even dimension l.

From Lemma 3.2, it is easy to conclude that

$$\sum_{r=k+1}^{m} \sum_{i,j=1}^{k_1} g(h(e_i, e_j), e_r) = 0. \tag{3.5}$$

Thus it follows that the trace of h due to $U_T^{k_1}$ becomes zero. Hence in view of the Definition 3.1, we obtain the following important result.

Theorem 3.3. *Let $U^k = U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ be a biwarped product submanifold isometrically immersed in a Kaehler manifold. Then U^k is $S-$ minimal.*

So, it is easy to conclude the following

$$\|H\|^2 = \frac{1}{k^2} \sum_{r=k+1}^{m} (h^r_{k_1+1k_1+1} + \cdots + h^r_{k_2k_2} + \cdots + h^r_{kk})^2, \tag{3.6}$$

where $\|H\|^2$ is the squared mean curvature.

4. Ricci curvature for biwarped product submanifold

In this section, we investigate Ricci curvature in terms of the squared norm of mean curvature and the warping functions as follows.

Theorem 4.1. *Let $U^k = U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ be a biwarped product submanifold isometrically immersed in a complex space form $\bar{U}(c)$. Then for each orthogonal unit vector field $\chi \in T_xU$, either tangent to $U_T^{k_1}$, $U_\perp^{k_2}$ or $U_\theta^{k_3}$, we have*

(1) The Ricci curvature satisfy the following inequalities

(i) If χ is tangent to $U_T^{k_1}$, then

$$\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_1k_3 - \frac{1}{2}). \tag{4.1}$$

(ii) If χ is tangent to $U_\perp^{k_2}$, then

$$\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_1k_3 + 1). \tag{4.2}$$

(iii) χ is tangent to $U_\theta^{k_2}$, then

$$\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_1k_3 + 1 - \frac{3}{2}\cos^2\theta). \tag{4.3}$$

(2) If $H(x) = 0$, then each point $x \in U^k$ there is a unit vector field χ which satisfies the equality case of (1) if and only if U^k is mixed totally geodesic and χ lies in the relative null space $\mathcal{N}_x$ at x.

(3) For the equality case we have

(a) The equality case of (4.1) holds identically for all unit vector fields tangent to $U_T^{k_1}$ at each $x \in U^k$ if and only if U^k is mixed totally geodesic and $S-$totally geodesic biwarped product submanifold in $\bar{U}^m(c)$.

(b) The equality case of (4.2) holds identically for all unit vector fields tangent to $U_\perp^{k_2}$ at each $x \in U^k$ if and only if U is mixed totally geodesic and either U^k is $S^\perp$- totally geodesic biwarped product or U^k is a $S^\perp$ totally umbilical in $\bar{S}^m(c)$ with $\dim S^\perp = 2$.

(c) The equality case of (4.3) holds identically for all unit vector fields tangent to $U_\theta^{k_3}$ at each $x \in U^k$ if and only if U is mixed totally geodesic and either U^k is S^θ- totally geodesic biwarped product submanifold or U^k is a S^θ totally umbilical in $\bar{U}^m(c)$ with $\dim S^\theta = 2$.

(d) The equality case of (1) holds identically for all unit tangent vectors to U^k at each $x \in U^k$ if and only if either U^k is totally geodesic submanifold or U^k is a mixed totally geodesic totally umbilical and $S-$ totally geodesic submanifold with $\dim U_\theta = 2$ and $\dim U_\perp = 2$

where k_1, k_2, and k_3 are the dimensions of $U_T^{k_1}$, $U_\perp^{k_2}$, and $U_\theta^{k_3}$ respectively.

Proof. Suppose that $U^k = U_T^{k_1} \times_{f_2} U_\perp^{k_2} \times_{f_3} U_\theta^{k_3}$ be a biwarped product submanifold of a complex space form. From Gauss equation, we have

$$k^2\|H\|^2 = 2\tau(U^k) + \|h\|^2 - 2\bar{\tau}(U^k). \tag{4.4}$$

Let $\{e_1,\dots,e_{k_1},e_{k_1+1},\dots,e_{k_2},\dots e_k\}$ be a local orthonormal frame of vector fields on U^k such that $\{e_1,\dots,e_{k_1}\}$ are tangent to $U_T^{k_1}$, $\{e_{k_1+1},\dots,e_{k_2}\}$ are tangent to $U_\perp^{k_2}$ and $\{e_{k_2+1},\dots,e_k\}$ are tangent to $U_\theta^{k_3}$. So, the unit tangent vector $\chi = e_A \in \{e_1,\dots,e_k\}$ can be expanded (4.4) as follows

$$k^2\|H\|^2 = 2\tau(U^k) + \frac{1}{2}\sum_{r=k+1}^{m}\{(h_{11}^r + \dots h_{k_2k_2}^r + \cdots + h_{kk}^r - h_{AA}^r)^2 + (h_{AA}^r)^2\}$$

$$-\sum_{r=k+1}^{m}\sum_{1\le i\neq j\le k} h_{ii}^r h_{jj}^r - 2\bar{\tau}(U^k). \tag{4.5}$$

The above expression can be written as follows

$$\begin{aligned} k^2\|H\|^2 = 2\tau(U^k) &+ \frac{1}{2}\sum_{r=k+1}^{m}\{(h_{11}^r + \dots h_{k_2k_2}^r + \cdots + h_{kk}^r)^2 \\ &+ (2h_{AA}^r - (h_{11}^r + \cdots + h_{kk}^r))^2\} + 2\sum_{r=k+1}^{m}\sum_{1\le i<j\le k}(h_{ij}^r)^2 \\ &- 2\sum_{r=k+1}^{m}\sum_{\substack{1\le i<j\le k\\ i,j\neq A}} h_{ii}^r h_{jj}^r - 2\bar{\tau}(U^k). \end{aligned}$$

In view of the Lemma 3.2, the preceding expression takes the form

$$\begin{aligned} k^2\|H\|^2 = 2\tau(U^k) &+ \frac{1}{2}\sum_{r=k+1}^{m}\{(h_{k_1+1k_1+1}^r + \dots h_{k_2k_2}^r + \cdots + h_{kk}^r)^2 \\ &+ \frac{1}{2}\sum_{r=k+1}^{m}(2h_{AA}^r - (h_{k_1+1k_1+1}^r + \dots h_{k_2k_2}^r + \cdots + h_{kk}^r))^2 \\ &+ 2\sum_{r=k+1}^{m}\sum_{1\le i<j\le k}(h_{ij}^r)^2 - 2\sum_{r=k+1}^{m}\sum_{\substack{1\le i<j\le k\\ i,j\neq A}} h_{ii}^r h_{jj}^r - 2\bar{\tau}(U^k). \end{aligned} \tag{4.6}$$

Considering unit tangent vector $\chi = e_A$, we have three choices χ is either tangent to the base manifold $U_T^{k_1}$ or to the fibers $U_\perp^{k_2}$ and $U_\theta^{k_3}$.

Case 1: If χ is tangent to $U_T^{k_1}$, then we need to choose a unit vector field from $\{e_1,\dots,e_{k_1}\}$. Let $\chi = e_1$.

Then from (2.14) and (3.5) we have

$$\begin{aligned}
k^2\|H\|^2 \geq & R(\chi) + \frac{1}{2}\sum_{r=k+1}^{m}\{(h^r_{k_1+1k_1+1} + \dots h^r_{k_2k_2} + \dots + h^r_{kk})^2 + \frac{k_2\Delta f_2}{f_2} \\
& + \frac{k_3\Delta f_3}{f_3} + \frac{1}{2}\sum_{r=k+1}^{m}(2h^r_{11} - (h^r_{k_1+1k_1+1} + \dots h^r_{k_2k_2} + \dots + h^r_{kk}))^2 \\
& + \sum_{r=k+1}^{m}\sum_{1\leq\alpha<\beta\leq k_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) \\
& + \sum_{r=k+1}^{m}\sum_{k_1+1\leq p<q\leq k_2}(h^r_{pp}h^r_{qq} - (h^r_{pq})^2) \\
& + \sum_{r=k+1}^{m}\sum_{k_2+1\leq s<t\leq k}(h^r_{ss}h^r_{tt} - (h^r_{st})^2) \\
& + \sum_{r=k+1}^{m}\sum_{1\leq i<j\leq k}(h^r_{ij})^2 - \sum_{r=k+1}^{m}\sum_{2\leq i<j\leq k}(h^r_{ii}h^r_{jj}) \\
& - 2\bar{\tau}(U) + \sum_{2\leq i<j\leq k}\bar{\kappa}(e_i, e_j) + \bar{\tau}(U^{k_1}_T) + \bar{\tau}(U^{k_2}_\perp) + \bar{\tau}(U^{k_3}_\theta).
\end{aligned} \tag{4.7}$$

Putting $V_1, V_4 = e_i$ and $V_2, V_3 = e_j$ in the formula (2.2), we have

$$2\bar{\tau}(U) = \frac{c}{4}[k(k-1) + 3k_1 + 3k_3\cos^2\theta] \tag{4.8}$$

$$\sum_{2\leq i<j\leq k}\bar{\kappa}(e_i, e_j) = \frac{c}{8}[(k-1)(k-2) + 3(k_1-1) + 3k_3\cos^2\theta]$$

$$\bar{\tau}(U^{k_1}_T) = \frac{c}{8}[k_1(k_1-1) + 3k_1]$$

$$\bar{\tau}(U^{k_2}_\perp) = \frac{c}{8}[k_2(k_2-1)]$$

$$\bar{\tau}(U^{k_3}_\theta) = \frac{c}{8}[k_3(k_3-1) + 3k_3\cos^2\theta].$$

Using these values in (4.7), we get

$$\begin{aligned} k^2\|H\|^2 \geq & R(\chi) + \frac{1}{2}k^2\|H\|^2 + \frac{1}{2}\sum_{r=k+1}^{m}(2h_{11}^r - (h_{k_1+1k_1+1}^r + \cdots + h_{kk}^r))^2 \\ & + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_2}(h_{ij}^r)^2 \\ & + \sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{k=k_2+1}^{k}(h_{ik}^r)^2 + \sum_{r=k+1}^{m}\sum_{\beta=2}^{k_1}h_{11}^r h_{\beta\beta}^r \\ & - \sum_{r=k+1}^{m}\sum_{i=2}^{k_1}\sum_{j=k_1+1}^{k_2}h_{ii}^r h_{jj}^r - \sum_{r=k+1}^{m}\sum_{i=2}^{k_1}\sum_{n=k_2+1}^{n}h_{ii}^r h_{nn}^r \\ & + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_3k_1 - \frac{1}{2}). \end{aligned} \tag{4.9}$$

In view of Lemma 3.1

$$\sum_{r=k+1}^{m}\sum_{\beta=2}^{k_1}h_{11}^r h_{\beta\beta}^r = \sum_{r=k+1}^{m}(h_{11}^r)^2$$

$$-\sum_{r=k+1}^{m}\sum_{i=2}^{k_1}[\sum_{j=k_1+1}^{k_2}h_{ii}^r h_{jj}^r + \sum_{n=k_2+1}^{k}h_{ii}^r h_{nn}^r] = \sum_{r=k+1}^{m}\sum_{j=k_1+1}^{n}h_{11}^r h_{jj}^r.$$

Utilizing in (4.9), we have

$$\begin{aligned} k^2\|H\|^2 \geq & R(\chi) + \frac{1}{2}k^2\|H\|^2 + \frac{1}{2}\sum_{r=k+1}^{m}(2h_{11}^r - (h_{k_1+1k_1+1}^r + \cdots + h_{kk}^r))^2 \\ & + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k_2}(h_{ij}^r)^2 \\ & + \sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{k=k_2+1}^{k}(h_{ik}^r)^2 - \sum_{r=k+1}^{m}(h_{11}^r)^2 + \sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k}h_{ii}^r h_{jj}^r \\ & + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_3k_1 - \frac{1}{2}). \end{aligned} \tag{4.10}$$

The third term on the right hand side can be written as

$$\begin{aligned} & \frac{1}{2}\sum_{r=k+1}^{m}(2h_{11}^r - (h_{k_1+1k_1+1}^r + \cdots + h_{k_2k_2}^r + \cdots + h_{kk}^r))^2 \\ & \quad = 2\sum_{r=k+1}^{m}(h_{11}^r)^2 + \frac{1}{2}k^2\|H\|^2 - 2\sum_{r=k+1}^{m}[\sum_{j=k_1+1}^{k_2}h_{11}^r h_{jj}^r \\ & \quad + \sum_{n=k_2+1}^{k}h_{11}^r h_{nn}^r]. \end{aligned} \tag{4.11}$$

Combining above two expressions, we have

$$\begin{aligned}\frac{1}{2}k^2\|H\|^2 \geq & R(\chi) + \sum_{r=k+1}^{m}(h^r_{11})^2 - \sum_{r=k+1}^{m}\sum_{j=k_1+1}^{k} h^r_{11}h^r_{jj} \\ & + \frac{1}{2}\sum_{r=k+1}^{m}(h^r_{k_1+1k_1+1} + \cdots + h^r_{k_2k_2} + \cdots + h^r_{kk})^2 \\ & + \sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k}(h^r_{ij})^2 + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} \\ & + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_3k_1 - \frac{1}{2}).\end{aligned} \tag{4.12}$$

Or equivalently

$$\begin{aligned}\frac{1}{4}k^2\|H\|^2 \geq & R(\chi) + \frac{1}{4}\sum_{r=k+1}^{m}(2h^r_{11} - (h^r_{k_1+1k_1+1} + \cdots + h^r_{k_2k_2} + \cdots + h^r_{kk}))^2 \\ & + \sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k}(h^r_{ij})^2 + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} \\ & + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_3k_1 - \frac{1}{2}),\end{aligned} \tag{4.13}$$

which gives the inequality (i) of (1).

Case 2. If χ is tangent to $U^{k_2}_{\perp}$, we chose the unit vector from $\{e_{k_1+1}, \ldots, e_{k_2}\}$. Suppose $\chi = e_{k_2}$, then from (4.6), we deduce

$$\begin{aligned}k^2\|H\|^2 \geq & R(\chi) + \frac{1}{2}\sum_{r=k+1}^{m}(h^r_{k_1+1k_1+1} + \ldots h^r_{k_2k_2} + \cdots + h^r_{kk})^2 + \frac{k_2\Delta f_2}{f_2} \\ & + \frac{k_3\Delta f_3}{f_3} + \frac{1}{2}\sum_{r=k+1}^{m}((h^r_{k_1+1k_1+1} + \ldots h^r_{k_2k_2} + \cdots + h^r_{kk}) - 2h^r_{k_2k_2})^2 \\ & + \sum_{r=k+1}^{m}\sum_{1\leq\alpha<\beta\leq k_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) + \sum_{r=k+1}^{m}\sum_{k_1+1\leq s<t\leq k_2}(h^r_{ss}h^r_{tt} - (h^r_{st})^2) \\ & + \sum_{r=k+1}^{m}\sum_{k_2+1\leq p<q\leq k}(h^r_{pp}h^r_{qq} - (h^r_{pq})^2) + \sum_{r=k+1}^{m}\sum_{1\leq i<j\leq k}(h^r_{ij})^2 \\ & - \sum_{r=k+1}^{m}\sum_{\substack{1\leq i<j\leq k\\ i,j\neq k_2}}(h^r_{ii}h^r_{jj}) - 2\bar{\tau}(U) + \sum_{\substack{1\leq i<j\leq k\\ i,j\neq k_2}}\bar{\kappa}(e_i, e_j) \\ & + \bar{\tau}(U^{k_1}_T) + \bar{\tau}(U^{k_2}_{\perp}) + \bar{\tau}(U^{k_3}_{\theta}).\end{aligned} \tag{4.14}$$

From (2.2) by putting $V_1, V_4 = e_i$ and $V_2, V_3 = e_j$, one can compute

$$\sum_{\substack{1\leq i<j\leq k\\ i,j\neq k_2}}\bar{\kappa}(e_i, e_j) = \frac{c}{8}[(k-1)(k-2) + 3k_1 + 3k_3\cos^2\theta]$$

$$\bar{\tau}(U_T^{k_1}) = \frac{c}{8}[k_1(k_1-1)+3k_1]$$

$$\bar{\tau}(U_\perp^{k_2}) = \frac{c}{8}[k_2(k_2-1)]$$

$$\bar{\tau}(U_\theta^{k_3}) = \frac{c}{8}[k_3(k_3-1)+3k_3\cos^2\theta].$$

Using these values together with (4.8) in (4.14) and applying similar techniques as in Case 1, we obtain

$$\begin{aligned} k^2\|H\|^2 \geq & R(\chi) + \frac{1}{2}\sum_{r=k+1}^{m}((h^r_{k_1+1k_1+1}+\dots h^r_{k_2k_2}+\cdots+h^r_{kk})-2h^r_{k_2k_2}))^2 \\ & + \frac{1}{2}k^2\|H\|^2 + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \sum_{r=k+1}^{m}\sum_{1\leq i<j\leq k}(h^r_{ij})^2 \\ & + \sum_{r=k+1}^{m}[\sum_{t=k_1+1}^{k_2-1}h^r_{k_2k_2}h^r_{tt} + \sum_{l=k_2+1}^{k}h^r_{k_2k_2}h^r_{ll}] \\ & \sum_{r=1}^{m}\sum_{i=1}^{k_1}[\sum_{j=k_1+1}^{k_2-1}h^r_{ii}h^r_{jj} + \sum_{n=k_2+1}^{k}h^r_{ii}h^r_{nn}] \\ & + \frac{c}{4}(k-k_1k_2-k_2k_3-k_3k_1+1). \end{aligned} \tag{4.15}$$

By the Lemma 3.1, one can conclude

$$\sum_{r=1}^{m}\sum_{i=1}^{k_1}[\sum_{j=k_1+1}^{k_2-1}h^r_{ii}h^r_{jj} + \sum_{n=k_2+1}^{k}h^r_{ii}h^r_{nn}] = 0.$$

The second and seventh terms on right hand side of (4.15) can be solved as follows

$$\begin{aligned} & \frac{1}{2}\sum_{r=k+1}^{m}((h^r_{k_1+1k_1+1}+\cdots+h^r_{kk})-2h^r_{k_2k_2}))^2 + \sum_{r=k+1}^{m}[\sum_{t=k_1+1}^{k_2-1}h^r_{k_2k_2}h^r_{tt} + \sum_{l=k_2+1}^{k}h^r_{k_2k_2}h^r_{ll}] \\ & = \frac{1}{2}\sum_{r=k+1}^{m}(h^r_{k_1+1k_1+1}+\cdots+h^r_{kk})^2 + 2\sum_{r=k+1}^{m}(h^r_{k_2k_2})^2 \\ & -2\sum_{r=k+1}^{m}\sum_{j=k_1+1}^{k}h^r_{k_2k_2}h^r_{jj} + \sum_{r=k+1}^{m}\sum_{t=k_1+1}^{k}h^r_{k_2k_2}h^r_{tt} - \sum_{r=k+1}^{m}(h^r_{k_2k_2})^2 \\ & = \frac{1}{2}\sum_{r=k+1}^{m}(h^r_{k_1+1k_1+1}+\cdots+h^r_{kk})^2 + \sum_{r=k+1}^{m}(h^r_{k_2k_2})^2 \\ & -\sum_{r=k+1}^{m}\sum_{j=k_1+1}^{k}h^r_{kk}h^r_{jj}. \end{aligned} \tag{4.16}$$

Utilizing these two values in (4.15), we arrive

$$\begin{aligned}\frac{1}{2}k^2\|H\|^2 \geq & R(\chi) + \sum_{r=k+1}^{m}(h^r_{k_2k_2})^2 - \sum_{r=k+1}^{m}\sum_{i=k_1+1}^{k} h^r_{kk}h^r_{jj}\\ & + \frac{1}{2}\sum_{r=k+1}^{m}(h^r_{k_1+1k_1+1} + \cdots + h^r_k k)^2 + \frac{1}{2}k^2\|H\|^2 + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3}\\ & + \sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k}(h^r_{ij})^2 + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_3k_1 + 1).\end{aligned} \tag{4.17}$$

By using similar steps as in Case 1, the above inequality can be written as

$$\begin{aligned}\frac{1}{4}k^2\|H\|^2 \geq & R(\chi) + \frac{1}{4}\sum_{r=k+1}^{m}(2h^r_{k_2k_2} - (h^r_{k_1+1k_1+1} + \cdots + h^r_{kk}))^2\\ & + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_1k_3 + 1).\end{aligned} \tag{4.18}$$

The last inequality leads to inequality (*ii*) of (1).
Case 3. If χ is tangent to $U^{k_3}_\theta$, then we choose the unit vector field from $\{e_{k_2+1}, \ldots, e_k\}$. Suppose the vector χ is e_k. Then from (4.6)

$$\begin{aligned}k^2\|H\|^2 \geq & R(\chi) + \frac{1}{2}\sum_{r=k+1}^{m}(h^r_{k_1+1k_1+1} + \ldots h^r_{k_2k_2} + \cdots + h^r_{kk})^2 + \frac{k_2\Delta f_2}{f_2}\\ & + \frac{k_3\Delta f_3}{f_3} + \frac{1}{2}\sum_{r=k+1}^{m}((h^r_{k_1+1k_1+1} + \ldots h^r_{k_2k_2} + \cdots + h^r_{kk}) - 2h^r_{kk})^2\\ & + \sum_{r=k+1}^{m}\sum_{1\leq\alpha<\beta\leq k_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) + \sum_{r=k+1}^{m}\sum_{k_1+1\leq s<t\leq k_2}(h^r_{ss}h^r_{tt} - (h^r_{st})^2)\\ & + \sum_{r=k+1}^{m}\sum_{k_2+1\leq p<q\leq k}(h^r_{pp}h^r_{qq} - (h^r_{pq})^2) + \sum_{r=k+1}^{m}\sum_{1\leq i<j\leq k}(h^r_{ij})^2\\ & - \sum_{r=k+1}^{m}\sum_{1\leq i<j\leq k-1}h^r_{ii}h^r_{jj} - 2\bar{\tau}(U) + \sum_{1\leq i<j\leq k-1}\bar{\kappa}(e_i, e_j)\\ & + \bar{\tau}(U^{k_1}_T) + \bar{\tau}(U^{k_2}_\perp) + \bar{\tau}(U^{k_3}_\theta).\end{aligned} \tag{4.19}$$

From (2.2), one can compute

$$\sum_{1\leq i<j\leq k-1}\bar{\kappa}(e_i, e_j) = \frac{c}{8}[(k-1)(k-2) + 3k_1 + 3(k_3 - 1)\cos^2\theta]$$

$$\bar{\tau}(U^{k_1}_T) = \frac{c}{8}[k_1(k_1 - 1) + 3k_1]$$

$$\bar{\tau}(U^{k_2}_\perp) = \frac{c}{8}[k_2(k_2 - 1)]$$

$$\bar{\tau}(U_{\theta}^{k_3}) = \frac{c}{8}[k_3(k_3 - 1) + 3k_3\cos^2\theta].$$

By usage of these values together with (4.8) in (4.19) and analogous to case 1 and case 2, we obtain

$$\begin{aligned}
k^2\|H\|^2 \geq & R(\chi) + \frac{1}{2}k^2\|H\|^2 + \frac{1}{2}\sum_{r=k+1}^{m}((h^r_{k_1+1k_1+1} + \dots h^r_{k_2k_2} + \cdots + h^r_{kk}) - 2h^r_{kk})^2 \\
& + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \sum_{r=k+1}^{m}\sum_{1\leq i<j\leq k}(h^r_{ij})^2 \\
& + \sum_{r=k+1}^{m}\sum_{q=k_1+1}^{k-1} h^r_{kk}h^r_{qq} - \sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k-1} h^r_{ii}h^r_{jj} \\
& + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_1k_3 + 1 - \frac{3}{2}\cos^2\theta).
\end{aligned} \tag{4.20}$$

On applying the Lemma 3.1, it is easy to verify

$$\sum_{r=k+1}^{m}\sum_{i=1}^{k_1}\sum_{j=k_1+1}^{k-1} h^r_{ii}h^r_{jj} = 0. \tag{4.21}$$

Using in (4.20), we obtain

$$\begin{aligned}
k^2\|H\|^2 \geq & R(\chi) + \frac{1}{2}k^2\|H\|^2 + \frac{1}{2}\sum_{r=k+1}^{m}((h^r_{k_1+1k_1+1} + \dots h^r_{k_2k_2} + \cdots + h^r_{kk}) - 2h^r_{kk})^2 \\
& + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \sum_{r=k+1}^{m}\sum_{1\leq i<j\leq k}(h^r_{ij})^2 + \sum_{r=k+1}^{m}\sum_{q=k_1+1}^{k-1} h^r_{kk}h^r_{qq} \\
& + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_1k_3 + 1 - \frac{3}{2}\cos^2\theta).
\end{aligned} \tag{4.22}$$

The third and seventh terms on the right hand side of (4.22) in a similar way as in case 1 and case 2 can be simplified as

$$\begin{aligned}
& \frac{1}{2}\sum_{r=k+1}^{m}((h^r_{k_1+1k_1+1} + \dots h^r_{k_2k_2} + \cdots + h^r_{kk}) - 2h^r_{kk})^2 + \sum_{r=k+1}^{m}\sum_{q=k_1+1}^{k-1} h^r_{k}h^r_{qq} \\
& = \frac{1}{2}\sum_{r=k+1}^{m}(h^r_{k_1+1k_1+1} + \dots h^r_{k_2k_2} + \cdots + h^r_{kk})^2 + \sum_{r=k+1}^{m}(h^r_{kk})^2 \\
& - \sum_{r=k+1}^{m}\sum_{j=k_1+1}^{k} h^r_{kk}h^r_{jj}.
\end{aligned} \tag{4.23}$$

By combining (4.22) and (4.23) and using similar techniques as used in case 1 and case 2, we can derive

$$\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{1}{4}\sum_{r=k+1}^{m}(2h^r_{kk} - (h^r_{k_1+1k_1+1} + \cdots + h^r_{kk}))^2 + \frac{k_2\Delta f_2}{f_2} + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_2 - k_2k_3 - k_1k_3 + 1 - \frac{3}{2}\cos^2\theta). \tag{4.24}$$

The last inequality leads to inequality (*iii*) in (1).

Next, we explore the equality cases of (1). First, we redefine the notion of the relative null space $\mathcal{N}_x$ of the submanifold U^k in the complex space form $\bar{U}^m(c)$ at any point $x \in U^k$, the relative null space was defined by B. Y. Chen [8], as follows

$$\mathcal{N}_x = \{V_1 \in T_xU^k : h(V_1, V_2) = 0, \forall V_2 \in T_xU^k\}.$$

For $A \in \{1, \ldots, k\}$ a unit vector field e_A tangent to U^k at x satisfies the equality sign of (4.1) identically if and only if

$$(i)\ \sum_{p=1}^{k_1}\sum_{q=k_1+1}^{k} h^r_{pq} = 0\ (ii)\ \sum_{b=1}^{k}\sum_{\substack{A=1\\ b\neq A}}^{n} h^r_{bA} = 0\ (iii)\ 2h^r_{AA} = \sum_{q=k_1+1}^{k} h^r_{qq}, \tag{4.25}$$

holds for $r \in \{k+1, \ldots m\}$, which implies that U^k is mixed totally geodesic biwarped product submanifold. Combining statements (*ii*) and (*iii*) with the fact that U^k is biwarped product submanifold, we get that the unit vector field $\chi = e_A$ belongs to the relative null space $\mathcal{N}_x$. The converse is trivial, this proves statement (2).

For a biwarped product submanifold, the equality sign of (4.1) holds identically for all unit tangent vector belong to U_T at x if and only if

$$(i)\sum_{p=1}^{k_1}\sum_{q=k_1+1}^{k} h^r_{pq} = 0\ \ (ii)\sum_{b=1}^{k}\sum_{\substack{A=1\\ b\neq A}}^{k_1} h^r_{bA} = 0\ \ (iii)\ 2h^r_{pp} = \sum_{q=k_1+1}^{k} h^r_{qq}, \tag{4.26}$$

where $p \in \{1, \ldots, k_1\}$ and $r \in \{k+1, \ldots, m\}$. Since U^k is biwarped product submanifold, the third condition implies that $h^r_{pp} = 0,\ \ p \in \{1, \ldots, k_1\}$. Using this in the condition (*ii*), we conclude that U^k is $S-$totally geodesic biwarped product submanifold in $\bar{U}^m(c)$ and mixed totally geodesicness follows from the condition (*i*). Which proves (*a*) in the statement (3).

For a biwarped product submanifold, the equality sign of (4.2) holds identically for all unit tangent vector fields tangent to $U_\perp$ at x if and only if

$$(i)\sum_{p=1}^{k_1}\sum_{q=k_1+1}^{n} h^r_{pq} = 0\ (ii)\sum_{b=1}^{k}\sum_{\substack{A=k_1+1\\ b\neq A}}^{k_2} h^r_{bA} = 0\ (iii)\ 2h^r_{KK} = \sum_{q=k_1+1}^{k} h^r_{qq}, \tag{4.27}$$

such that $K \in \{k_1+1, \ldots, k_2\}$ and $r \in \{k+1, \ldots, m\}$. From the condition (*iii*) two cases emerge, that is

$$h^r_{LL} = 0,\ \ \forall L \in \{k_1+1, \ldots, k_2\}\ \text{ and }\ r \in \{k+1, \ldots, m\}\ \text{ or }\ dim\ U_\perp = 2. \tag{4.28}$$

If the first case of (4.27) satisfies, then by virtue of condition (ii), it is easy to conclude that U^k is a $S^\perp-$ totally geodesic biwarped product submanifold in $\bar{U}^m(c)$. This is the first case of part (b) of statement (3).

For a biwarped product submanifold, the equality sign of (4.3) holds identically for all unit tangent vector fields tangent to $U_\theta^{k_3}$ at x if and only if

$$(i)\sum_{p=1}^{k_1}\sum_{q=k_1+1}^{k} h^r_{pq} = 0\ (ii)\sum_{b=1}^{n}\sum_{\substack{A=k_2+1\\ b\neq A}}^{k_3} h^r_{bA} = 0\ (iii)\ 2h^r_{LL} = \sum_{q=k_1+1}^{n} h^r_{qq}, \tag{4.29}$$

such that $L \in \{k_2 + 1, \dots, k\}$ and $r \in \{k + 1, \dots, m\}$. From the condition ($iii$) two cases arise, that is

$$h^r_{LL} = 0,\ \ \forall L \in \{k_2 + 1, \dots, n\}\ \text{ and }\ r \in \{k + 1, \dots, m\}\ \text{ or }\ dim\ U_\theta = 2. \tag{4.30}$$

If the first case of (4.29) satisfies, then by virtue of condition (ii), it is easy to conclude that U^k is a $S^\theta-$ totally geodesic biwarped product submanifold in $\bar{U}^m(c)$. This is the first case of part (c) of statement (3).

For the other case, assume that U^k is not $S^\theta-$totally geodesic biwarped product submanifold and dim $U_\theta = 2$. Then condition (ii) of (4.29) implies that U^k is $S^\theta-$ totally umbilical biwarped product submanifold in $\bar{U}(c)$, which is second case of this part. This verifies part (c) of (3).

To prove (d) using parts (a), (b) and (c) of (3), we combine (4.26), (4.27) and (4.29). For the first case of this part, assume that $dimU_\perp \neq 2$ and $dimU_\theta \neq 2$. Since from parts (a), (b) and (c) of statement (3) we conclude that U^k is $S-$totally geodesic, $S^\perp-$ totally geodesic and $S^\theta-$ totally geodesic submanifolds in $\bar{U}^m(c)$. Hence U^k is a totally geodesic submanifold in $\bar{U}^m(c)$.

For another case, suppose that first case does not satisfy. Then parts (a), (b) and (c) provide that U^k is mixed totally geodesic and $S-$ totally geodesic submanifold of $\bar{U}^m(c)$ with $dimU_\perp = 2$ and $dimU_\theta = 2$. From the conditions (b) and (c) it follows that U^k is $S^\perp-$ and $D^\theta-$totally umbilical biwarped product submanifolds and from (a) it is $S-$totally geodesic, which is part (d). This proves the theorem. □

If $U_\perp^{k_2} = \{0\}$, then the biwarped product submanifold becomes the Point wise semi-slant warped product submanifold that is $U^k = U_T^{k_1} \times_{f_2} U_\theta^{k_3}$. Now, we have the following corollary which can be deduced from the Theorem 4.2.

Corollary 4.2. *Let $U^k = U_T^{k_1} \times_{f_3} U_\theta^{k_3}$ be a pointwise semi-slant warped product submanifold isometrically immersed in a complex space form $\bar{U}(c)$. Then for each orthogonal unit vector field $\chi \in T_xU$, either tangent to $U_T^{k_1}$ or $U_\theta^{k_3}$, we have*

(1) The Ricci curvature satisfy the following inequalities

(i) If χ is tangent to $U_T^{k_1}$, then

$$\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_3 - \frac{1}{2}). \tag{4.31}$$

(ii) χ is tangent to $U_\theta^{k_3}$, then

$$\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{k_3\Delta f_3}{f_3} + \frac{c}{4}(k - k_1k_3 + 1 - \frac{3}{2}\cos^2\theta). \tag{4.32}$$

(2) If $H(x) = 0$, then each point $x \in U^k$ there is a unit vector field χ which satisfies the equality case of (1) if and only if U^k is mixed totally geodesic and χ lies in the relative null space $\mathcal{N}_x$ at x.

(3) For the equality case we have

(a) The equality case of (4.31) holds identically for all unit vector fields tangent to U_T at each $x \in U^k$ if and only if U^k is mixed totally geodesic and $S-$totally geodesic point wise semi slant warped product submanifold in $\bar{U}^m(c)$.

(b) The equality case of (4.32) holds identically for all unit vector fields tangent to $U_\theta^{k_3}$ at each $x \in U^k$ if and only if S is mixed totally geodesic and either U^k is D^θ- totally geodesic point wise semi slant warped product submanifold or U^k is a S^θ totally umbilical in $\bar{U}^m(c)$ with dim $S^\theta = 2$.

(c) The equality case of (1) holds identically for all unit tangent vectors to U^k at each $x \in U^k$ if and only if either U^k is totally geodesic submanifold or U^k is a mixed totally geodesic totally umbilical and $S-$ totally geodesic submanifold with dim $U_\theta = 2$.

where k_1 and k_3 are the dimensions of $U_T^{k_1}$ and $U_\theta^{k_3}$ respectively.

Now, we have another case that is if $U_\theta^{k_3} = \{0\}$ then the biwarped product submanifold becomes the CR-warped product submanifold. In this case we have the following corollary.

Corollary 4.3. *Let $U^k = U_T^{k_1} \times_{f_2} U_\perp^{k_2}$ be a CR-warped product submanifold isometrically immersed in a complex space form $\bar{U}^m(c)$. Then for each orthogonal unit vector field $\chi \in T_xU$, either tangent to $U_T^{k_1}$ or $U_\perp^{k_2}$, we have*

(1) The Ricci curvature satisfy the following inequalities

(i) If χ is tangent to $U_T^{k_1}$, then

$$\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{U_2\Delta f_2}{f_2} + \frac{c}{4}(k - k_1k_2 - \frac{1}{2}). \tag{4.33}$$

(ii) If χ is tangent to $U_\perp^{k_2}$, then

$$\frac{1}{4}k^2\|H\|^2 \geq R(\chi) + \frac{k_2\Delta f_2}{f_2} + \frac{c}{4}(k - k_1k_2 + 1). \tag{4.34}$$

(2) If $H(x) = 0$, then each point $x \in U^k$ there is a unit vector field χ which satisfies the equality case of (1) if and only if U^k is mixed totally geodesic and χ lies in the relative null space $\mathcal{N}_x$ at x.

(3) For the equality case we have

(a) The equality case of (4.33) holds identically for all unit vector fields tangent to U_T at each $x \in U^k$ if and only if U^k is mixed totally geodesic and $S-$totally geodesic CR-warped product submanifold in $\bar{U}^m(c)$.

(b) The equality case of (4.34) holds identically for all unit vector fields tangent to $U_\perp^{k_2}$ at each $x \in U^k$ if and only if U is mixed totally geodesic and either U^k is $S^\perp$- totally geodesic biwarped product or U^k is a $S^\perp$ totally umbilical in $\bar{U}^m(c)$ with dim $S^\perp = 2$.

(c) The equality case of (1) holds identically for all unit tangent vectors to U^k at each $x \in U^k$ if and only if either U^k is totally geodesic submanifold or U^k is a mixed totally geodesic totally umbilical and $S-$ totally geodesic submanifold with dim $U_\perp = 2$.

where k_1 and k_2 are the dimensions of $U_T^{k_1}$ and $U_\perp^{k_2}$ respectively.

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Conflict of interest

The authors declare that they have no any conflict of interest.

References

1. R. L. Bishop, B. O'Neil, Manifolds of negative curvature, *T. Am. Math. Soc.*, **145** (1969), 1–9.
2. J. K. Beem, P. Ehrlich, T. G. Powell, Warped product manifolds in relativity, In: Th. M Rassias, G. M. Rassias, Editor, *Selected Studies: Physics-Astrophysics, Mathematics, History of Science*, Amsterdam: North-Holland, 1982, 41–56.
3. S. W. Hawkings, G. F. R. Ellis, *The large scale structure of space-time*, Cambridge: Cambridge Univ. Press, 1973.
4. B. O'Neill, *Semi-Riemannian geometry with application to relativity*, Academic Press, 1983.
5. B. Y. Chen, CR-submanifolds of a Kaehler manifold I, *J. Differ. Geom.*, **16** (1981), 305–323.
6. B. Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds I, *Monatsh Math.*, **133** (2001), 177–195.
7. B. Y. Chen, A survey on geometry of warped product submanifolds, *arXiv:1307.0236*.
8. B. Y. Chen, Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimension, *Glasg. Math. J.*, **41** (1999), 33–41.
9. K. Arslan, R. Ezentas, I. Mihai, C. Ozgur, Certain inequalities for submanifolds in $(k, \mu)-$contact space form, *B. Aust. Math. Soc.*, **64** (2001), 201–212.
10. D. Cioroboiu, B. Y. Chen, Inequalities for semi-slant submanifolds in Sasakian space forms, *IJMMS*, **27** (2003), 1731–1738.
11. A. Mihai, C. Ozgur, Chen inequalities for submanifolds of real space forms with a semi-symmetric metric connection, *Taiwan. J. Math.*, **14** (2010), 1465–1477.
12. D. W. Yoon, Inequality for Ricci curvature of slant submanifolds in cosymplectic space forms, *Turk. J. Math.*, **30** (2006), 43–56.
13. M. Aquib, J. W. Lee, G. E. Vilcu, W. Yoon, Classification of Casorati ideal Lagrangian submanifolds in complex space forms, *Differ. Geom. Appl.*, **63** (2019), 30–49.
14. M. M. Tripathi, Improved Chen-Ricci inequality for curvature-like tensors and its applications, *Differ. Geom. Appl.*, **29** (2011), 685–698.
15. A. Mustafa, S. Uddin, F. R. Al-Solamy, Chen-Ricci inequality for warped products in Kenmotsu space forms and its applications, *RACSAM*, **113** (2019), 1–18.

16. B. Y. Chen, F. Dillen, Optimal inequality for multiply warped product submanifolds, *IEJG*, **1** (2008), 1–11

17. J. P. Baker, *Twice warped products*, M. Sc. Thesis, University of Missouri-Columbia, Columbia, 1997.

18. H. M. Tastan, Biwarped product submanifolds of a Kaehler manifold, *Filomat*, **32** (2018), 2349–2365.

19. M. A. Khan, K. Khan, Biwarped product submanifolds of complex space forms, *Int. J. Geom. Methods Mod. Phys.*, **16** (2019), 1950072.