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Research article

Accelerated life tests for modified Kies exponential lifetime distribution: binomial removal, transformers turn insulation application and numerical results

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Abstract: This paper is concerned with statistical inference of multiple constant-stress testing for progressive type-II censored data with binomial removal. The failure times of the test units are assumed to be independent and follow the modified Kies exponential (MKEx) distribution. The maximum likelihood method as well as Bayes method are used to derive both point and interval estimates of the parameters. Furthermore, a real data application for transformers turn insulation is used to illustrate the proposed methods. Moreover, this real data set is used to show that MKEx distribution can be a possible alternative model to the exponential, generalized exponential and Weibull distributions. Finally, simulation studies are carried out to investigate the accuracy of the different estimation methods.

Keywords: modified Kies family; constant-stress; progressive type-II censoring; maximum likelihood; Bayesian estimation; modified Kolmogorov-Smirnov
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1. Introduction

The purpose of a life testing is to analyze failure times of test units that obtained under normal operating conditions. As it is well known, the modern products are designed to last longer, so for these products, collecting failure data under ordinary circumstances is entirely difficult or even impractical. In such situations, items should be exposed to stress higher than the manufacture levels of stress, in order to obtain data about their failure times. We can call this kind of life test under tough conditions accelerated life testing (ALT), and the failure times of the test units from such ALT can be used to determine life characteristics under normal usage conditions.

There are many models to apply acceleration in the experiments such as Arrhenius model and inverse power model. Choosing the appropriate model of acceleration depends on the type of stress (voltage, pressure, or temperature) that the researcher wants to apply. Arrhenius model and inverse power model are usually used for the thermal and non-thermal stresses, respectively. The common kinds of ALTs are constant-stress ALT (CSALT), step-stress ALT (SSALT) and progressive-stress ALT (PSALT). In the CSALT the stress is constant during the whole time of the experiment, but the stress is increased gradually at specific times, in the SSALT. On the Contrast, the stress is increasing function of the time in the PSALT. ALTs can be divided based on the number of stress levels into two types: Simple and multiple level ALT. Simple ALT only has two levels of stress while, multiple ALT contains more than wo levels of stress.

Several authors have investigated the CSALT, Mohie El-Din et al. [32] considered the estimation problem of the CSALT for the extension of the exponential distribution under progressive type-II censoring. Mohie El-Din et al. [33] discussed the problem of obtaining the optimal CSALT designs for the Lindley distribution. The concern of estimation for CSALT for generalized half normal distribution under complete sampling was addressed by Wang and Shi [39]. Abd El-Raheem [6] developed CSALT's optimal designs for extension of the exponential distribution. The problem of the optimum configuration of CSALT to extension of the exponential distribution under censoring was considered by Abd El-Raheem [7]. For further references on the CSALT, see Abd El-Raheem [8–10].

The SSALTs are discussed by several authors see for example, Balakrishnan and Han [17], Ismail [24], Mohie El-Din et al. [30,31], Chandra and Khan [37] and Hafez et al. [22]. For more reading about PSALTs , we can refer to Abdel-Hamid and Al-Hussaini [2–4], AL-Hussaini et al. [12], Abdel-Hamid and Abushul [5], Mohie El-Din et al. [34, 35] and Abd El-Raheem [11].

Censoring appears normally in reliability examinations. It happens when lifetimes are known only for a portion of the units under examination, the residual lifetimes being known only to exceed determined values. There two main types of censoring schemes which are type-I and type-II censoring schemes (CSs). The fundamental difference between type-I and type-II CSs is that the first depends on the time of ending the experiment and the second depends on the number of failures. The test units in these two CSs cannot be excluded from the test until the end the test. A progressive type-II (PT-II) censoring is an extension of type-II censoring. It encourages the experimenter to remove experimented units at different times throughout the test. For more reading about PT-II CS, we can refer to Balakrishnan and Aggarwala [16], Almetwally et al. [13], Alshenawy et al. [14] and El-Sherpieny et al. [21].

Due to the great importance of PT-II censoring in reliability experiments, many researchers addressed the issue of statistical inference of ALTs under PT-II censored data, we can refer to Abdel-Hamid [1], Abdel-Hamid and AL-Hussaini [2, 4], Jaheen et al. [25] and Mohie El-Din et al. [29].

The motivation and contribution in this research is that we apply CSALT for the lifetime of units that follow the MKEx distribution under PT-II censored sample with binomial removal, also we estimate the parameters using Bayesian and classical methods. Moreover, we use a real life censored data of transforms insulation to illustrate the proposed methods.

The paper is organized as follows: The MKEx distribution and test assumptions for CSALT are discussed in Section 2. The maximum likelihood estimation (MLE) for the model parameters are provided in Section 3. Bayes estimates (BEs) under different loss functions are discussed in Section 4. In Section 5, asymptotic and bootstrap confidence intervals (CIs) are presented. Section 6 contains the simulation numerical results. Section 7 contains the transformers turn insulation application. Section 8 contains the conclusion of the paper and the major findings in this research.

2. Lifetime distribution and test assumptions

2.1. MKEx distribution

In 2013, Kumar and Dharmaja [26] introduced and studied the reduced Kies distribution. In many references, the reduced Kies distribution is known as modified Kies (MK) distribution. Kumar and Dharmaja [26] find out that the MK distribution can perform better than common lifetime distributions such as Weibull distribution and some of its extensions in modelling lifetime data. Kumar and Dharmaja [27] presented the exponentiated MK distribution and introduced its statistical properties. Dey et al. [20] estimated the distribution parameters of MK distribution under progressive type-II censoring and introduced the recurrence relations for the moments of the MK distribution. Based on MK distribution and the $T-X$ family, Al-Babtain et al. [15] introduced a new lifetime distribution, they called it modified Kies exponential (MKEx) distribution. Aljohani et al. [40] discussed the parameter estimation of the MKEx distribution using the MLE method, based on ranked set sampling. It has bathtub shape, increasing and decreasing failure rate. Furthermore, It has the ability to model negatively and positively skewed data. Moreover, it has a closed form cumulative distribution function (CDF) and very easy to handle which make the distribution is candidate to use in different fields such as life testing, reliability, biomedical studies and survival analysis. In this context, Al-Babtain et al. [15] used two different types of real data to show that this distribution may be a good alternative to many popular distributions such as Weibull, Marshall-Olkin exponential, Kumaraswamy exponential, beta exponential, gamma, and exponentiated exponential distribution. The CDF of the MKEx distribution is

$$F(x;\alpha,\sigma) = 1 - e^{-(e^{\sigma x}-1)^{\alpha}}, x > 0,\ \alpha,\sigma > 0. \tag{2.1}$$

The corresponding probability density function (PDF) of (2.1) is given by

$$f(x;\alpha,\sigma) = \alpha\sigma\, e^{\alpha\sigma x}\left(1 - e^{-\sigma x}\right)^{\alpha-1} e^{-(e^{\sigma x}-1)^{\alpha}}, x > 0,\ \alpha,\sigma > 0. \tag{2.2}$$

2.2. Multiple CSALT's assumptions

In this subsection, we introduce the assumption of CSALT under PT-II CS with binomial removal. Suppose an ALT contains number of stress levels $L \geq 2$ such that the stress is arranged ascendingly where $\phi_1 < \phi_2 < ... < \phi_L$, within the level l, $l = 1, 2, ..., L$, identical n_l units are exposed to an accelerated condition, so that the number of units under the lifetime experiment are $\sum_{l=1}^{L} n_l = n$, where n is the whole number of tested items in the test. In each stress level, ϕ_l, $l = 1, 2, ..., L$, at the time of the first failure $x_{l1:m_l:n_l}$, R_{l1} of the $n_l - 1$ remaining units are randomly excluded from the test. In the same manner R_{l2} of the surviving units, $n_l - R_{l1} - 1$, are randomly excluded from the test after the second failure $x_{l2:m_l:n_l}$ is detected. This mechanism continues until the failure of m_l^{th} occurs. The remaining surviving units $R_{lm_l} = n_l - m_l - \sum_{j=1}^{m_l-1} R_{lj}$ are excluded from the test after the m_l^{th} occurs, and the test is terminated. Suppose that the elimination of an individual unit from the test is independent of the others but with the same probability of removal P. Then, the number of units withdrawn at each failure time has a binomial distribution. That is $R_1 \sim binomial(n_l - m_l, P)$, $R_{lj} \sim binomial(n_l - m_l - \sum_{l=1}^{m_l-1} R_{lj}, P), l = 2, ..., m_l$ and $R_{lm_l} = n_l - m_l - \sum_{j=1}^{m_l-1} R_{lj}$. In this context, the assumptions of multiple CSALT are as follows:

1. In each stress level ϕ_l, the lifetime of the experimental units follow MKEx(α, σ_l) distribution.
2. The scale parameter in each stress level σ_l and the stress level ϕ_l is linked by the following relation.

$$\log(\sigma_l) = \zeta + \beta\eta_l, \; l = 0, 1, ..., L, \tag{2.3}$$

where $\zeta \in (-\infty, \infty)$ and $\beta > 0$ are the unknown model parameters and $\eta_l = \eta(\phi_l)$ is an increasing function of ϕ.

(a) If $\eta(\phi_l) = \log(\phi_l)$, the model in (2.3) becomes the inverse power model.
(b) If $\eta(\phi_l) = \frac{1}{-\phi_l}$, the model in (2.3) becomes Arrhenius model.
(c) If $\eta(\phi_l) = \phi_l$, the model in (2.3) becomes exponential model.

For more extensive reading about acceleration and its different models we can refer to the book of Nelson [36], specifically Chapter 2.

From Eq (2.3), we have

$$\sigma_l = \sigma_0 \exp\{\beta(\eta_l - \eta_0)\} = \sigma_0\theta^{z_l} > 0, \qquad l = 0, 1, ..., L, \tag{2.4}$$

where $\theta = \exp\{\beta(\eta_1 - \eta_0)\} = \frac{\sigma_1}{\sigma_0} > 1$, $\sigma_0 > 0$ is the scale parameter of the MKEx distribution under usage conditions ϕ_0, and

$$z_l = \frac{\eta_l - \eta_0}{\eta_1 - \eta_0}, \qquad l = 1, 2, ..., L, \text{ satisfying } z_L > z_{L-1} > ... > z_1 = 1.$$

The transformation from the parameters $(\alpha, \sigma_l) = (\alpha, \zeta, \beta)$ to the new parameters $(\alpha, \sigma_0, \theta)$ is an one-to-one mapping. Since the Jacobian determinant from (α, ζ, β) to $(\alpha, \sigma_0, \theta)$ does not equal zero. Thus, the unknown parameters should be estimated are α, σ_0 and θ.

3. Estimation via maximum likelihood method

In this section, the classical estimates of the parameters of MKEx distribution under PT-II CS with binomial removal are obtained. As we mentioned later that the R_{lj} has a binomial distribution, then the PDF of R_{lj}, $l = 1, 2, ..., L$ is given as follows:

$$Pr(R_{l1} = r_{l1}) = \binom{n_l - m_l}{r_{l1}} P^{r_{l1}} (1 - P)^{(n_l - m_l - r_{l1})},$$

while, for $j = 2, 3, ..., m_l - 1$:

$$Pr(R_{lj} = r_{lj} | R_{(l-1)j}) = \binom{n_l - m_l - \sum_{j=1}^{l-1} r_{lj}}{r_{lj}} P^{r_{lj}} (1 - P)^{(n_l - m_l - \sum_{j=1}^{l} r_{lj})},$$

where $0 \leq r_{lj} \leq n_l - m_l - \sum_{j=1}^{l-1} r_{lj}$. Furthermore, we suppose that R_{lj} is independent of $X_{lj:m_l:n_l}$ for all l, $l = 1, 2, ..., L$. Therefore, the likelihood function α, σ_0, θ under PT-II censoring with binomial removal is given by

$$L^* = \prod_{l=1}^{L} C_l \prod_{j=1}^{m_l} f(x_{lj}; \alpha, \sigma_0, \theta) \left(1 - F(x_{lj}; \alpha, \sigma_0, \theta)\right)^{R_{lj}} Pr(R_{lj} = r_{lj}), \tag{3.1}$$

since $X_{lj:m_l:n_l}$ and R_{lj} for all $l = 1, 2, ..., L$ are independent, then the MLE of P can be derived by maximizing $Pr(R_{lj} = r_{lj})$ directly. Hence the MLE of P is given by

$$\hat{P} = \sum_{l=1}^{L} \frac{\sum_{l=1}^{m_l-1} r_{lj}}{(m_l - 1)(n_l - m_l) - \sum_{l=1}^{m_l-1} (m_l - l - 1) r_{lj}}.$$

The likelihood function under PT-II censoring after estimating P, have the following form:

$$L^*(\alpha, \sigma_0, \theta) = \prod_{l=1}^{L} C_l \prod_{j=1}^{m_l} \alpha \sigma_0 \theta^{z_l} \ e^{\alpha \sigma_0 \theta^{z_l} \tau_{lj}} \left(1 - e^{-\sigma_0 \theta^{z_l} \tau_{lj}}\right)^{\alpha - 1} e^{-\left(1 + R_{lj}\right)\left[e^{\sigma_0 \theta^{z_l} \tau_{lj}} - 1\right]^{\alpha}}, \tag{3.2}$$

where $\tau_{lj} = x_{lj:m_l:n_l}$, $C_l = n_l \left(n_l - 1 - R_{l1}\right) \left(n_l - 2 - R_{l1} - R_{l2}\right) ... \left(n_l - m_l + 1 - \sum_{j=1}^{m_l-1} R_{lj}\right)$.

The log-likelihood function after removing the normalizing constant, can be formed as in Eq (3.3)

$$\begin{aligned} \ell(\alpha, \sigma_0, \theta) = [\log(\alpha) + \log(\sigma_0)] \sum_{l=1}^{L} m_l + log(\theta) \sum_{l=1}^{L} m_l z_l + \alpha \sigma_0 \sum_{l=1}^{L} \sum_{j=1}^{m_l} \theta^{z_l} \tau_{lj} + \\ (\alpha - 1) \sum_{l=1}^{L} \sum_{j=1}^{m_l} \log\left(1 - e^{-\sigma_0 \theta^{z_l} \tau_{lj}}\right) - \sum_{l=1}^{L} \sum_{j=1}^{m_l} \left(1 + R_{lj}\right) \left[e^{\sigma_0 \theta^{z_l} \tau_{lj}} - 1\right]^{\alpha}. \end{aligned} \tag{3.3}$$

By finding partial derivatives of ℓ with respect to the distribution parameters, we have

$$\frac{\partial \ell}{\partial \alpha} = \frac{\sum_{l=1}^{L} m_l}{\alpha} + \sigma_0 \sum_{l=1}^{L} \sum_{j=1}^{m_l} \theta^{z_l} + \sum_{l=1}^{L} \sum_{j=1}^{m_l} \log\left(1 - e^{-\epsilon_{lj}}\right) - \sum_{l=1}^{L} \sum_{j=1}^{m_l} \left(1 + R_{lj}\right) \left[e^{\epsilon_{lj}} - 1\right]^{\alpha} \log\left[e^{\epsilon_{lj}} - 1\right], \tag{3.4}$$

$$\frac{\partial \ell}{\partial \sigma_0} = \frac{\sum_{l=1}^{L} m_l}{\sigma_0} + \sigma_0 \sum_{l=1}^{L} \sum_{j=1}^{m_l} \theta^{z_l} \tau_{lj} + (\alpha - 1) \sum_{l=1}^{L} \sum_{j=1}^{m_l} \frac{\theta^{z_l} \tau_{lj} e^{-\epsilon_{lj}}}{1 - e^{-\epsilon_{lj}}} - \alpha \sum_{l=1}^{L} \sum_{j=1}^{m_l} \theta^{z_l} \tau_{lj} e^{\epsilon_{lj}} \left(1 + R_{lj}\right) \left[e^{\epsilon_{lj}} - 1\right]^{\alpha - 1}, \tag{3.5}$$

$$\begin{aligned} \frac{\partial \ell}{\partial \theta} = \frac{1}{\theta} \sum_{l=1}^{L} m_l z_l + \alpha \sigma_0 \sum_{l=1}^{L} \sum_{j=1}^{m_l} z_l \theta^{z_l - 1} \tau_{lj} - \alpha \sigma_0 \sum_{l=1}^{L} \sum_{j=1}^{m_l} \left(1 + R_{lj}\right) e^{\epsilon_{lj}} z_l \theta^{z_l - 1} \tau_{lj} \left[e^{\epsilon_{lj}} - 1\right]^{\alpha - 1} + \\ (\alpha - 1) \sum_{l=1}^{L} \sum_{j=1}^{m_l} \frac{\sigma_0 z_l \theta^{z_l - 1} \tau_{lj} e^{-\epsilon_{lj}}}{1 - e^{-\epsilon_{lj}}}, \end{aligned} \tag{3.6}$$

where $\epsilon_{lj} = \sigma_0 \theta^{z_l} \tau_{lj}$. The MLE of $(\alpha, \sigma_0, \theta)$ is $(\hat{\alpha}, \hat{\sigma_0}, \hat{\theta})$, which may be derived simultaneously by solving Eqs (3.4)–(3.6). Regrettably, solving these equations will be very hard, so we have to use numerical techniques like the Newton-Raphson method.

4. Bayesian estimation

This section includes the BEs of α, σ_0 and θ. We assume that α, σ_0 and θ are independent and have gamma priors. Gamma prior for the acceleration factor $\theta > 1$ was first considered by DeGroot and Goel [19]. They stated that in most problems of accelerated life testing the accelerating factor θ will be greater than 1. However, in order to not restrict the applicability of the acceleration model we shall consider prior distributions for θ that assign positive density to all positive values of θ. If the experimenter is almost certain that $\theta > 1$, then he can choose a gamma prior distribution that assigns a suitably small probability to the interval $0 < \theta < 1$. For more details about this point, see Section 3 of DeGroot and Goel [19]. The gamma priors for distribution parameters are as follows:

$$\mathbb{C}_1(\alpha) \propto \alpha^{\eta_1-1} e^{-\mathcal{H}_1\alpha}, \quad \alpha > 0,\ \eta_1, \mathcal{H}_1 > 0, \tag{4.1}$$

$$\mathbb{C}_2(\sigma_0) \propto \sigma_0^{\eta_2-1} e^{-\mathcal{H}_2\sigma_0}, \quad \sigma_0 > 0,\ \eta_2, \mathcal{H}_2 > 0, \tag{4.2}$$

$$\mathbb{C}_3(\theta) \propto \theta^{\eta_3-1} e^{-\mathcal{H}_3\theta}, \quad \theta > 1,\ \eta_3, \mathcal{H}_3 > 0. \tag{4.3}$$

The joint prior of α, σ_0 and θ is obtained as

$$\mathbb{C}(\alpha, \sigma_0, \theta) \propto \alpha^{\eta_1-1}\sigma_0^{\eta_2-1}\theta^{\eta_3-1} \exp\{-(\alpha\mathcal{H}_1 + \sigma_0\mathcal{H}_2 + \mathcal{H}_3\theta)\}, \quad \alpha, \sigma_0 > 0, \theta > 1. \tag{4.4}$$

To determine suitable and superior values of the hyper-parameters of the independent joint prior, we can use estimate and variance-covariance matrix of MLE method. By equating mean and variance of gamma priors as the following equations, the estimated hyper-parameters can be computed as

$$\eta_j = \frac{\left[\frac{1}{B}\sum_{i=1}^{B}\hat{\Omega}_i^j\right]^2}{\frac{1}{B-1}\sum_{i=1}^{B}\left[\hat{\Omega}_i^j - \frac{1}{B}\sum_{i=1}^{B}\hat{\Omega}_i^j\right]^2},\ j = 1, 2, 3, \tag{4.5}$$

$$\mathcal{H}_j = \frac{\frac{1}{B}\sum_{i=1}^{B}\hat{\Omega}_i^j}{\frac{1}{B-1}\sum_{i=1}^{B}\left[\hat{\Omega}_i^j - \frac{1}{B}\sum_{i=1}^{B}\hat{\Omega}_i^j\right]^2},\ j = 1, 2, 3, \tag{4.6}$$

where, B is the number of iteration and $\hat{\Omega}^1 = \hat{\alpha}$, $\hat{\Omega}^2 = \hat{\sigma_0}$, and $\hat{\Omega}^3 = \hat{\theta}$.

By multiplying Eq (3.2) by (4.4), and making some simplifications the posterior distribution $\mathbb{C}^*(\alpha, \sigma_0, \theta)$ is formed as follows

$$\begin{aligned}\mathbb{C}^*(\alpha, \sigma_0, \theta) \propto\ & \alpha^{\eta_1+\sum_{l=1}^{L} m_l-1}\sigma_0^{\eta_2+\sum_{l=1}^{L} m_l-1}\theta^{\eta_3-1} \exp\{-(\alpha\mathcal{H}_1 + \sigma_0\mathcal{H}_2 + \mathcal{H}_3\theta)\} \\ & \prod_{l=1}^{L}\prod_{j=1}^{m_l}\theta^{z_l}\ e^{\alpha\sigma_0\theta^{z_l}\tau_{lj}}\left(1 - e^{-\sigma_0\theta^{z_l}\tau_{lj}}\right)^{\alpha-1} e^{-(1+R_{lj})\left[e^{\sigma_0\theta^{z_l}\tau_{lj}}-1\right]^{\alpha}}.\end{aligned} \tag{4.7}$$

The BEs of $u(\Theta) = u(\alpha, \sigma_0, \theta)$ using squared error (SE) and LINEX loss functions are as follows

$$\widetilde{u}_{SE}(\Theta) = E(u(\Theta)), \tag{4.8}$$

and

$$\widetilde{u}_{LINEX}(\Theta) = -\frac{1}{c}\log[E(e^{-c\,u(\Theta)})], \quad c \neq 0. \tag{4.9}$$

It is obvious that both BEs of $u(\alpha, \sigma_0, \theta)$ in (4.8) and (4.9) are considered as the division of more than one integration over each other. As we know multiple integrals is very tough to be solved analytically or even mathematically by hand. Therefore, we have to use the Markov Chain Monte Carlo (MCMC) technique to find an approximate value of integrals. An important methods of the MCMC technique, is the Metropolis-Hastings (MH) algorithm, some times they call it the random walk algorithm. Its similar to acceptance-rejection sampling, the MH algorithm consider for each iteration of the algorithm, a candidate value can be generated from normal proposal distribution.

The MH algorithm produces a series of draws from MKEx distribution as follows:

1. initiate with $\alpha^{(0)} = \hat{\alpha}$, $\sigma_0^{(0)} = \hat{\sigma_0}$, $\theta^{(0)} = \hat{\theta}$.
2. Set $i = 1$.
3. Simulate α^* from proposal distribution $\mathbf{N}(\alpha^{(i-1)}, var(\alpha^{(i-1)}))$.
4. Evaluate the acceptance probability $A(\alpha^{(i-1)}|\alpha^*) = \min\left[1, \frac{\mathbb{C}^*(\alpha^*|\sigma_0^{(i-1)}, \theta^{(i-1)})}{\mathbb{C}^*(\alpha^{(i-1)}|\sigma_0^{(i-1)}, \theta^{(i-1)})}\right]$.
5. Draw $U \sim U(0, 1)$.
6. If $U \le A(\alpha^{(i-1)}|\alpha^*)$, put $\alpha^{(i)} = \alpha^*$, else put $\alpha^{(i)} = \alpha^{(i-1)}$.
7. Do the Steps 2–6 for σ_0 and θ.
8. Put $i = i + 1$.
9. Repeat Steps 3–8, N times to obtain $(\alpha^{(1)}, \sigma_0^{(1)}, \theta^{(1)})$, ..., $(\alpha^{(N)}, \sigma_0^{(N)}, \theta^{(N)})$.

Then, the BEs of $u(\alpha, \sigma_0, \theta)$ using MCMC under SE, and LINEX loss functions are respectively

$$\tilde{u}_{SE} = \frac{1}{N-M} \sum_{i=M+1}^{N} u(\alpha^{(i)}, \sigma_0^{(i)}, \theta^{(i)}),$$

$$\tilde{u}_{LINEX} = -\frac{1}{c} \log\left[\frac{1}{N-M} \sum_{i=M+1}^{N} \exp\{-c\, u(\alpha^{(i)}, \sigma_0^{(i)}, \theta^{(i)})\}\right],$$

where M is the burn-in period.

The conditional posterior distributions used in MH algorithm are given as follows:

$$\mathbb{C}^*(\alpha|\sigma_0, \theta) \propto \exp\{-\alpha\mathcal{H}_1\}\alpha^{\eta_1+\sum_{l=1}^{L} m_l-1} \prod_{l=1}^{L}\prod_{j=1}^{m_l} e^{\alpha\sigma_0\theta^{z_l}\tau_{lj}} \left(1 - e^{-\sigma_0\theta^{z_l}\tau_{lj}}\right)^{\alpha-1} e^{-\left(1+R_{lj}\right)\left[e^{\sigma_0\theta^{z_l}\tau_{lj}}-1\right]^{\alpha}}, \quad (4.10)$$

$$\mathbb{C}^*(\sigma_0|\alpha, \theta) \propto \sigma_0^{\eta_2+\sum_{l=1}^{L} m_l-1} \exp\{-\sigma_0\mathcal{H}_2\} \prod_{l=1}^{L}\prod_{j=1}^{m_l} e^{\alpha\sigma_0\theta^{z_l}\tau_{lj}} \left(1 - e^{-\sigma_0\theta^{z_l}\tau_{lj}}\right)^{\alpha-1} e^{-\left(1+R_{lj}\right)\left[e^{\sigma_0\theta^{z_l}\tau_{lj}}-1\right]^{\alpha}}, \quad (4.11)$$

and

$$\mathbb{C}^*(\theta|\alpha, \sigma_0) \propto \theta^{\eta_3-1} \exp\{-\mathcal{H}_3\theta\} \prod_{l=1}^{L}\prod_{j=1}^{m_l} \theta^{z_l}\, e^{\alpha\sigma_0\theta^{z_l}\tau_{lj}} \left(1 - e^{-\sigma_0\theta^{z_l}\tau_{lj}}\right)^{\alpha-1} e^{-\left(1+R_{lj}\right)\left[e^{\sigma_0\theta^{z_l}\tau_{lj}}-1\right]^{\alpha}}. \quad (4.12)$$

The Bayesian estimates have CIs which are called the credible intervals or some times we call it the highest posterior density (HPD) intervals, for more information see, Chen and Shao [18]. They performed a technique that was used extensively to generate the HPD intervals of unknown parameters of the distribution. In this method, samples are drawn with the proposed MH algorithm that are used to generate estimates, for the HPD algorithm see Chen and Shao [18].

5. Confidence intervals

In this section, the asymptotic, percentile Bootstrap and Bootstrap-t confidence intervals (CIs) for the unknown distribution parameters α, σ_0, θ are obtained.

5.1. Asymptotic confidence intervals

Asymptotic CI is the most popular approach to establish the approximate confidence limits for parameters, we use the MLE to obtain the observed Fisher information matrix $I(\hat{\Omega})$, which consists of of the negative second derivative of the natural logarithm of the likelihood function evaluated at $\hat{\Omega} = (\hat{\alpha}, \hat{\sigma_0}, \hat{\theta})$, where

$$I(\hat{\Omega}) = \begin{bmatrix} I_{\hat{\alpha}\hat{\alpha}} & & \\ I_{\hat{\sigma_0}\hat{\alpha}} & I_{\hat{\sigma_0}\hat{\sigma_0}} & \\ I_{\hat{\theta}\hat{\alpha}} & I_{\hat{\theta}\hat{\sigma_0}} & I_{\hat{\theta}\hat{\theta}} \end{bmatrix}$$

and by inverseing this matrix we can find the asymptotic variance-covariance matrix. Now we can find the asymptotic variance-covariance matrix of the parameter vector Ω is $V(\hat{\Omega}) = I^{-1}(\hat{\Omega})$.
So the $100(1-\gamma)\%$ asymptotic confidence intervals for parameters α, σ_0 and θ can be established as follows:

$$\left(\hat{\vartheta}_l,\ \hat{\vartheta}_u\right) = \hat{\vartheta} \pm Z_{1-\gamma/2}\sqrt{V(\hat{\vartheta})}, \tag{5.1}$$

where ϑ is $\alpha,\ \sigma_0$ or θ, and Z_q is the $100q-th$ percentile of a standard normal distribution.

5.2. Bootstrap confidence interval

In this subsection, two parametric bootstrap methods: Percentile bootstrap (B-P) and the bootstrap-t (B-T) [23] are considered to obtain CIs for α, σ_0, and θ.
The percentile bootstrap CIs can be obtained in such a way.

1. Find the values of the MLE of MKEx distribution.
2. To find the estimates of the bootstrap, we must generate a bootstrap samples, $(\alpha^b, \sigma_0^b, \theta^b)$, using MLEs of $(\alpha, \sigma_0, \theta)$.
3. Repeat step number (2) $\mathcal{B}$ times to have $(\alpha^{b(1)}, \alpha^{b(2)}, \ldots, \alpha^{b(\mathcal{B})}), (\sigma_0^{b(1)}, \sigma_0^{b(2)}, \ldots \sigma_0^{b(\mathcal{B})})$ and $(\theta^{b(1)}, \theta^{b(2)}, \ldots, \theta^{b(\mathcal{B})})$.
4. Arrange $(\alpha^{b(1)}, \alpha^{b(2)}, \ldots, \alpha^{b(\mathcal{B})}), (\sigma_0^{b(1)}, \sigma_0^{b(2)}, \ldots \sigma_0^{b(\mathcal{B})})$ and $(\theta^{b(1)}, \theta^{b(2)}, \ldots, \theta^{b(\mathcal{B})})$ from smallest to the biggest, $(\alpha^{b[1]}, \alpha^{b[2]}, \ldots, \alpha^{b([\mathcal{B}])}), (\sigma_0^{b[1]}, \sigma_0^{b[2]}, \ldots \sigma_0^{b[\mathcal{B}]})$ and $(\theta^{b[1]}, \theta^{b[2]}, \ldots, \theta^{b[\mathcal{B}]})$.
5. The two side $100(1-\gamma)\%$ percentile bootstrap confidence interval for α, σ_0 and θ are evaluated by $[\alpha^{b([\mathcal{B}\frac{\gamma}{2}])}, \alpha^{b([\mathcal{B}(1-\frac{\gamma}{2})])}], [\sigma_0^{b([\mathcal{B}\frac{\gamma}{2}])}, \sigma_0^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$ and $[\theta^{b([\mathcal{B}\frac{\gamma}{2}])}, \theta^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$.

The bootstrap-t CIs can be obtained in such a way.

1. Repeat the first two steps in the percentile bootstrap algorithm.
2. Evaluate $T = \frac{\hat{\vartheta}^b - \hat{\vartheta}}{\sqrt{V(\hat{\vartheta}^b)}}$, where ϑ is $\alpha,\ \sigma_0$ or θ, and $V(\hat{\vartheta}^b)$ is asymptotic variances of $\hat{\vartheta}^b$.
3. Repeat the above two steps $\mathcal{B}$ times and rearrange $(T^{(1)}, T^{(2)}, \ldots, T^{(\mathcal{B})})$ in ascending order.

4. A two side $100(1-\gamma)\%$ bootstrap-t CI for α, σ_0 and θ are given by $[\alpha - T_{\alpha}^{b([\mathcal{B}\frac{\gamma}{2}])}\sqrt{V(\alpha^b)}, \alpha + \sqrt{V(\alpha^b)}T_{\alpha}^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$, $[\sigma_0 - T_{\sigma_0}^{b([\mathcal{B}\frac{\gamma}{2}])}\sqrt{V(\sigma_0^b)}, \sigma_0 + \sqrt{V(\sigma_0^b)}T_{\sigma_0}^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$ and $[\theta - T_{\theta}^{b([\mathcal{B}\frac{\gamma}{2}])}\sqrt{V(\theta^b)}, \theta + \sqrt{V(\theta^b)}T_{\theta}^{b([\mathcal{B}(1-\frac{\gamma}{2})])}]$

6. Simulation study

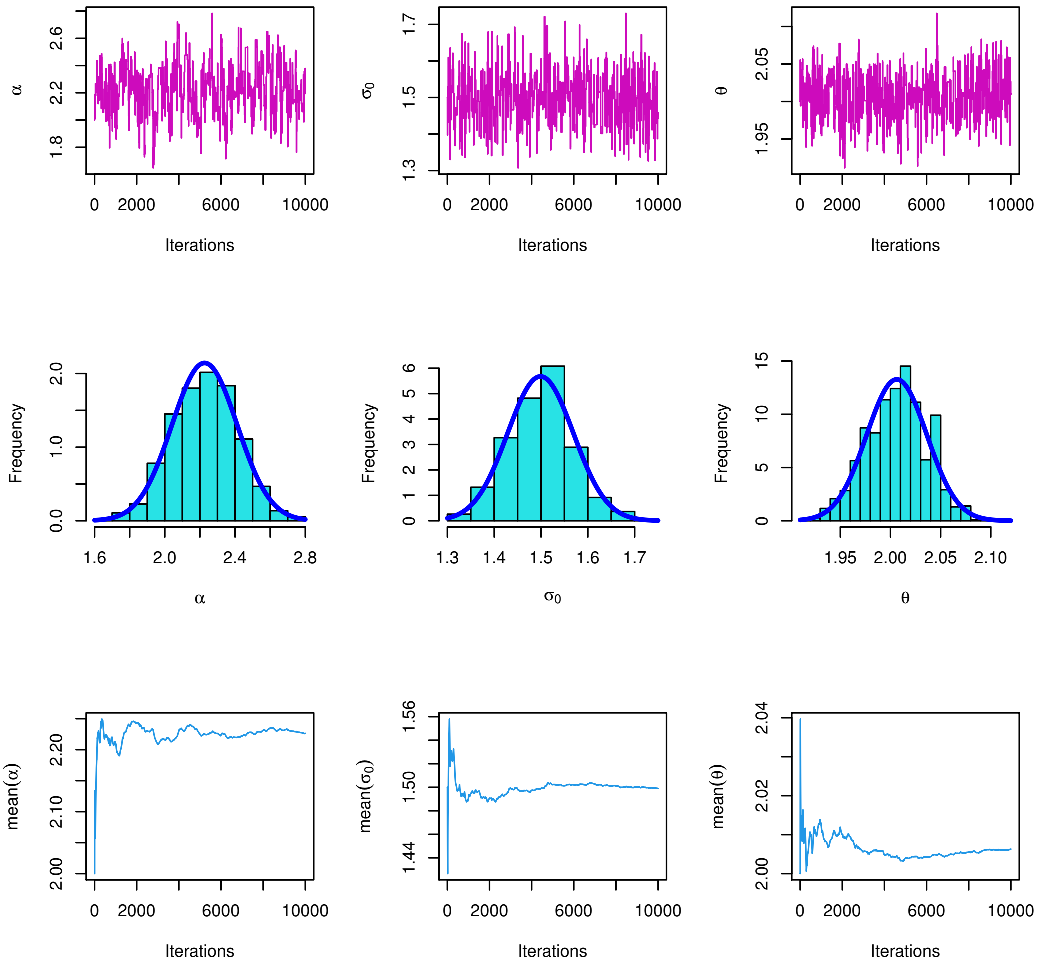

Figure 1. MCMC iterations and the kernel histograms of the posterior samples for each parameter with true values: $\alpha = 2, \sigma_0 = 1.5, \theta = 2, n_1 = 60; n_2 = 40; n_3 = 20; n_4 = 10, m_1 = 54; m_2 = 36; m_3 = 18; m_4 = 9, and P = 0.8$.

In this section, we conduct a Monte Carlo simulation to find the estimates of the distribution parameters with different sample sizes n_l and different censoring schemes R_l with different probability of binomial removal P. We compare the performance of the MLEs and the BEs under different loss functions in terms of their relative absolute bias (RABias), and the mean square error (MSE). Furthermore, we estimate the length of asymptotic CI (L.CI), length of B-P CI (L.BP), length of B-T

CI (L.BT) and for Bayesian estimation method we estimate length of credible interval (L.Cr). In the simulation, we consider different sample sizes, n_l, different number of failures, m_l, and different ratio of failures, r_l, where $r_l = m_l/n_l$. In addition, probability of binomial removal P is considered to be 0.35, and 0.8 for each stress level l, $l = 1, 2, ..., L$. The true values of the parameters used in the simulation study are ($\alpha = 2$, $\sigma_0 = 1.5$, $\theta = 2$) and ($\alpha = 0.8$, $\sigma_0 = 1.5$, $\theta = 2$). The simulation study is done using 10,000 iterations, and the average of the results of these iterations are tabulated in Tables 1–9.

The MCMC iterations and the kernel histograms of the posterior samples of the parameters α, σ_0, and θ are plotted in Figure 1.

Figure 1 shows that the MCMC samples are well mixed and stationary achieved. Also, it indicates that posterior distributions of the three parameters are symmetric.

Table 1. RABias and MSE of the MLE for binomial removal parameter P based on CSALT.

L	2				4			
P	n_1, n_2	m_1, m_2	RABias	MSE	n_1, n_2, n_3, n_4	m_1, m_2, m_3, m_4	RABias	MSE
0.35	8, 7	6, 5	1.7083	0.3684	6, 4, 3, 2	4, 3, 2, 1	1.5257	0.3297
		7, 6	1.6929	0.3658		5, 4, 3, 2	0.6704	0.1678
0.8		6, 5	0.1837	0.0343		4, 3, 2, 1	0.2269	0.1164
		7, 6	0.1257	0.0250		5, 4, 3, 2	0.1656	0.0467
0.35	20, 18	14, 13	1.6814	0.3582	15, 10, 8, 5	10, 7, 6, 4	1.4933	0.3442
		18, 16	1.6087	0.3244		14, 9, 7, 4	1.4403	0.3191
0.8		14, 13	0.1242	0.0251		10, 7, 6, 4	0.1811	0.0300
		18, 16	0.1169	0.0209		14, 9, 7, 4	0.1526	0.0242
0.35	40, 27	28, 19	1.6577	0.3449	30, 20, 10, 7	21, 14, 7, 5	0.9262	0.1096
		36, 24	1.5523	0.3014		27, 18, 9, 6	0.8184	0.0877
0.8		28, 19	0.1172	0.0236		21, 14, 7, 5	0.1622	0.0262
		36, 24	0.0933	0.0152		27, 18, 9, 6	0.1310	0.0231
0.35	75, 55	52, 38	1.6159	0.3266	60, 40, 20, 10	42, 28, 14, 7	0.8864	0.1011
		68, 50	1.5166	0.2869		53, 36, 18, 9	0.7818	0.0805
0.8		52, 38	0.1068	0.0199		42, 28, 14, 7	0.1513	0.0249
		68, 50	0.0032	0.0189		53, 36, 18, 9	0.1140	0.0229

The following observations are conducted from the simulation study:

1. For fixed m_l, P, η_l and true values of the parameters, the RABias, MSE and length of CIs decrease as n_l increases.
2. The best method of estimation is the Bayesian estimation according to the the values of the MSE.
3. The BEs under SE loss function are better than the corresponding estimates under LINEX loss function with $c = -0.5$ for all parameters.
4. BEs under LINEX loss function for α and σ_0 with $c = 0.5$ are better than the corresponding estimates under SE loss function for all cases.
5. BEs under LINEX loss function for θ, with $c = 0.5$ is better than SE loss function when α is less than 1, while, when α is greater than 1, BEs under SE loss function are better.

Table 2. RABias and MSE of the MLE, and BEs based on SE and LINEX loss functions for MKEx distribution when $L = 2$, $\alpha = 2$, $\sigma_0 = 1.5$ and $\theta = 2$.

Point Estimation				$\alpha = 2; \sigma_0 = 1.5; \theta = 2$							
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100$							
				Sample 1: $n_1 = 8; n_2 = 7$ Sample 2: $n_1 = 20; n_2 = 18$							
$L = 2$				MLE		SE		LINEX (c = -0.5)		LINEX (c = 0.5)	
n	P	m_1, m_2		RABias	MSE	RABias	MSE	RABias	MSE	RABias	MSE
15	0.35	6, 5	α	0.2807	0.7270	0.0813	0.2016	0.1219	0.2767	0.0428	0.1474
			σ_0	0.0136	0.2230	0.0157	0.0189	0.0041	0.0206	0.0269	0.0181
			θ	0.0100	0.1531	0.0154	0.0072	0.0119	0.0070	0.0188	0.0075
		7, 6	α	0.2068	0.4976	0.0515	0.1240	0.0845	0.1650	0.0201	0.0959
			σ_0	0.0103	0.1609	0.0100	0.0185	0.0215	0.0217	0.0011	0.0164
			θ	0.0039	0.1081	0.0084	0.0056	0.0053	0.0055	0.0114	0.0057
	0.8	6, 5	α	0.3184	0.8187	0.10372	0.20867	0.14484	0.28773	0.06465	0.15069
			σ_0	0.0057	0.1727	0.00148	0.01665	0.01312	0.01933	0.00981	0.01499
			θ	0.0003	0.1158	0.00776	0.00556	0.00440	0.00562	0.01109	0.00561
		7, 6	α	0.2290	0.5483	0.07143	0.14796	0.10564	0.20064	0.03904	0.11035
			σ_0	0.0254	0.1427	0.01641	0.01869	0.02780	0.02235	0.00540	0.01604
			θ	0.0029	0.0858	0.00545	0.00474	0.00247	0.00476	0.00841	0.00480
38	0.35	14, 13	α	0.1809	0.2897	0.06951	0.07044	0.08387	0.08417	0.05554	0.05901
			σ_0	0.0195	0.0544	0.01144	0.00761	0.00652	0.00779	0.01630	0.00757
			θ	0.0024	0.0189	0.00641	0.00247	0.00505	0.00243	0.00778	0.00252
		18, 16	α	0.0858	0.1227	0.02433	0.02919	0.03482	0.03376	0.01407	0.02573
			σ_0	0.0103	0.0426	0.00725	0.00693	0.01174	0.00745	0.00282	0.00653
			θ	0.0001	0.0148	0.00318	0.00217	0.00198	0.00217	0.00437	0.00219
	0.8	14, 13	α	0.1644	0.2671	0.06245	0.06461	0.07617	0.07703	0.04910	0.05441
			σ_0	0.0014	0.0541	0.00018	0.00746	0.00533	0.00793	0.00491	0.00714
			θ	0.0007	0.0190	0.00441	0.00249	0.00302	0.00248	0.00580	0.00252
		18, 16	α	0.0868	0.1340	0.02659	0.03344	0.03722	0.03849	0.01617	0.02951
			σ_0	0.0084	0.0421	0.00626	0.00675	0.01078	0.00722	0.00179	0.00639
			θ	0.0017	0.0142	0.00185	0.00210	0.00063	0.00211	0.00306	0.00210

Table 2 (continued).

Point Estimation				$\alpha = 2; \sigma_0 = 1.5; \theta = 2$							
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100$							
				Sample 1: $n_1 = 40; n_2 = 27$ Sample 2: $n_1 = 75; n_2 = 55$							
$L = 2$				MLE		SE		LINEX (c = -0.5)		LINEX (c = 0.5)	
n	P	m_1, m_2		RABias	MSE	RABias	MSE	RABias	MSE	RABias	MSE
67	0.35	28, 19	α	0.1150	0.1262	0.0451	0.0306	0.0523	0.0346	0.0380	0.0272
			σ_0	0.0145	0.0244	0.0063	0.0036	0.0036	0.0036	0.0090	0.0036
			θ	0.0002	0.0103	0.0052	0.0015	0.0043	0.0015	0.0060	0.0016
		36, 24	α	0.0560	0.0709	0.0181	0.0178	0.0237	0.0194	0.0125	0.0164
			σ_0	0.0053	0.0197	0.0049	0.0033	0.0072	0.0034	0.0026	0.0032
			θ	0.0007	0.0079	0.0014	0.0012	0.0006	0.0013	0.0021	0.0013
	0.8	28, 19	α	0.1203	0.1206	0.0519	0.0347	0.0591	0.0390	0.0449	0.0308
			σ_0	0.0071	0.0260	0.0006	0.0039	0.0022	0.0041	0.0034	0.0039
			θ	0.0026	0.0109	0.0028	0.0017	0.0019	0.0017	0.0037	0.0017
		36, 24	α	0.0463	0.0625	0.0123	0.0145	0.0179	0.0159	0.0067	0.0135
			σ_0	0.0023	0.0224	0.0040	0.0036	0.0064	0.0037	0.0016	0.0035
			θ	0.0025	0.0084	0.0012	0.0013	0.0005	0.0013	0.0019	0.0013
130	0.35	53, 39	α	0.0865	0.0667	0.0375	0.0173	0.0409	0.0186	0.0341	0.0160
			σ_0	0.0118	0.0139	0.0047	0.0020	0.0032	0.0020	0.0061	0.0020
			θ	0.0015	0.0054	0.0019	0.0008	0.0015	0.0008	0.0024	0.0008
		68, 50	α	0.0263	0.0268	0.0082	0.0065	0.0109	0.0068	0.0055	0.0062
			σ_0	0.0003	0.0110	0.0011	0.0018	0.0023	0.0019	0.0001	0.0018
			θ	0.0015	0.0043	0.0001	0.0007	0.0003	0.0007	0.0005	0.0007
	0.8	53, 39	α	0.0757	0.0606	0.0329	0.0155	0.0364	0.0168	0.0295	0.0144
			σ_0	0.0108	0.0204	0.0029	0.0021	0.0014	0.0021	0.0044	0.0021
			θ	0.0001	0.0207	0.0011	0.0008	0.0007	0.0008	0.0016	0.0008
		68, 50	α	0.0225	0.0250	0.0068	0.0066	0.0095	0.0069	0.0041	0.0064
			σ_0	0.0020	0.0110	0.0018	0.0018	0.0030	0.0018	0.0005	0.0018
			θ	0.000946	0.0042	0.0004	0.0007	0.000002	0.00070	0.00073	0.00070

Table 3. RABias and MSE of the MLE, and BE based on SE and LINEX loss functions for MKEx distribution when $L = 4$, $\alpha = 2$, $\sigma_0 = 1.5$ and $\theta = 2$.

Point Estimation				$\alpha = 2; \sigma_0 = 1.5; \theta = 2$							
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100; \eta_3 = 130; \eta_4 = 170$							
				Sample 1: $n_1 = 6; n_2 = 4; n_3 = 3; n_4 = 2$ Sample 2: $n_1 = 15; n_2 = 10; n_3 = 8; n_4 = 5$							
$L = 4$				MLE		SE		LINEX (c = -0.5)		LINEX (c = 0.5)	
n	P	r_l		RABias	MSE	RABias	MSE	RABias	MSE	RABias	MSE
15	0.35	0.7	α	0.2247	0.6369	0.0476	0.1663	0.0869	0.2233	0.0102	0.1278
			σ_0	0.0633	0.1460	0.0405	0.0223	0.0293	0.0232	0.0514	0.0223
			θ	0.0022	0.0210	0.0176	0.0037	0.0160	0.0035	0.0191	0.0038
		0.9	α	0.1733	0.4048	0.0394	0.1077	0.0672	0.1366	0.0127	0.0876
			σ_0	0.0070	0.0973	0.0095	0.0176	0.0187	0.0205	0.0006	0.0155
			θ	0.0125	0.0132	0.0009	0.0016	0.0000	0.0017	0.0019	0.0016
	0.8	0.7	α	0.2771	0.7357	0.0886	0.2122	0.1317	0.2934	0.0472	0.1527
			σ_0	0.0062	0.1505	0.0220	0.0179	0.0107	0.0194	0.0329	0.0173
			θ	0.0157	0.0193	0.0198	0.0038	0.0184	0.0036	0.0211	0.0040
		0.9	α	0.1673	0.3860	0.0353	0.0941	0.0624	0.1184	0.0093	0.0779
			σ_0	0.0022	0.0852	0.0114	0.0158	0.0207	0.0182	0.0024	0.0141
			θ	0.0087	0.0112	0.0015	0.0016	0.0006	0.0016	0.0025	0.0016
38	0.35	0.7	α	0.1408	0.2127	0.0461	0.0511	0.0595	0.0604	0.0330	0.0437
			σ_0	0.0429	0.0434	0.0281	0.0077	0.0241	0.0075	0.0321	0.0080
			θ	0.0021	0.0054	0.0081	0.0010	0.0076	0.0010	0.0085	0.0011
		0.9	α	0.0822	0.1173	0.0241	0.0313	0.0345	0.0361	0.0140	0.0276
			σ_0	0.0150	0.0329	0.0027	0.0054	0.0062	0.0056	0.0006	0.0052
			θ	0.0069	0.0046	0.0058	0.0009	0.0055	0.0008	0.0062	0.0009
	0.8	0.7	α	0.1618	0.2389	0.0595	0.0629	0.0727	0.0741	0.0466	0.0537
			σ_0	0.0293	0.0403	0.0142	0.0063	0.0103	0.0063	0.0181	0.0063
			θ	0.0022	0.0052	0.0044	0.0008	0.0040	0.0008	0.0049	0.0008
		0.9	α	0.0815	0.1145	0.0249	0.0316	0.0351	0.0359	0.0150	0.0283
			σ_0	0.0104	0.0308	0.0054	0.0055	0.0088	0.0058	0.0020	0.0052
			θ	0.0033	0.0041	0.0042	0.0007	0.0039	0.0007	0.0046	0.0008

Table 3 (continued).

Point Estimation				$\alpha = 2; \sigma_0 = 1.5; \theta = 2$							
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100; \eta_3 = 130; \eta_4 = 170$							
				Sample 1: $n_1 = 30; n_2 = 20; n_3 = 10; n_4 = 7$ Sample 2: $n_1 = 60; n_2 = 40; n_3 = 20; n_4 = 10$							
$L = 4$				MLE		SE		LINEX (c = -0.5)		LINEX (c = 0.5)	
n	P	r		RABias	MSE	RABias	MSE	RABias	MSE	RABias	MSE
67	0.35	0.7	α	0.1147	0.1226	0.0468	0.0329	0.0540	0.0369	0.0398	0.0293
			σ_0	0.0019	0.0216	0.0102	0.0035	0.0081	0.0035	0.0124	0.0035
			θ	0.0125	0.0043	0.0099	0.0010	0.0096	0.0010	0.0102	0.0010
		0.9	α	0.0500	0.0562	0.0176	0.0163	0.0231	0.0178	0.0121	0.0152
			σ_0	0.0055	0.0164	0.0014	0.0029	0.0032	0.0029	0.0004	0.0028
			θ	0.0031	0.0028	0.0032	0.0005	0.0030	0.0005	0.0035	0.0005
	0.8	0.7	α	0.1203	0.1206	0.0519	0.0347	0.0591	0.0390	0.0449	0.0308
			σ_0	0.0102	0.0200	0.0057	0.0036	0.0036	0.0036	0.0078	0.0036
			θ	0.0035	0.0031	0.0052	0.0006	0.0050	0.0006	0.0055	0.0006
		0.9	α	0.0456	0.0526	0.0173	0.0167	0.0226	0.0182	0.0120	0.0156
			σ_0	0.0038	0.0168	0.0031	0.0029	0.0049	0.0030	0.0013	0.0029
			θ	0.0006	0.0028	0.0021	0.0005	0.0018	0.0005	0.0023	0.0005
130	0.35	0.7	α	0.0694	0.0538	0.0288	0.0132	0.0322	0.0143	0.0253	0.0122
			σ_0	0.0040	0.0102	0.0058	0.0017	0.0047	0.0017	0.0069	0.0018
			θ	0.0074	0.0020	0.0058	0.0004	0.0056	0.0004	0.0059	0.0004
		0.9	α	0.0323	0.0310	0.0107	0.0082	0.0133	0.0086	0.0080	0.0079
			σ_0	0.0025	0.0084	0.0023	0.0015	0.0032	0.0015	0.0014	0.0015
			θ	0.0012	0.0015	0.0016	0.0003	0.0014	0.0003	0.0017	0.0003
	0.8	0.7	α	0.0577	0.0472	0.0227	0.0115	0.0261	0.0124	0.0193	0.0107
			σ_0	0.0051	0.0097	0.0029	0.0017	0.0018	0.0017	0.0040	0.0017
			θ	0.0029	0.0017	0.0032	0.0003	0.0030	0.0003	0.0033	0.0003
		0.9	α	0.0248	0.0273	0.0065	0.0073	0.0091	0.0076	0.0039	0.0071
			σ_0	0.0030	0.0075	0.0001	0.0014	0.0011	0.0014	0.0008	0.0014
			θ	0.0010	0.0013	0.0008	0.0002	0.0006	0.0002	0.0009	0.0002

Table 4. RABias and MSE of the MLE, and BEs based on SE and LINEX loss functions for MKEx distribution when $L = 2$, $\alpha = 0.8$, $\sigma_0 = 1.5$ and $\theta = 2$.

Point Estimation				$\alpha = 0.8; \sigma_0 = 1.5; \theta = 2$							
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100$							
				Sample 1: $n_1 = 8; n_2 = 7$ Sample 2: $n_1 = 20; n_2 = 18$							
$L = 2$				MLE		SE		LINEX (c = -0.5)		LINEX (c = 0.5)	
n	P	m_1, m_2		RABias	MSE	RABias	MSE	RABias	MSE	RABias	MSE
15	0.35	6, 5	α	0.1278	0.1722	0.0313	0.0311	0.0387	0.0362	0.0242	0.0269
			σ_0	0.1002	1.1995	0.0302	0.1361	0.0857	0.2044	0.0202	0.0986
			θ	0.0229	0.3389	0.0380	0.0347	0.0217	0.0348	0.0536	0.0374
		7, 6	α	0.0952	0.1180	0.0229	0.0240	0.0289	0.0274	0.0170	0.0211
			σ_0	0.1602	1.0521	0.0777	0.1758	0.1336	0.2682	0.0266	0.1166
			θ	0.0231	0.2521	0.0103	0.0350	0.0040	0.0395	0.0240	0.0333
	0.8	6, 5	α	0.1497	0.2026	0.0430	0.0366	0.0504	0.0422	0.0359	0.0318
			σ_0	0.1600	1.3241	0.0769	0.2322	0.1356	0.3572	0.0224	0.1451
			θ	0.0404	0.3357	0.0116	0.0420	0.0043	0.0478	0.0269	0.0392
		7, 6	α	0.1110	0.1340	0.0322	0.0280	0.0384	0.0319	0.0262	0.0247
			σ_0	0.1252	0.8346	0.0749	0.1401	0.1266	0.2076	0.0275	0.0955
			θ	0.0382	0.2447	0.0016	0.0310	0.0124	0.0357	0.0150	0.0285
38	0.35	14, 13	α	0.0680	0.0470	0.02411	0.01026	0.02655	0.01108	0.02171	0.00952
			σ_0	0.0118	0.3393	0.01797	0.05235	0.04424	0.06923	0.00659	0.04177
			θ	0.0104	0.1031	0.01428	0.01387	0.00764	0.01404	0.02081	0.01411
		18, 16	α	0.0324	0.0255	0.00806	0.00568	0.00997	0.00600	0.00616	0.00541
			σ_0	0.0458	0.2617	0.03866	0.04876	0.06114	0.06205	0.01733	0.03929
			θ	0.0080	0.1082	0.00217	0.01202	0.00350	0.01261	0.00776	0.01172
	0.8	14, 13	α	0.0630	0.0471	0.02304	0.01048	0.02544	0.01126	0.02067	0.00976
			σ_0	0.0186	0.2695	0.02483	0.04619	0.05040	0.06075	0.00079	0.03693
			θ	0.0165	0.0948	0.00611	0.01373	0.00058	0.01440	0.01267	0.01349
		18, 16	α	0.0326	0.0252	0.00941	0.00581	0.01134	0.00615	0.00750	0.00551
			σ_0	0.0667	0.2473	0.05112	0.05710	0.07450	0.07329	0.02895	0.04524
			θ	0.0065	0.0667	0.00305	0.01101	0.00251	0.01145	0.00853	0.01085

Table 4 (continued).

Point Estimation				$\alpha = 0.8; \sigma_0 = 1.5; \theta = 2$							
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100$							
				Sample 1: $n_1 = 40; n_2 = 27$ Sample 2: $n_1 = 75; n_2 = 55$							
$L = 2$				MLE		SE		LINEX (c = -0.5)		LINEX (c = 0.5)	
n	P	m_1, m_2		RABias	MSE	RABias	MSE	RABias	MSE	RABias	MSE
67	0.35	28, 19	α	0.1302	0.0284	0.0355	0.0067	0.0444	0.0070	0.0084	0.0064
			σ_0	0.0171	0.1258	0.0839	0.0219	0.0559	0.0255	0.0146	0.0194
			θ	0.0103	0.0520	0.0260	0.0079	0.0130	0.0080	0.0040	0.0080
		36, 24	α	0.0547	0.0118	0.0148	0.0027	0.0185	0.0028	0.0033	0.0026
			σ_0	0.0154	0.1067	0.0712	0.0193	0.0474	0.0225	0.0128	0.0169
			θ	0.0039	0.0515	0.0257	0.0062	0.0129	0.0063	0.0031	0.0062
	0.8	28, 19	α	0.1213	0.0287	0.0359	0.0071	0.0448	0.0075	0.0089	0.0068
			σ_0	0.0061	0.1314	0.0876	0.0240	0.0584	0.0284	0.0160	0.0210
			θ	0.0134	0.0489	0.0245	0.0081	0.0122	0.0083	0.0040	0.0080
		36, 24	α	0.0433	0.0119	0.0149	0.0029	0.0186	0.0030	0.0036	0.0028
			σ_0	0.0320	0.1113	0.0742	0.0211	0.0495	0.0248	0.0141	0.0183
			θ	0.0038	0.0377	0.0188	0.0064	0.0094	0.0065	0.0032	0.0063
130	0.35	53, 39	α	0.0863	0.0129	0.01616	0.00313	0.02020	0.00323	0.00391	0.00303
			σ_0	0.0087	0.0630	0.04198	0.01045	0.02799	0.01139	0.00696	0.00979
			θ	0.0051	0.0247	0.01237	0.00398	0.00619	0.00401	0.00199	0.00399
		68, 50	α	0.0204	0.0054	0.00677	0.00129	0.00847	0.00131	0.00161	0.00127
			σ_0	0.0152	0.0564	0.03761	0.01023	0.02507	0.01120	0.00682	0.00947
			θ	0.0031	0.0188	0.00938	0.00321	0.00469	0.00324	0.00160	0.00319
	0.8	53, 39	α	0.0715	0.0115	0.01432	0.00277	0.01790	0.00285	0.00346	0.00269
			σ_0	0.0087	0.0630	0.04198	0.01045	0.02799	0.01139	0.00696	0.00979
			θ	0.0051	0.0247	0.01237	0.00398	0.00619	0.00401	0.00199	0.00399
		68, 50	α	0.0204	0.0054	0.00677	0.00129	0.00847	0.00131	0.00161	0.00127
			σ_0	0.0152	0.0564	0.03761	0.01023	0.02507	0.01120	0.00682	0.00947
			θ	0.0031	0.0188	0.00938	0.00321	0.00469	0.00324	0.00160	0.00319

Table 5. RABias and MSE of the MLE, and BEs based on SE and LINEX loss functions for MKEx distribution when $L = 4$, $\alpha = 0.8$, $\sigma_0 = 1.5$ and $\theta = 2$.

Point Estimation				$\alpha = 0.8; \sigma_0 = 1.5; \theta = 2$							
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100; \eta_3 = 130; \eta_4 = 170$							
				$n_1 = 6; n_2 = 4; n_3 = 3; n_4 = 2$ $n_1 = 15; n_2 = 10; n_3 = 8; n_4 = 5$							
$L = 4$				MLE		SE		LINEX (c = -0.5)		LINEX (c = 0.5)	
n	P	r_l		RABias	MSE	RABias	MSE	RABias	MSE	RABias	MSE
15	0.35	0.7	α	0.3488	0.2183	0.2729	0.0383	0.3411	0.0444	0.0479	0.0331
			σ_0	0.0573	0.8454	0.5636	0.1499	0.3757	0.2403	0.0999	0.1047
			θ	0.0200	0.1297	0.0648	0.0255	0.0324	0.0266	0.0127	0.0264
		0.9	α	0.1845	0.1066	0.1332	0.0206	0.1665	0.0230	0.0257	0.0186
			σ_0	0.0551	0.7860	0.5240	0.2074	0.3494	0.2965	0.1383	0.1458
			θ	0.0489	0.0956	0.0478	0.0128	0.0239	0.0141	0.0064	0.0120
	0.8	0.7	α	0.3929	0.2463	0.3078	0.0546	0.3848	0.0633	0.0682	0.0468
			σ_0	0.0694	0.8392	0.5594	0.1381	0.3730	0.2078	0.0921	0.1005
			θ	0.0204	0.1148	0.0574	0.0211	0.0287	0.0199	0.0106	0.0230
		0.9	α	0.1813	0.0960	0.1200	0.0188	0.1499	0.0210	0.0235	0.0171
			σ_0	0.1096	0.6549	0.4366	0.2021	0.2911	0.2910	0.1348	0.1406
			θ	0.0260	0.0649	0.0324	0.0110	0.0162	0.0116	0.0055	0.0106
38	0.35	0.7	α	0.1571	0.0498	0.0622	0.0109	0.0778	0.0117	0.0137	0.0102
			σ_0	0.0583	0.2391	0.1594	0.0372	0.1062	0.0441	0.0248	0.0341
			θ	0.0017	0.0306	0.0153	0.0062	0.0077	0.0059	0.0031	0.0065
		0.9	α	0.0703	0.0255	0.0319	0.0060	0.0399	0.0063	0.0075	0.0057
			σ_0	0.0695	0.1907	0.1272	0.0410	0.0848	0.0515	0.0274	0.0336
			θ	0.0150	0.0225	0.0112	0.0047	0.0056	0.0046	0.0024	0.0049
	0.8	0.7	α	0.1907	0.0575	0.0718	0.0129	0.0898	0.0139	0.0161	0.0119
			σ_0	0.0125	0.2257	0.1505	0.0497	0.1003	0.0641	0.0331	0.0405
			θ	0.0005	0.0242	0.0121	0.0046	0.0060	0.0045	0.0023	0.0047
		0.9	α	0.0831	0.0283	0.0354	0.0065	0.0442	0.0069	0.0082	0.0062
			σ_0	0.0602	0.1982	0.1321	0.0431	0.0881	0.0541	0.0287	0.0351
			θ	0.0056	0.0195	0.0098	0.0039	0.0049	0.0038	0.0019	0.0040

Table 5 (continued).

Point Estimation				$\alpha = 0.8; \sigma_0 = 1.5; \theta = 2$							
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100; \eta_3 = 130; \eta_4 = 170$							
				Sample 1: $n_1 = 30; n_2 = 20; n_3 = 10; n_4 = 7$ Sample 2: $n_1 = 60; n_2 = 40; n_3 = 20; n_4 = 10$							
$L = 4$				MLE		SE		LINEX (c = -0.5)		LINEX (c = 0.5)	
n	P	r_l		RABias	MSE	RABias	MSE	RABias	MSE	RABias	MSE
67	0.35	0.7	α	0.1201	0.0247	0.0309	0.0058	0.0386	0.0061	0.0072	0.0055
			σ_0	0.0023	0.1043	0.0696	0.0186	0.0464	0.0208	0.0124	0.0172
			θ	0.0235	0.0185	0.0092	0.0046	0.0046	0.0044	0.0023	0.0048
		0.9	α	0.0431	0.0132	0.0165	0.0031	0.0206	0.0032	0.0038	0.0030
			σ_0	0.0290	0.0829	0.0553	0.0171	0.0369	0.0195	0.0114	0.0153
			θ	0.0056	0.0109	0.0055	0.0023	0.0027	0.0023	0.0012	0.0023
	0.8	0.7	α	0.1213	0.0287	0.0359	0.0071	0.0448	0.0075	0.0089	0.0068
			σ_0	0.0072	0.1084	0.0723	0.0209	0.0482	0.0239	0.0139	0.0187
			θ	0.0074	0.0155	0.0077	0.0031	0.0039	0.0030	0.0015	0.0032
		0.9	α	0.0376	0.0133	0.01657	0.00319	0.02071	0.00328	0.00399	0.00311
			σ_0	0.0101	0.0790	0.05268	0.01694	0.03512	0.01915	0.01129	0.01528
			θ	0.0050	0.0124	0.00621	0.00243	0.00311	0.00244	0.00122	0.00244
130	0.35	0.7	α	0.0691	0.0101	0.01262	0.00238	0.01577	0.00246	0.00297	0.00230
			σ_0	0.0092	0.0548	0.03650	0.00955	0.02434	0.00999	0.00637	0.00927
			θ	0.0115	0.0101	0.00508	0.00219	0.00254	0.00214	0.00110	0.00225
		0.9	α	0.0171	0.0058	0.00724	0.00144	0.00906	0.00146	0.00180	0.00142
			σ_0	0.0089	0.0367	0.02445	0.00688	0.01630	0.00734	0.00459	0.00655
			θ	0.0005	0.0066	0.00330	0.00129	0.00165	0.00128	0.00064	0.00129
	0.8	0.7	α	0.0691	0.0101	0.01262	0.00238	0.01577	0.00246	0.00297	0.00230
			σ_0	0.0038	0.0526	0.03508	0.01014	0.02339	0.01090	0.00676	0.00957
			θ	0.0035	0.0081	0.00407	0.00160	0.00203	0.00157	0.00080	0.00163
		0.9	α	0.0201	0.0061	0.00765	0.00145	0.00956	0.00148	0.00182	0.00143
			σ_0	0.0121	0.0400	0.02668	0.00848	0.01779	0.00913	0.00565	0.00795
			θ	0.0008	0.0059	0.00296	0.001172	0.00148	0.001169	0.00059	0.00118

Table 6. Lengths of asymptotic, percentile bootstrap, bootstrap-t, and credible CIs when $L = 2$, $\alpha = 2$, $\sigma_0 = 1.5$ and $\theta = 2$.

Interval Estimation				$\alpha = 2; \sigma_0 = 1.5; \theta = 2$											
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100$											
				Sample 1: $n_1 = 8; n_2 = 7$ Sample 2: $n_1 = 20; n_2 = 18$											
$L = 2$				MLE			SE			LINEX (c = -0.5)			LINEX (c = 0.5)		
n	P	m_1, m_2		L.CI	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT
15	0.35	6, 5	α	2.5173	0.0792	0.0794	1.6412	0.0540	0.0533	1.8278	0.0579	0.0579	1.4676	0.0468	0.0460
			σ_0	1.8502	0.0549	0.0562	0.5315	0.0166	0.0162	0.5621	0.0171	0.0173	0.5039	0.0168	0.0169
			θ	1.5327	0.0489	0.0476	0.3094	0.0101	0.0101	0.3147	0.0099	0.0100	0.3046	0.0098	0.0099
		7, 6	α	2.2411	0.0751	0.0750	1.3207	0.0426	0.0425	1.4486	0.0494	0.0494	1.2044	0.0366	0.0366
			σ_0	1.5719	0.0502	0.0501	0.5307	0.0169	0.0169	0.5640	0.0181	0.0181	0.5015	0.0148	0.0150
			θ	1.2890	0.0411	0.0403	0.2852	0.0092	0.0092	0.2891	0.0090	0.0090	0.2815	0.0090	0.0091
	0.8	6, 5	α	2.5208	0.0825	0.0821	1.5962	0.0499	0.0505	1.7706	0.0570	0.0577	1.4355	0.0462	0.0466
			σ_0	1.6294	0.0496	0.0501	0.5060	0.0159	0.0160	0.5397	0.0169	0.0166	0.4767	0.0154	0.0152
			θ	1.3349	0.0424	0.0426	0.2861	0.0091	0.0091	0.2921	0.0090	0.0090	0.2806	0.0085	0.0085
		7, 6	α	2.2818	0.0770	0.0764	1.4007	0.0443	0.0449	1.5490	0.0496	0.0499	1.2663	0.0409	0.0403
			σ_0	1.4738	0.0477	0.0474	0.5274	0.0171	0.0168	0.5631	0.0184	0.0182	0.4957	0.0158	0.0156
			θ	1.1484	0.0375	0.0378	0.2665	0.0088	0.0087	0.2698	0.0091	0.0091	0.2636	0.0082	0.0083
38	0.35	14, 13	α	1.5632	0.0486	0.0503	0.8867	0.0261	0.0266	0.9284	0.0305	0.0303	0.8473	0.0272	0.0272
			σ_0	0.9079	0.0296	0.0297	0.3354	0.0104	0.0103	0.3440	0.0114	0.0111	0.3274	0.0104	0.0105
			θ	0.5389	0.0175	0.0171	0.1882	0.0057	0.0057	0.1894	0.0059	0.0059	0.1870	0.0060	0.0061
		18, 16	α	1.1977	0.0366	0.0363	0.6423	0.0205	0.0203	0.6669	0.0215	0.0210	0.6193	0.0185	0.0189
			σ_0	0.8076	0.0255	0.0252	0.3237	0.0099	0.0099	0.3313	0.0102	0.0102	0.3165	0.0107	0.0107
			θ	0.4776	0.0148	0.0149	0.1811	0.0054	0.0054	0.1820	0.0057	0.0058	0.1802	0.0059	0.0058
	0.8	14, 13	α	1.5640	0.0509	0.0509	0.8682	0.0270	0.0271	0.9099	0.0296	0.0294	0.8298	0.0256	0.0253
			σ_0	0.9123	0.0290	0.0291	0.3388	0.0107	0.0107	0.3478	0.0115	0.0112	0.3302	0.0107	0.0104
			θ	0.5409	0.0170	0.0171	0.1927	0.0061	0.0061	0.1940	0.0060	0.0060	0.1914	0.0062	0.0059
		18, 16	α	1.2636	0.0382	0.0381	0.6862	0.0219	0.0222	0.7119	0.0228	0.0226	0.6617	0.0217	0.0220
			σ_0	0.8028	0.0255	0.0252	0.3201	0.0093	0.0095	0.3272	0.0099	0.0099	0.3132	0.0103	0.0102
			θ	0.4674	0.0148	0.0148	0.1790	0.0059	0.0058	0.1800	0.0056	0.0056	0.1780	0.0057	0.0057

Table 6 (Continued).

Interval Estimation				$\alpha = 2; \sigma_0 = 1.5; \theta = 2$											
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100$											
				Sample 1: $n_1 = 40; n_2 = 27$ Sample 2: $n_1 = 75; n_2 = 55$											
$L = 2$				MLE			SE			LINEX (c = -0.5)			LINEX (c = 0.5)		
n	P	m_1, m_2		L.CI	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT
67	0.35	28, 19	α	1.0621	0.0344	0.0346	0.5881	0.0186	0.0187	0.6029	0.0187	0.0190	0.5738	0.0189	0.0191
			σ_0	0.6071	0.0195	0.0194	0.2317	0.0076	0.0076	0.2347	0.0076	0.0076	0.2288	0.0076	0.0076
			θ	0.3979	0.0128	0.0128	0.1485	0.0049	0.0049	0.1491	0.0048	0.0048	0.1478	0.0047	0.0046
		36, 24	α	0.9476	0.0303	0.0305	0.5033	0.0161	0.0161	0.5139	0.0160	0.0158	0.4932	0.0164	0.0163
			σ_0	0.5490	0.0167	0.0171	0.2231	0.0071	0.0071	0.2255	0.0068	0.0069	0.2207	0.0066	0.0067
			θ	0.3494	0.0112	0.0112	0.1383	0.0044	0.0044	0.1388	0.0046	0.0047	0.1377	0.0045	0.0044
	0.8	28, 19	α	0.9826	0.0307	0.0306	0.6060	0.0193	0.0195	0.6207	0.0196	0.0197	0.5920	0.0186	0.0186
			σ_0	0.6308	0.0199	0.0200	0.2462	0.0075	0.0076	0.2498	0.0077	0.0077	0.2428	0.0078	0.0078
			θ	0.4086	0.0125	0.0125	0.1595	0.0053	0.0052	0.1601	0.0049	0.0049	0.1588	0.0051	0.0052
		36, 24	α	0.9110	0.0296	0.0293	0.4628	0.0151	0.0151	0.4735	0.0145	0.0149	0.4528	0.0142	0.0142
			σ_0	0.5867	0.0188	0.0188	0.2343	0.0072	0.0070	0.2371	0.0073	0.0073	0.2316	0.0070	0.0070
			θ	0.3598	0.0109	0.0109	0.1403	0.0046	0.0046	0.1408	0.0045	0.0045	0.1399	0.0046	0.0046
130	0.35	53, 39	α	0.7519	0.0237	0.0238	0.4228	0.0131	0.0133	0.4279	0.0139	0.0140	0.4179	0.0125	0.0124
			σ_0	0.4571	0.0148	0.0149	0.1736	0.0056	0.0056	0.1748	0.0053	0.0053	0.1725	0.0054	0.0053
			θ	0.2877	0.0091	0.0091	0.1122	0.0036	0.0036	0.1124	0.0037	0.0037	0.1119	0.0037	0.0037
		68, 50	α	0.6085	0.0195	0.0191	0.3096	0.0099	0.0101	0.3130	0.0095	0.0095	0.3064	0.0099	0.0097
			σ_0	0.4108	0.0138	0.0137	0.1684	0.0054	0.0054	0.1692	0.0054	0.0054	0.1675	0.0053	0.0053
			θ	0.2573	0.0081	0.0082	0.1056	0.0033	0.0033	0.1058	0.0034	0.0034	0.1055	0.0033	0.0033
	0.8	53, 39	α	0.7616	0.0233	0.0233	0.4147	0.0130	0.0130	0.4201	0.0135	0.0134	0.4094	0.0131	0.0132
			σ_0	0.5570	0.0187	0.0180	0.1778	0.0057	0.0057	0.1791	0.0058	0.0057	0.1766	0.0058	0.0058
			θ	0.5645	0.0257	0.0179	0.1087	0.0032	0.0032	0.1090	0.0036	0.0035	0.1085	0.0034	0.0034
		68, 50	α	0.5948	0.0182	0.0184	0.3150	0.0101	0.0100	0.3182	0.0101	0.0102	0.3119	0.0098	0.0098
			σ_0	0.4111	0.0129	0.0128	0.1657	0.0052	0.0052	0.1666	0.0053	0.0053	0.1648	0.0053	0.0053
			θ	0.2555	0.0075	0.0079	0.1037	0.0032	0.0031	0.1038	0.0034	0.0034	0.1035	0.0032	0.0033

Table 7. Lengths of asymptotic, percentile bootstrap, bootstrap-t, and credible CIs when $L = 4$, $\alpha = 2$, $\sigma_0 = 1.5$ and $\theta = 2$.

Interval Estimation				$\alpha = 2; \sigma_0 = 1.5; \theta = 2$											
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100; \eta_3 = 130; \eta_4 = 170$											
				Sample 1: $n_1 = 6; n_2 = 4; n_3 = 3; n_4 = 2$ Sample 2: $n_1 = 15; n_2 = 10; n_3 = 8; n_4 = 5$											
$L = 4$				MLE			SE			LINEX (c = -0.5)			LINEX (c = 0.5)		
n	P	r_l		L.CI	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT
15	0.35	0.7	α	2.5862	0.0809	0.0801	1.5552	0.0501	0.0494	1.7233	0.0533	0.0536	1.3997	0.0432	0.0426
			σ_0	1.4515	0.0458	0.0460	0.5344	0.0168	0.0167	0.5724	0.0177	0.0184	0.5018	0.0153	0.0153
			θ	0.5679	0.0175	0.0179	0.1931	0.0062	0.0062	0.1947	0.0061	0.0064	0.1917	0.0058	0.0059
		0.9	α	2.0923	0.0705	0.0705	1.2496	0.0381	0.0381	1.3506	0.0435	0.0432	1.1565	0.0380	0.0380
			σ_0	1.2229	0.0391	0.0390	0.5166	0.0171	0.0170	0.5501	0.0181	0.0180	0.4877	0.0163	0.0165
			θ	0.4396	0.0143	0.0142	0.1590	0.0050	0.0050	0.1597	0.0050	0.0051	0.1583	0.0051	0.0051
	0.8	0.7	α	2.5672	0.0778	0.0778	1.6679	0.0556	0.0520	1.8563	0.0611	0.0601	1.4872	0.0472	0.0472
			σ_0	1.5210	0.0494	0.0493	0.5079	0.0165	0.0164	0.5422	0.0177	0.0178	0.4783	0.0160	0.0157
			θ	0.5306	0.0163	0.0163	0.1864	0.0056	0.0058	0.1871	0.0060	0.0059	0.1858	0.0058	0.0058
		0.9	α	2.0533	0.0633	0.0644	1.1707	0.0377	0.0388	1.2574	0.0399	0.0399	1.0920	0.0341	0.0340
			σ_0	1.1444	0.0345	0.0348	0.4890	0.0156	0.0155	0.5147	0.0168	0.0167	0.4657	0.0145	0.0146
			θ	0.4099	0.0133	0.0133	0.1583	0.0051	0.0050	0.1590	0.0048	0.0048	0.1576	0.0050	0.0050
38	0.35	0.7	α	1.4320	0.0431	0.0420	0.8095	0.0269	0.0269	0.8432	0.0266	0.0266	0.7780	0.0240	0.0241
			σ_0	0.7776	0.0252	0.0255	0.3016	0.0092	0.0092	0.3085	0.0096	0.0097	0.2952	0.0096	0.0097
			θ	0.2872	0.0093	0.0091	0.1091	0.0034	0.0035	0.1092	0.0035	0.0035	0.1089	0.0034	0.0034
		0.9	α	1.1780	0.0373	0.0374	0.6670	0.0219	0.0216	0.6947	0.0216	0.0216	0.6421	0.0205	0.0205
			σ_0	0.7060	0.0223	0.0224	0.2875	0.0090	0.0091	0.2923	0.0090	0.0091	0.2829	0.0092	0.0092
			θ	0.2618	0.0083	0.0083	0.1050	0.0032	0.0033	0.1052	0.0035	0.0035	0.1047	0.0034	0.0034
	0.8	0.7	α	1.4367	0.0458	0.0458	0.8657	0.0266	0.0264	0.9022	0.0284	0.0286	0.8319	0.0275	0.0272
			σ_0	0.7688	0.0230	0.0229	0.2991	0.0097	0.0096	0.3060	0.0098	0.0098	0.2927	0.0091	0.0091
			θ	0.2818	0.0088	0.0088	0.1076	0.0032	0.0033	0.1078	0.0033	0.0033	0.1074	0.0033	0.0034
		0.9	α	1.1629	0.0371	0.0366	0.6690	0.0225	0.0226	0.6903	0.0220	0.0222	0.6491	0.0211	0.0210
			σ_0	0.6855	0.0210	0.0203	0.2883	0.0094	0.0095	0.2934	0.0087	0.0087	0.2833	0.0083	0.0084
			θ	0.2500	0.0082	0.0082	0.1017	0.0032	0.0032	0.1019	0.0032	0.0032	0.1014	0.0033	0.0033

Table 7 (continued).

Interval Estimation				$\alpha = 2; \sigma_0 = 1.5; \theta = 2$											
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100; \eta_3 = 130; \eta_4 = 170$											
				Sample 1: $n_1 = 30; n_2 = 20; n_3 = 10; n_4 = 7$ Sample 2: $n_1 = 60; n_2 = 40; n_3 = 20; n_4 = 10$											
$L = 4$				MLE			SE			LINEX (c = -0.5)			LINEX (c = 0.5)		
n	P	r_l		L.CI	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT
67	0.35	0.7	α	1.0373	0.0340	0.0342	0.6089	0.0194	0.0193	0.6237	0.0198	0.0200	0.5948	0.0183	0.0185
			σ_0	0.5766	0.0184	0.0184	0.2228	0.0069	0.0070	0.2257	0.0072	0.0072	0.2200	0.0069	0.0068
			θ	0.2380	0.0077	0.0076	0.0960	0.0030	0.0030	0.0961	0.0031	0.0030	0.0959	0.0029	0.0029
		0.9	α	0.8428	0.0273	0.0271	0.4817	0.0149	0.0147	0.4906	0.0152	0.0152	0.4734	0.0149	0.0149
			σ_0	0.5018	0.0159	0.0159	0.2097	0.0066	0.0066	0.2115	0.0065	0.0065	0.2079	0.0069	0.0068
			θ	0.2049	0.0062	0.0061	0.0865	0.0026	0.0026	0.0867	0.0027	0.0028	0.0864	0.0028	0.0027
	0.8	0.7	α	0.9826	0.0307	0.0306	0.6060	0.0193	0.0195	0.6207	0.0196	0.0197	0.5920	0.0186	0.0186
			σ_0	0.5515	0.0164	0.0164	0.2331	0.0072	0.0074	0.2356	0.0073	0.0073	0.2306	0.0072	0.0072
			θ	0.2166	0.0068	0.0066	0.0897	0.0029	0.0028	0.0899	0.0029	0.0029	0.0896	0.0029	0.0029
		0.9	α	0.8255	0.0264	0.0265	0.4888	0.0156	0.0157	0.4980	0.0146	0.0148	0.4800	0.0161	0.0161
			σ_0	0.5074	0.0157	0.0156	0.2118	0.0065	0.0065	0.2140	0.0068	0.0069	0.2098	0.0066	0.0066
			θ	0.2089	0.0063	0.0065	0.0893	0.0029	0.0028	0.0893	0.0026	0.0026	0.0892	0.0028	0.0028
130	0.35	0.7	α	0.7290	0.0230	0.0233	0.3905	0.0119	0.0119	0.3953	0.0122	0.0121	0.3859	0.0124	0.0124
			σ_0	0.3963	0.0122	0.0122	0.1604	0.0050	0.0050	0.1613	0.0052	0.0052	0.1595	0.0052	0.0052
			θ	0.1669	0.0052	0.0052	0.0691	0.0021	0.0021	0.0692	0.0022	0.0022	0.0690	0.0021	0.0020
		0.9	α	0.6425	0.0222	0.0223	0.3462	0.0111	0.0110	0.3493	0.0114	0.0113	0.3432	0.0112	0.0112
			σ_0	0.3590	0.0108	0.0110	0.1521	0.0047	0.0047	0.1529	0.0050	0.0050	0.1514	0.0047	0.0047
			θ	0.1518	0.0048	0.0048	0.0659	0.0020	0.0020	0.0660	0.0021	0.0021	0.0658	0.0021	0.0021
	0.8	0.7	α	0.7224	0.0225	0.0224	0.3809	0.0124	0.0125	0.3859	0.0125	0.0121	0.3762	0.0120	0.0119
			σ_0	0.3848	0.0116	0.0117	0.1616	0.0055	0.0055	0.1625	0.0051	0.0050	0.1607	0.0049	0.0049
			θ	0.1619	0.0049	0.0050	0.0670	0.0021	0.0021	0.0671	0.0021	0.0021	0.0670	0.0022	0.0022
		0.9	α	0.6182	0.0197	0.0196	0.3321	0.0104	0.0103	0.3352	0.0107	0.0107	0.3291	0.0106	0.0104
			σ_0	0.3396	0.0110	0.0110	0.1467	0.0049	0.0049	0.1474	0.0048	0.0048	0.1461	0.0047	0.0047
			θ	0.1395	0.0043	0.0043	0.0591	0.0019	0.0019	0.0591	0.0018	0.0018	0.0590	0.0020	0.0020

Table 8. Lengths of asymptotic, percentile bootstrap, bootstrap-t, and credible CIs when $L = 2$, $\alpha = 0.8$, $\sigma_0 = 1.5$ and $\theta = 2$.

Interval Estimation				$\alpha = 0.8; \sigma_0 = 1.5; \theta = 2$											
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100$											
				Sample 1: $n_1 = 8; n_2 = 7$ Sample 2: $n_1 = 20; n_2 = 18$											
$L = 2$				MLE			SE			LINEX (c = -0.5)			LINEX (c = 0.5)		
n	P	m_1, m_2		L.CI	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT
15	0.35	6, 5	α	1.2818	0.0391	0.0391	0.6468	0.0209	0.0209	0.6816	0.0210	0.0211	0.6141	0.0196	0.0201
			σ_0	4.2547	0.1357	0.1365	1.4359	0.0459	0.0460	1.7001	0.0531	0.0526	1.2259	0.0384	0.0384
			θ	2.2759	0.0724	0.0718	0.6676	0.0219	0.0213	0.7113	0.0236	0.0234	0.6315	0.0192	0.0194
		7, 6	α	1.1211	0.0357	0.0355	0.5804	0.0195	0.0194	0.6082	0.0198	0.0198	0.5540	0.0176	0.0176
			σ_0	3.9110	0.1198	0.1198	1.5795	0.0522	0.0520	1.8729	0.0632	0.0624	1.3303	0.0424	0.0424
			θ	1.9608	0.0641	0.0653	0.7292	0.0223	0.0223	0.7784	0.0244	0.0250	0.6902	0.0212	0.0213
	0.8	6, 5	α	1.3181	0.0432	0.0432	0.6705	0.0223	0.0223	0.7018	0.0225	0.0228	0.6407	0.0204	0.0205
			σ_0	4.4137	0.1384	0.1384	1.8351	0.0564	0.0572	2.2040	0.0723	0.0709	1.4882	0.0475	0.0476
			θ	2.2501	0.0726	0.0707	0.7987	0.0264	0.0260	0.8565	0.0285	0.0286	0.7475	0.0233	0.0232
		7, 6	α	1.1413	0.0364	0.0365	0.6061	0.0191	0.0196	0.6327	0.0204	0.0208	0.5808	0.0192	0.0186
			σ_0	3.5064	0.1147	0.1141	1.4000	0.0431	0.0429	1.6242	0.0530	0.0532	1.2009	0.0390	0.0388
			θ	1.9169	0.0582	0.0584	0.6901	0.0209	0.0208	0.7345	0.0227	0.0224	0.6516	0.0196	0.0197
38	0.35	14, 13	α	0.6620	0.0198	0.0199	0.3494	0.0113	0.0114	0.3564	0.0111	0.0109	0.3427	0.0110	0.0110
			σ_0	2.2835	0.0764	0.0752	0.8911	0.0295	0.0295	0.9986	0.0309	0.0308	0.8006	0.0250	0.0246
			θ	1.2565	0.0391	0.0396	0.4482	0.0137	0.0138	0.4608	0.0142	0.0142	0.4364	0.0140	0.0137
		18, 16	α	0.5722	0.0189	0.0189	0.2889	0.0095	0.0094	0.2935	0.0098	0.0098	0.2843	0.0084	0.0084
			σ_0	1.9883	0.0642	0.0635	0.8356	0.0272	0.0271	0.9083	0.0278	0.0274	0.7707	0.0249	0.0249
			θ	1.2885	0.0430	0.0437	0.4297	0.0132	0.0132	0.4396	0.0139	0.0141	0.4201	0.0133	0.0135
	0.8	14, 13	α	0.6925	0.0224	0.0228	0.3585	0.0113	0.0113	0.3652	0.0116	0.0114	0.3519	0.0120	0.0122
			σ_0	2.0331	0.0682	0.0681	0.8301	0.0263	0.0264	0.9201	0.0286	0.0291	0.7537	0.0241	0.0239
			θ	1.2007	0.0377	0.0377	0.4571	0.0141	0.0141	0.4706	0.0145	0.0144	0.4445	0.0137	0.0137
		18, 16	α	0.5670	0.0178	0.0188	0.2896	0.0089	0.0088	0.2943	0.0094	0.0093	0.2850	0.0088	0.0087
			σ_0	1.9103	0.0604	0.0605	0.8876	0.0285	0.0287	0.9671	0.0309	0.0309	0.8167	0.0259	0.0260
			θ	1.0117	0.0330	0.0330	0.4109	0.0135	0.0136	0.4192	0.0139	0.0139	0.4031	0.0124	0.0124

Table 8 (Continued).

Interval Estimation				$\alpha = 0.8; \sigma_0 = 1.5; \theta = 2$											
				$\eta_0 = 40; \eta_1 = 60; \eta_2 = 100$											
				Sample 1: $n_1 = 40; n_2 = 27$ Sample 2: $n_1 = 75; n_2 = 55$											
$L = 2$				MLE			SE			LINEX (c = -0.5)			LINEX (c = 0.5)		
n	P	m_1, m_2		L.CI	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT
67	0.35	28, 19	α	0.5196	0.0173	0.0173	0.2747	0.0088	0.0089	0.2776	0.0090	0.0089	0.2719	0.0087	0.0087
			σ_0	1.3876	0.0442	0.0445	0.5786	0.0184	0.0183	0.6153	0.0200	0.0200	0.5452	0.0163	0.0169
			θ	0.8908	0.0287	0.0288	0.3431	0.0105	0.0104	0.3486	0.0107	0.0105	0.3378	0.0107	0.0108
		36, 24	α	0.3905	0.0124	0.0122	0.1949	0.0062	0.0062	0.1967	0.0063	0.0063	0.1932	0.0060	0.0060
			σ_0	1.2781	0.0396	0.0396	0.5295	0.0165	0.0164	0.5556	0.0183	0.0181	0.5058	0.0158	0.0158
			θ	0.8892	0.0284	0.0282	0.3075	0.0102	0.0104	0.3117	0.0097	0.0097	0.3035	0.0089	0.0090
	0.8	28, 19	α	0.5444	0.0174	0.0174	0.2899	0.0095	0.0094	0.2929	0.0093	0.0092	0.2870	0.0091	0.0090
			σ_0	1.4211	0.0464	0.0462	0.6031	0.0191	0.0190	0.6427	0.0204	0.0203	0.5685	0.0183	0.0189
			θ	0.8611	0.0269	0.0268	0.3513	0.0113	0.0113	0.3573	0.0113	0.0113	0.3456	0.0109	0.0109
		36, 24	α	0.4054	0.0132	0.0133	0.2078	0.0067	0.0067	0.2098	0.0065	0.0065	0.2059	0.0069	0.0069
			σ_0	1.2950	0.0418	0.0420	0.5479	0.0178	0.0176	0.5745	0.0173	0.0174	0.5236	0.0166	0.0165
			θ	0.7605	0.0252	0.0245	0.3119	0.0100	0.0100	0.3164	0.0099	0.0098	0.3077	0.0096	0.0100
130	0.35	53, 39	α	0.3544	0.0108	0.0108	0.1860	0.0059	0.0059	0.1870	0.0061	0.0060	0.1850	0.0057	0.0057
			σ_0	0.9829	0.0330	0.0315	0.3996	0.0121	0.0122	0.4121	0.0130	0.0131	0.3879	0.0122	0.0120
			θ	0.6156	0.0199	0.0202	0.2460	0.0075	0.0075	0.2480	0.0077	0.0077	0.2441	0.0076	0.0076
		68, 50	α	0.2815	0.0093	0.0093	0.1399	0.0044	0.0044	0.1405	0.0045	0.0045	0.1393	0.0045	0.0046
			σ_0	0.9273	0.0293	0.0293	0.3891	0.0123	0.0121	0.3993	0.0125	0.0124	0.3794	0.0120	0.0120
			θ	0.5365	0.0173	0.0172	0.2220	0.0068	0.0068	0.2233	0.0073	0.0073	0.2208	0.0070	0.0070
	0.8	53, 39	α	0.3549	0.0105	0.0107	0.1838	0.0055	0.0056	0.1848	0.0058	0.0058	0.1829	0.0058	0.0060
			σ_0	0.9829	0.0330	0.0315	0.3996	0.0121	0.0122	0.4121	0.0130	0.0131	0.3879	0.0122	0.0120
			θ	0.6156	0.0199	0.0202	0.2460	0.0075	0.0075	0.2480	0.0077	0.0077	0.2441	0.0076	0.0076
		68, 50	α	0.2815	0.0093	0.0093	0.1399	0.0044	0.0044	0.1405	0.0045	0.0045	0.1393	0.0045	0.0046
			σ_0	0.9273	0.0293	0.0293	0.3891	0.0123	0.0121	0.3993	0.0125	0.0124	0.3794	0.0120	0.0120
			θ	0.5365	0.0173	0.0172	0.2220	0.0068	0.0068	0.2233	0.0073	0.0073	0.2208	0.0070	0.0070

Table 9. Lengths of asymptotic, percentile bootstrap, bootstrap-t, and credible CIs when $L = 4$, $\alpha = 0.8$, $\sigma_0 = 1.5$ and $\theta = 2$.

Interval Estimation				$\alpha = 0.8; \sigma 0 = 1.5; \theta = 2$ $\eta_0 = 40; \eta_1 = 60; \eta_2 = 100; \eta_3 = 130; \eta_4 = 170$ Sample 1: $n_1 = 6; n_2 = 4; n_3 = 3; n_4 = 2$ Sample 2: $n_1 = 15; n_2 = 10; n_3 = 8; n_4 = 5$											
$L = 4$				MLE			SE			LINEX (c = -0.5)			LINEX (c = 0.5)		
n	P	r_l		L.CI	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT
15	0.35	0.7	α	1.4698	0.0462	0.0467	0.7239	0.0236	0.0232	0.7627	0.0247	0.0248	0.6868	0.0213	0.0212
			σ_0	3.5902	0.1156	0.1146	1.7286	0.0530	0.0527	2.0040	0.0676	0.0656	1.4862	0.0479	0.0479
			θ	1.4036	0.0448	0.0448	0.5356	0.0169	0.0170	0.5837	0.0182	0.0184	0.5071	0.0164	0.0162
		0.9	α	1.1418	0.0350	0.0350	0.5520	0.0169	0.0167	0.5758	0.0191	0.0191	0.5299	0.0169	0.0169
			σ_0	3.4620	0.1133	0.1134	1.5179	0.0480	0.0484	1.9046	0.0640	0.0629	1.2278	0.0393	0.0400
			θ	1.1501	0.0355	0.0354	0.4442	0.0142	0.0142	0.4623	0.0147	0.0152	0.4285	0.0136	0.0138
	0.8	0.7	α	1.5060	0.0484	0.0474	0.8394	0.0267	0.0268	0.8866	0.0286	0.0288	0.7927	0.0252	0.0249
			σ_0	3.5695	0.0975	0.0977	1.6700	0.0531	0.0531	1.9328	0.0587	0.0589	1.4413	0.0475	0.0479
			θ	1.3192	0.0397	0.0399	0.4541	0.0141	0.0138	0.4718	0.0148	0.0150	0.4413	0.0142	0.0142
		0.9	α	1.0735	0.0357	0.0351	0.5226	0.0162	0.0164	0.5419	0.0177	0.0176	0.5044	0.0157	0.0153
			σ_0	3.1076	0.1160	0.1133	1.4556	0.0459	0.0456	1.7433	0.0528	0.0523	1.2233	0.0387	0.0381
			θ	0.9778	0.0307	0.0308	0.4108	0.0130	0.0130	0.4216	0.0126	0.0124	0.4015	0.0119	0.0120
38	0.35	0.7	α	0.7228	0.0236	0.0235	0.3757	0.0117	0.0119	0.3830	0.0123	0.0124	0.3687	0.0114	0.0116
			σ_0	1.8867	0.0622	0.0625	0.7685	0.0247	0.0246	0.8331	0.0279	0.0280	0.7126	0.0224	0.0223
			θ	0.6863	0.0216	0.0217	0.2607	0.0082	0.0084	0.2633	0.0083	0.0083	0.2585	0.0086	0.0084
		0.9	α	0.5868	0.0186	0.0188	0.2986	0.0093	0.0094	0.3037	0.0097	0.0096	0.2936	0.0099	0.0098
			σ_0	1.6633	0.0516	0.0516	0.7435	0.0240	0.0241	0.8232	0.0260	0.0263	0.6782	0.0204	0.0204
			θ	0.5759	0.0188	0.0183	0.2460	0.0076	0.0076	0.2485	0.0080	0.0080	0.2437	0.0081	0.0080
	0.8	0.7	α	0.7254	0.0236	0.0231	0.3822	0.0118	0.0119	0.3900	0.0123	0.0123	0.3746	0.0119	0.0120
			σ_0	1.8619	0.0601	0.0598	0.8725	0.0279	0.0280	0.9773	0.0310	0.0310	0.7863	0.0252	0.0252
			θ	0.6098	0.0212	0.0213	0.2452	0.0077	0.0077	0.2478	0.0079	0.0078	0.2428	0.0076	0.0075
		0.9	α	0.6061	0.0193	0.0193	0.3077	0.0094	0.0094	0.3124	0.0097	0.0097	0.3031	0.0095	0.0094
			σ_0	1.7097	0.0532	0.0537	0.7779	0.0248	0.0251	0.8414	0.0282	0.0284	0.7223	0.0225	0.0224
			θ	0.5460	0.0179	0.0179	0.2318	0.0073	0.0073	0.2338	0.0079	0.0079	0.2299	0.0071	0.0070

Table 9 (Continued).

Interval Estimation				$\alpha = 0.8; \sigma_0 = 1.5; \theta = 2$; $\eta_0 = 40; \eta_1 = 60; \eta_2 = 100; \eta_3 = 130; \eta_4 = 170$; Sample 1: $n_1 = 30; n_2 = 20; n_3 = 10; n_4 = 7$; Sample 2: $n_1 = 60; n_2 = 40; n_3 = 20; n_4 = 10$											
$L = 4$				MLE			SE			LINEX (c = -0.5)			LINEX (c = 0.5)		
n	P	r_l		L.CI	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT	L.Cr	L.BP	L.BT
67	0.35	0.7	α	0.4878	0.0151	0.0146	0.2564	0.0082	0.0080	0.2593	0.0085	0.0085	0.2535	0.0081	0.0080
			σ_0	1.2667	0.0387	0.0389	0.5344	0.0172	0.0172	0.5636	0.0185	0.0186	0.5079	0.0156	0.0160
			θ	0.5004	0.0160	0.0160	0.2055	0.0064	0.0064	0.2067	0.0066	0.0066	0.2044	0.0064	0.0064
		0.9	α	0.4294	0.0139	0.0139	0.2141	0.0066	0.0066	0.2159	0.0067	0.0067	0.2124	0.0072	0.0071
			σ_0	1.1164	0.0338	0.0343	0.5006	0.0148	0.0148	0.5214	0.0161	0.0161	0.4817	0.0151	0.0152
			θ	0.4073	0.0127	0.0127	0.1787	0.0059	0.0059	0.1797	0.0058	0.0058	0.1777	0.0059	0.0059
	0.8	0.7	α	0.5444	0.0174	0.0174	0.2899	0.0095	0.0094	0.2929	0.0093	0.0092	0.2870	0.0091	0.0090
			σ_0	1.2908	0.0426	0.0427	0.5627	0.0180	0.0180	0.5912	0.0187	0.0185	0.5363	0.0168	0.0166
			θ	0.4844	0.0157	0.0157	0.1971	0.0066	0.0066	0.1985	0.0064	0.0064	0.1958	0.0062	0.0062
		0.9	α	0.4359	0.0138	0.0136	0.2196	0.0068	0.0070	0.2215	0.0070	0.0070	0.2177	0.0068	0.0068
			σ_0	1.1008	0.0348	0.0348	0.5018	0.0162	0.0161	0.5222	0.0159	0.0159	0.4830	0.0155	0.0153
			θ	0.4354	0.0132	0.0132	0.1920	0.0061	0.0061	0.1931	0.0059	0.0059	0.1909	0.0059	0.0058
130	0.35	0.7	α	0.3291	0.0105	0.0106	0.1689	0.0054	0.0052	0.1697	0.0052	0.0053	0.1680	0.0055	0.0055
			σ_0	0.9161	0.0292	0.0295	0.3825	0.0121	0.0122	0.3920	0.0125	0.0125	0.3734	0.0119	0.0118
			θ	0.3847	0.0119	0.0119	0.1591	0.0052	0.0050	0.1598	0.0049	0.0049	0.1585	0.0049	0.0050
		0.9	α	0.2937	0.0093	0.0093	0.1483	0.0047	0.0046	0.1490	0.0045	0.0045	0.1476	0.0047	0.0048
			σ_0	0.7493	0.0242	0.0243	0.3227	0.0099	0.0099	0.3288	0.0108	0.0108	0.3170	0.0099	0.0098
			θ	0.3187	0.0110	0.0106	0.1387	0.0042	0.0042	0.1392	0.0045	0.0045	0.1382	0.0041	0.0041
	0.8	0.7	α	0.3291	0.0105	0.0106	0.1689	0.0054	0.0052	0.1697	0.0052	0.0053	0.1680	0.0055	0.0055
			σ_0	0.8994	0.0289	0.0283	0.3941	0.0125	0.0123	0.4051	0.0128	0.0127	0.3836	0.0124	0.0124
			θ	0.3526	0.0113	0.0113	0.1486	0.0051	0.0051	0.1493	0.0048	0.0048	0.1480	0.0044	0.0044
		0.9	α	0.3001	0.0098	0.0095	0.1483	0.0045	0.0045	0.1490	0.0048	0.0048	0.1477	0.0047	0.0047
			σ_0	0.7814	0.0230	0.0234	0.3542	0.0113	0.0112	0.3616	0.0115	0.0115	0.3471	0.0108	0.0107
			θ	0.3018	0.0101	0.0099	0.1330	0.0044	0.0044	0.1334	0.0043	0.0043	0.1327	0.0042	0.0041

6. The BEs under LINEX loss function with parameter $c = 0.5$ have MSE less than corresponding estimates with $c = -0.5$, for all parameters.
7. Lengths of credible CIs are always shorter than the corresponding lengths of asymptotic CIs.
8. The percentile bootstrap CI has the shortest length among all considered CIs.

7. Application of transformer insulation

In this section, a real data set is analyzed to illustrate the proposed methods in the previous sections. Furthermore, this data set is used to show that the MKEx distribution can be a possible alternative to widely known distributions such as exponential distribution, generalized exponential distribution and Weibull distribution.

Nelson [36] presented in Chapter three of his book the results of a constant-stress accelerated life test of a transformer insulation. The test consisted of three levels of constant voltage, which are respectively $35.4kv$, $42.4kv$ and $46.7kv$ with normal voltage is $14.4kv$. The results of such test are presented in Table 10. In this table, the sign "+" refers to the censored data.

Table 10. Failure times of transformers insulation.

Voltage	Failure times
$35.4kv$	0.1, 59.4, 71.2, 166.5, 204.7, 229.7, 308.3, 537.9, 1002.3+, 1002.3+
$42.4kv$	0.6, 13.4, 15.2, 19.9, 25.0, 30.2, 32.8, 44.4, 50.2+, 56.2
$46.7kv$	3.1, 8.3, 8.9, 9.0, 13.6, 14.9, 16.1, 16.9, 21.3, 48.1+

For each level of this test, we suggested using the following progressives CSs as follows:

1. For $\phi_1 = 35.4kv$: $n_1 = 10$, $m_1 = 8$ and $R_{1j} = 0$, $j = 1, ..., 7$, $R_{18} = 2$, with $P = 0.0277$.
2. For $\phi_2 = 42.4kv$: $n_2 = 10$, $m_2 = 9$ and $R_{2j} = 0$, $j = 1, ..., 7$, $R_{28} = 1$, $R_{29} = 0$, with $P = 0.0123$.
3. For $\phi_3 = 46.7kv$: $n_3 = 10$, $m_3 = 9$ and $R_{3j} = 0$, $j = 1, ..., 8$, $R_{39} = 1$, with $P = 0.0123$.

In the following subsection, we explain how to perform a goodness of fit test for the data in Table 10 and the proposed MKEx distribution.

7.1. Modified KS algorithm for fitting progressive censored data

When the data is PT-II censored data, we have to use modified KolmogorovSmirnov (KS) goodness of fit test. The modified Kolmogorov-Smirnov statistic for PT-II censored data was originally introduced by Pakyari and Balakrishnan [38]. This algorithm is based on several steps, first, find the estimates of the parameters for the proposed distribution and next transforming the data to normality, then testing the goodness of fit of the transformed data to normality. Let $\tau_{1:m:n} < \tau_{2:m:n} < ... < \tau_{m:m:n}$ be a PT-II censored sample with CS $(R_1, R_2, ..., R_m)$ from a distribution function $F(t; \theta)$, then the modified KS statistic for PT-II censored data is

$$D_{m:n} = \max\{D^+_{m:n},\ D^-_{m:n}\}, \tag{7.1}$$

where

$$D^+_{m:n} = \max_{i=1,2,...,m} \{v_{i:m:n} - u_{i:m:n}\},$$

and

$$D^-_{m:n} = \max_{i=1,2,...,m} \{u_{i:m:n} - \nu_{i-1:m:n}\},$$

where $\nu_{i:m:n} = E(U_{i:m:n})$ is the expected value of the ith PT-II censored order statistic from the $U(0,1)$ distribution, given by

$$\nu_{i:m:n} = 1 - \prod_{j=m-i+1}^{m} \left(\frac{j + R_{m-j+1} + ... + R_m}{j + 1 + R_{m-j+1} + ... + R_m} \right),$$

and $u_{i:m:n} = F(t_{i:m:n}; \hat{\theta})$ for $i = 1, 2, ..., m$.

The following algorithm was proposed by Pakyari and Balakrishnan [38] to apply the KS test for PT-II censored data.

1. Find the MLE of the parameter θ, and calculate $\alpha_{i:m:n} = F(t_{i:m:n}; \hat{\theta})$ for $i = 1, 2, ..., m$.
2. Evaluate $y_{i:m:n} = \Phi^{-1}(\alpha_{i:m:n})$ for $i = 1, 2, ..., m$.
3. Considering $y_{1:m:n}, y_{2:m:n}, ..., y_{m:m:n}$ as a PT-II censored data from a normal distribution with mean μ and standard deviation σ, calculate the MLEs $\hat{\mu}$ and $\hat{\sigma}$.
4. Evaluate $u_{i:m:n} = \Phi\{(y_{i:m:n} - \hat{\mu})/\hat{\sigma}\}$ for $i = 1, 2, ..., m$.
5. Evaluate $D_{m:n}$ according to (7.1).
6. Evaluate the P-value of the test, and reject the null hypothesis H_0 at significance level 0.05 if P-value less than 0.05.

Table 11 contains the MLEs and BEs using SE and LINEX loss functions of α, σ_0 and θ for the real data set.

Table 11. Different estimates of unknown parameters for the real data set.

MLE			SE			LINEX (c = -0.1)		
$\hat{\alpha}$	$\hat{\sigma_0}$	$\hat{\theta}$	$\hat{\alpha}_{SE}$	$\hat{\sigma}_{0SE}$	$\hat{\theta}_{SE}$	$\hat{\alpha}_{LINEX}$	$\hat{\sigma}_{0LINEX}$	$\hat{\theta}_{LINEX}$
0.699051	0.000151778	34.9999	1.15775	0.000298505	39.4009	1.15805	0.000298505	39.416

Based on the results of MLEs, the summary of results of the modified KS test for MKEx distribution are displayed in Table 12.

Table 12. Summary of the results of the modified KS test for MKEx distribution.

Distribution	Stress level	Test statistic	P-value
MKEx	ϕ_1	0.225534	0.248
MKEx	ϕ_2	0.217075	0.854
MKEx	ϕ_3	0.168044	0.500

Table 13 contains the value of the life characteristics σ_l, $l = 0, 1, 2, 3$ for each stress level, also it contains the mean time to failure (MTTF) for each stress level.

Figures 2–5 display the reliability function of transformers insulation under different stress levels.

Table 13. The values of life characteristic and MTTF for each stress level for the MKEx distribution.

Stress level	The life characteristics (σ_l)	MTTF
ϕ_0	0.00015	4106.75
ϕ_1	0.0053	117.336
ϕ_2	0.0108	57.5029
ϕ_3	0.0158	39.2529

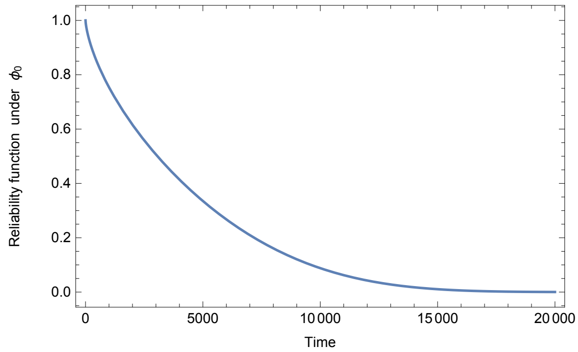

Figure 2. Reliability function under ϕ_0.

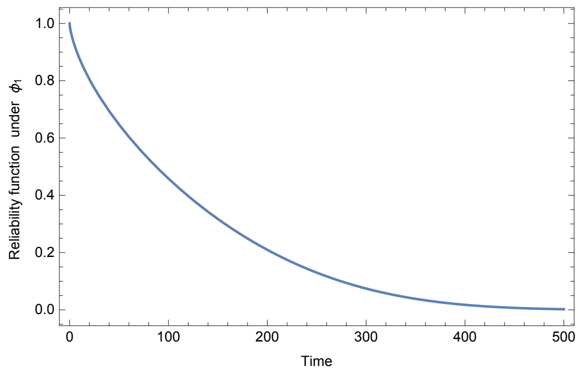

Figure 3. Reliability function under ϕ_1.

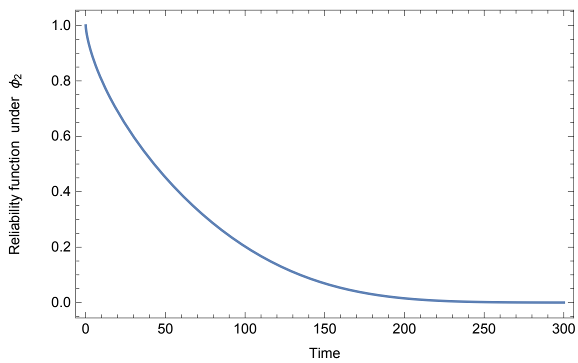

Figure 4. Reliability function under ϕ_2.

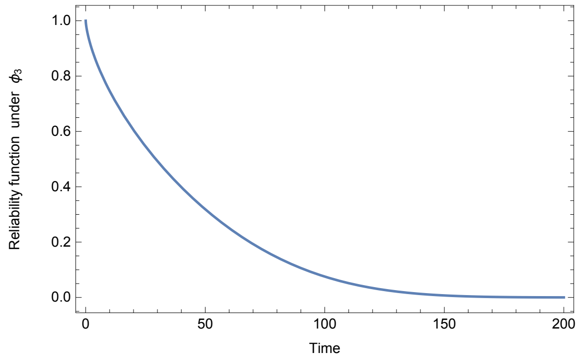

Figure 5. Reliability function under ϕ_3.

From Figures 2–5, we can note that with the increase in the stress value, the reliability function tends to zero faster.

The remainder of this section deals with the comparison of the proposed model, MKEx, generalized exponential (GE), Weibull, and exponential distributions. The comparison is performed using the real data shown in Table 10 and the PT-II CSs previously presented in this section. To clarify which of these distributions is more suitable for the data in Table 10, the parameters for the four distributions are estimated and the Akaike information criteria (AIC) is calculated for each distribution. The results of these calculations are summarized in Table 14. Furthermore, the results of modified KS test for the GE, Weibull, and exponential distributions are presented in Table 15.

Table 14. MLEs of the parameters of MKEx, GE, Weibull, and exponential distributions with AIC.

Distribution	AIC	Estimated parameters		
		$\hat{\alpha}$	$\hat{\sigma_0}$	$\hat{\theta}$
MKEx	293.330	0.699051	0.000151	34.9999
GE	294.138	0.34446	0.000171	17.0303
Weibull	298.298	1.17927	0.000690	10.0043
Exponential	299.563	-	0.004015	2.02216

From Tables 14 and 15, we can see that the MKEx distribution provides a better fit to the given data compared to exponential, GE, and Weibull regarding AIC.

8. Conclusions

This paper discussed the statistical inference of CSALT under PT-II censoring with binomial removal when the lifetimes of test units follow the MKEx distribution. In this context, we obtained the point and interval estimates for the unknown parameters using both classical and Bayesian methods. We concluded that the Bayesian method was better than the classical method according to MSE and relative absolute bias of the estimates. Regarding the interval estimates, we noted that the

Table 15. Summary of the results of the modified KS test for GE , Weibull, and exponential distributions.

Distribution	Stress level	Test statistic	P-value
GE	ϕ_1	0.1871	0.570
	ϕ_2	0.2224	0.525
	ϕ_3	0.1957	0.571
Weibull	ϕ_1	0.190144	0.749
	ϕ_2	0.193394	0.871
	ϕ_3	0.178979	0.462
Exponential	ϕ_1	0.187407	0.849
	ϕ_2	0.20754	0.95
	ϕ_3	0.188103	0.384

percentile bootstrap interval was the best one according to the shortness of the interval length. Furthermore, An application about the insulation of transformers was discussed and used to illustrate the theoretical results. Moreover, the data of insulation of transformers was used to show that the suggested model, MKEx, can be a possible alternative to some well known distributions.

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Conflict of interest

The authors declare no conflict of interest.

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