Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(5): 5176–5192.
DOI: 10.3934/math.2021307
Received: 13 November 2020
Accepted: 03 March 2021
Published: 10 March 2021

Research article

Fully nonlocal stochastic control problems with fractional Brownian motions and Poisson jumps

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Abstract: In this paper, we establish a representation of mild solutions to fully nonlocal stochastic evolution problems. Through the iterative technique and energy estimates, we obtain the existence and uniqueness of mild solution. Furthermore, we prove the existence of optimal control for fully nonlocal stochastic control problems with a non-convex cost function. Two examples are given at the end.

Keywords: fully nonlocal; fractional Brownian motion; Poisson jump; mild solution; optimal control
Mathematics Subject Classification: 60H15, 35A01, 47H06

1. Introduction

The theory of fractional derivatives is available for describing the tailing phenomena in time and nonlocal appearance in space. These characteristics ensure that fractional derivatives can be widely used in engineering, such as material memory, fluid dynamics and so on, see [1] and reference therein. A fractional Brownian motion (fBm) B^H is a centered Guassian process. Especially, it is neither a semi-martingale nor a Markov process when $H \neq 1/2$. Fractional Brownian motions are widely used in modelling of fractal phenomena and stock markets, see [2, 3]. Hence, it is significant to study the complex systems in $\mathbb{R}^d$ as follows

$$\begin{cases} {}_0^cD_t^\beta u(t,x) + (-\Delta)^{\frac{\alpha}{2}} u(t,x) = & f(t,x,u(t,x)) + \mathcal{A}v(x) + g(t,x)\frac{dB^H(t)}{dt} \\ & + \int_Y h(t,x,u(t,x);y)\tilde{\eta}(dy),\ t\in(0,\infty),\ x\in\mathbb{R}^d, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}^d, & \end{cases} \tag{1.1}$$

where parameters $\beta \in (1/2, 1)$, $\alpha \in (1, 2)$ and $H \in (1/2, 1)$. ${}_0^cD_t^\beta$ denotes the β-order Caputo fractional derivative and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian. f and h are nonlinear functions. B^H is a fBm with Hurst index H and $\tilde{\eta}$ is a Poisson martingale measure. Let V be a Hilbert space, $v \in V$ and $\mathcal{A} \in \mathcal{L}(V; L^2(\mathbb{R}^d))$. $\mathcal{A}v$ is the control term.

In problem (1.1), there are fractional derivatives in time and space, involving both fBm term and Poisson jump. There are two difficulties to study this type of problem. One is mathematical methods to deal with fBms and Poisson jumps driven time-space fractional stochastic systems. Another is L^p-estimates of the Green functions caused by the inconvenience of fractional calculus. There have been a series of researches on this topic:

1. The fully nonlocal deterministic problems.

$$\begin{cases} {}_0^cD_t^{\beta} u(t,x) + (-\Delta)^{\frac{\alpha}{2}} u(t,x) = f(t,x,u(t,x)),\ t \in (0,\infty),\ x \in \mathbb{R}^d, \\ u(0,x) = u_0,\ x \in \mathbb{R}^d. \end{cases} \tag{1.2}$$

   Kemppainen, Siljander and Zacher [4] dealt with the representation of solutions and the L^2-decay of mild solutions for (1.2) when right hand term is $f(t,x)$. Li, Liu and Wang [5] worked on the situation of $f(t,x,u) = \nabla \cdot (u(Bu))$ and obtained rich results.
2. The stochastic evolution problems with fBms and Poisson jumps, i.e., $\beta = 1$ and $\alpha = 2$.
   It is well known that the smaller H is, the harder the problem is, since the lack of regularity. Tindel, Tudor and Viens [6] discussed a class of linear stochastic evolution equations driven by fBms and obtained the well-posedness of solutions under Hurst index $H \in (1/2,1)$ and $H \in (0,1/2)$, respectively. Caraballo, Garrido-Atienza and Taniguchi [7] investigated the abstract stochastic evolution equations with fBms in Hilbert space and considered the existence of weak solutions when $H \in (1/2,1)$. Duc et al. [8] discussed the case of $H \in (1/2,1)$ as well and obtained the exponential stability. Tang and Meng [9] studied stochastic evolution equations with Poisson jumps and obtained the existence and uniqueness of solutions by the Gälerkin approximation under Lipschitz type conditions, in addition, they also considered the optimal control problem of the equations.
3. The fractional stochastic problems.
   Gu et al. [10] dealt with the problems of space-fractional stochastic reaction diffusion equations. They proved the existence, uniqueness and compactness of solution. Chen, Hu and Nualart [11] studied the time-space fractional stochastic equations with Wiener processes and obtained the well-posedness of solutions. Recently, Ahmed et al. considered several stochastic integrodifferential equations with fractional power operators. For examples, in [12] and [13] stochastic integrodifferential equations with Hilfer fractional derivatives driven by fBms were discussed. Ahmed [14] considered an abstract nonlocal stochastic integrodifferential system with Caputo fractional derivatives driven by fBms and Poisson jumps and obtained the existence of mild solutions by fixed point theory provided the order of Caputo fractional derivative ranges from 1 to 2.

We use iterative technique to deal with problem (1.1), where the order of Caputo fractional derivative $\beta \in (1/2,1)$, under Lipschitz nonlinearities as well as proper assumptions on noise terms.

We are also interested in optimal control problems. Optimal control is a fundamental topic in control theory. Due to wildly applications in engineering and medical science, optimal control problems are studied by a lot of researchers, see [9, 15, 16] and references therein. The convexity of cost functions and the convexity of the set of control actions are two important factors in optimal control problems. Recently, Benner and Trautwein [17] discussed the existence of optimal control of stochastic heat equations with a convex cost function. Fuhrman, Hu and Tessitore [18] proved the stochastic maximum principle of stochastic partial differential equations and applied it to obtain the optimal control. Durga and Muthukumar [19] investigated optimal control problem of fractional stochastic equations with Poisson jumps. Inspired by the papers above, we study the optimal control problem of (1.1) with an abstract non-convex cost function. Under suitable assumptions on cost function and the compactness of the admissible set, we prove the existence of optimal control.

The paper is organized as follows. In Section 2, we present basic notions and relative results. In Section 3, we show the representation of mild solutions and prove the existence and uniqueness by iterative technique. In Section 4, we discuss the optimal control problem. In Section 5, we give two applications. In Section 6, we give conclusions.

2. Preliminaries

The β-order Caputo fractional derivative of u is defined by

$$ {}_0^cD_t^\beta u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{d}{ds} u(s) ds, $$

where Γ is Gamma function. The fractional Laplacian is defined by

$$ (-\Delta)^{\frac{\alpha}{2}} u(x) = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F} u(\xi))(x), \ \alpha \in (0,2), $$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. Let the function space

$$ W^{\frac{\alpha}{2},2}(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d), \ \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{\alpha}{2}} \mathcal{F} u] \in L^2(\mathbb{R}^d)\} $$

be endowed with the norm

$$ \|u\|_{W^{\frac{\alpha}{2},2}} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{\alpha}{2}} \mathcal{F} u]\|_2, $$

where $\|\cdot\|_2$ denotes the norm of $L^2(\mathbb{R}^d)$.

Remark 1. *Fractional Laplacian can also be defined through singular integral*

$$ (-\Delta)^{\frac{\alpha}{2}} \varphi(x) = C(d,\alpha) \text{P.V.} \int_{\mathbb{R}^d} \frac{\varphi(x) - \varphi(y)}{|x-y|^{d+\alpha}} dy, \ x \in \mathbb{R}^d, $$

where $\alpha \in (0,2)$. *These two approaches are equivalent on some levels, see [20].*

The bi-parameters Mittag-Leffler function is defined by

$$ M_{k,l}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(nk+l)}, \ z \in \mathbb{C}. $$

Function $M_{k,l}(z)$ is the extension of exponential function and $M_{1,1}(z) = e^z$. Let

$$ \mathcal{F}\Phi(t,z) = M_{\beta,1}(-|\xi|^\alpha t^\beta), \ \mathcal{F} J(t,z) = M_{\beta,\beta}(-|\xi|^\alpha t^\beta) $$

and

$$ \begin{aligned} R_\alpha^\beta(t) F &= \Phi(t,z) * F, \\ S_\alpha^\beta(t) F &= t^{\beta-1} J(t,z) * F \triangleq \Psi(t,z) * F, \end{aligned} $$

where $\alpha \in (1, 2]$, $\beta \in (0, 1)$ and $*$ denotes convolution. In fact $R^{\beta}_{\alpha}(t)u_0$ is a formal solution of problem

$$\begin{cases} {}^{c}_{0}D^{\beta}_{t}\, u(t, x) + (-\Delta)^{\frac{\alpha}{2}} u(t, x) = 0, & t \in (0, \infty),\ x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Therefore, by Laplace transform and Duhamel's formula, the mild solution of (1.2) can be expressed as

$$\begin{aligned} u =& R^{\beta}_{\alpha}(t)u_0 + \int_0^t S^{\beta}_{\alpha}(t - s)f(s, u)ds \\ =& \int_{\mathbb{R}^d} \Phi(t, x - z)u_0(z)dz + \int_0^t \int_{\mathbb{R}^d} \Psi(t - s, x - z)f(s, z, u(s, z))dzds, \end{aligned}$$

if each integral is well defined.

We recall the continuity and L^p estimates of operators Φ, Ψ, R^{β}_{α} and S^{β}_{α}. Let

$$\chi_1 = \begin{cases} \frac{d}{d-\alpha}, & d > \alpha, \\ \infty, & \text{otherwise}, \end{cases} \quad \chi_2 = \begin{cases} \frac{d}{d-2\alpha}, & d > 2\alpha, \\ \infty, & \text{otherwise}, \end{cases}$$

for $\beta \in (0, 1)$ and $\alpha \in (1, 2)$.

Lemma 1. *[5] Assume $\beta \in (0, 1)$ and $\alpha \in (1, 2)$.*

1. *Let $p \in [1, \chi_1)$. Then $\Phi(t, x) \in C((0, \infty), L^p(\mathbb{R}^d))$ and there exists a real number $C > 0$ such that*

$$\|\Phi(t)\|_p \le Ct^{-\frac{d\beta}{\alpha}(1-\frac{1}{p})}.$$

2. *Let $p \in [1, \chi_2)$. Then $\Psi(t, x) \in C((0, \infty), L^p(\mathbb{R}^d))$ and there exists a real number $C > 0$ such that*

$$\|\Psi(t)\|_p \le Ct^{-\frac{d\beta}{\alpha}(1-\frac{1}{p})+\beta-1}.$$

Let $q \in [1, \infty)$ and

$$\kappa_1 = \begin{cases} \frac{qd}{d-q\alpha}, & d > q\alpha, \\ \infty, & \text{otherwise}, \end{cases} \quad \kappa_2 = \begin{cases} \frac{qd}{d-2q\alpha}, & d > 2q\alpha, \\ \infty, & \text{otherwise}. \end{cases}$$

Lemma 2. *[5] Assume $\beta \in (0, 1)$, $\alpha \in (1, 2)$.*

1. *Let $p \in [1, \kappa_1)$. Then*

$$\|R^{\beta}_{\alpha}(t)u\|_p \le Ct^{-\frac{\beta d}{\alpha}(\frac{1}{q}-\frac{1}{p})}\|u\|_q.$$

2. *Let $p \in [1, \kappa_2)$. Then*

$$\|S^{\beta}_{\alpha}(t)u\|_p \le Ct^{-\frac{\beta d}{\alpha}(\frac{1}{q}-\frac{1}{p})+\beta-1}\|u\|_q.$$

3. *Let $r \in [1, \infty)$ and $u \in L^r(\mathbb{R}^d)$. Then the mapping*

$$t \mapsto R^{\beta}_{\alpha}(t)u \ \in C([0, \infty),\ L^r(\mathbb{R}^d)).$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space endowed with a usual filtration $\{\mathcal{F}_t\}_{0\le t<\infty}$ and $L_2^{\mathcal{F}_t}([0,T];L^2(\mathbb{R}^d))$ be the space of all $\mathcal{F}_t$-adapted random processes such that

$$\|u\|^2_{L_2^{\mathcal{F}_t}} = \mathbb{E}[\sup_{t\in[0,T]} \|u(t)\|_2^2] < \infty.$$

Denote $L_2(\Omega; L^2(\mathbb{R}^d))$ the family of all random variables such that

$$\|\zeta\|^2_{L_2(\Omega;L^2(\mathbb{R}^d))} = \mathbb{E}\|\zeta\|_2^2 < \infty.$$

We introduce a general concept of fBms. Let $B^H(t)$ be a fBm on $(\Omega, \mathcal{F}, \mathbb{P})$ and U be a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^\infty$. The space of continuous bounded linear operators on U is denoted by $\mathcal{L}(U)$. Let $Q \in \mathcal{L}(U)$ be a symmetric operator and $\{\lambda_i\}_{i=1}^\infty$ be a sequence of eigenvalues of Q. Further assume that Q is nonnegative and tr $Q < \infty$. Then the Q-fBm on U is defined by

$$B_Q^H(t) = \sum_{i=1}^\infty \sqrt{\lambda_i} e_i \beta_i^H(t),$$

where $\{\beta_i^H(t)\}_{t\in[0,T]}$ is a sequence of 1-dimensional fBms. B_Q^H is a Gaussian process and its mean and covariance are

$$\mathbb{E}\langle B_Q^H(t), y\rangle_U = 0 \text{ and } \mathbb{E}\langle B_Q^H(t), x\rangle_U \langle B_Q^H(s), y\rangle_U = \rho(t,s)\langle Qx, y\rangle_U,$$

where $t,\ s \in [0,T]$, $x,\ y \in U$ and $\rho(t,s)$ is the covariance operator,

$$\rho(t,s) = \mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Besides, B_Q^H becomes standard Brownian motion when Q is an identical operator and Hurst index $H = 1/2$.

We recall relative results of stochastic calculus with respect to fBms, see [6, 21] and reference therein. First, we call ψ an element of Hilbert-Schmidt space $\mathcal{L}_2^0(U, L^2(\mathbb{R}^d))$ if $\psi \in \mathcal{L}(U, L^2(\mathbb{R}^d))$ and

$$\|\psi\|^2_{\mathcal{L}_2^0} = \sum_{i=1}^\infty \|\sqrt{\lambda_i}\psi e_i\|_2^2 < \infty.$$

Lemma 3. *([7]) Let $\phi : [0,T] \to \mathcal{L}_2^0(U, L^2(\mathbb{R}^d))$ satisfy $\int_0^T \|\phi(s)\|^2_{\mathcal{L}_2^0} ds < \infty$. Then integral $\int_0^t \phi(s) dB^H(s)$ is a well defined random variable which takes values in $L^2(\mathbb{R}^d)$ and*

$$\mathbb{E}\|\int_0^t \phi(s) dB^H(s)\|_2^2 \le 2Ht^{2H-1}\int_0^t \|\phi(s)\|^2_{\mathcal{L}_2^0} ds,\ \forall t \in [0,T]. \tag{2.1}$$

Let $(Y, \mathcal{B}(Y), \pi)$ be a σ-finite measurable space on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{k_t\}_{t\ge0}$ be a stationary Poisson point process on $(Y, \mathcal{B}(Y), \pi)$. Let $\eta(dy, t)$ be a Poisson counting measure which is generated by $\{k_t\}_{t\ge0}$. The Poisson martingale measure $\tilde{\eta}$ on $(Y, \mathcal{B}(Y), \pi)$ is given by

$$\tilde{\eta}(dy, t) = \eta(dy, t) - \pi(dy)t.$$

Let $S^{\pi,2}(Y, L^2(\mathbb{R}^d))$ be the space of all measurable functions p defined on $(Y, \mathcal{B}(Y), \pi)$ with

$$\|p\|^2_{S^{\pi,2}} = \int_Y \|p(x, u; y)\|_2^2 \pi(dy) < \infty.$$

We use notation $S^{\pi,2}_{\mathcal{F}}([0, T] \times Y, L^2(\mathbb{R}^d))$ to denote the space of all $\mathcal{F} \times \mathcal{B}(Y)$-measurable processes q with

$$\|q\|^2_{S^{\pi,2}_{\mathcal{F}}} = \mathbb{E} \int_0^T \int_Y \|q(s, x, u; y)\|_2^2 \pi(dy) ds < \infty.$$

The integral

$$\int_0^{t+} \int_Y q(s, x, u; y) \tilde{\eta}(dy, ds), \ \forall t \in [0, T]$$

is well defined when $q \in S^{\pi,2}_{\mathcal{F}}([0, T] \times Y, L^2(\mathbb{R}^d))$. Further, if $q \in S^{\pi,2}_{\mathcal{F}}([0, T] \times Y, L^2(\mathbb{R}^d))$, then for all $t \in [0, T]$, $\int_0^{t+} \int_Y q(s, u; y) \tilde{\eta}(dy, ds)$ belongs to $L_2(\Omega; L^2(\mathbb{R}^d))$ and

$$\mathbb{E} \| \int_0^{t+} \int_Y q(s, u; y) \tilde{\eta}(dy, ds) \|_2^2 = \mathbb{E} \int_0^t \int_Y \|q(s, u; y)\|_2^2 \pi(dy) ds. \tag{2.2}$$

See [22, 23] for details.

3. Mild solutions

In this section, we give the representation of mild solutions of problem (1.1). Under suitable assumptions, we obtain the existence and uniqueness of mild solutions by iterative technique.

Let $\mathcal{D}_T$ be the set of all random processes having the following properties:

(1) u is $\mathcal{F}_t$ adapted for any $t \in [0, T]$.

(2) $u \in L_2^{\mathcal{F}_t}([0, T], L^2(\mathbb{R}^d))$.

(3) u is right continuous with left limit for all $x \in \mathbb{R}^d$ and a.s. $\omega \in \Omega$, that is,

$$u(t-, x) = \lim_{s \uparrow t} u(s, x).$$

Definition 1. *We call $u \in \mathcal{D}_T$ a mild solution of (1.1) if u satisfies the following formula a.s.*

$$\begin{aligned} u =& R_\alpha^\beta(t) u_0 + \int_0^t S_\alpha^\beta(t-s) f(s, u) ds + \int_0^t S_\alpha^\beta(t-s) \mathcal{A} v ds \\ &+ \int_0^t S_\alpha^\beta(t-s) g(s) dB^H(s) + \int_0^{t+} \int_Y S_\alpha^\beta(t-s) h(s, u; y) \tilde{\eta}(dy, ds) \end{aligned} \tag{3.1}$$

or equivalently

$$\begin{aligned} u(t, x) =& \int_{\mathbb{R}^d} \Phi(t, x-z) u_0(z) dz + \int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-z) f(s, z, u(s, z)) dz ds \\ &+ \int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-z) \mathcal{A} v(z) dz ds + \int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-z) g(s, z) dz dB^H(s) \\ &+ \int_0^{t+} \int_Y \int_{\mathbb{R}^d} \Psi(t-s, x-z) h(s, z, u(s, z); y) dz \tilde{\eta}(dy, ds). \end{aligned} \tag{3.2}$$

Remark 2. *In fact, a mild solution given by Definition 1 is a formal solution of problem (1.1). Furthermore, if u is a classical solution of problem (1.1), i.e., u is sufficiently smooth and u satisfies problem (1.1) for all* $(t,x) \in [0,\infty) \times \mathbb{R}^d$*, then u must be a mild solution of problem (1.1).*

The assumptions are given as follows.

(A1): There exists a positive function $\Lambda(x) \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that

$$|f(t,x,u(t,x))|^2 + \int_Y |h(t,x,u(t,x);y)|^2 \pi(dy) \le \Lambda(x)(1 + |u(t,x)|).$$

(A2): There exists a positive real number $L > 0$ such that

$$\begin{aligned}&|f(t,x,\tilde{u}(t,x)) - f(t,x,\tilde{\tilde{u}}(t,x))|^2 + \int_Y |h(t,x,\tilde{u}(t,x);y) - h(t,x,\tilde{\tilde{u}}(t,x);y)|^2 \pi(dy)\\ \le & L|\tilde{u}(t,x) - \tilde{\tilde{u}}(t,x)|^2.\end{aligned}$$

(G): Function $g : [0,\infty) \to \mathcal{L}_2^0(U, L^2(\mathbb{R}^d))$ and there exists a constant $r > 1/(2\beta - 1)$ such that

$$\int_0^T \|g(s,\cdot)\|_{\mathcal{L}_2^0}^{2r} ds < C(T), \quad \forall T > 0,$$

where $C(T)$ denotes the constant dependent on T.

(H): Function $h \in S_{\mathcal{F}}^{\pi,2}([0,\infty) \times Y, L^2(\mathbb{R}^d))$.

We show the regularity of stochastic integral terms before introduce the main result.

Lemma 4. *Let* $\beta \in (1/2, 1)$, $\alpha \in (1,2)$ *and* $H \in (1/2, 1)$. *If (A1), (G) and (H) hold, then for any* $u \in \mathcal{D}_T$,

$$\int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-z) g(s,z) dz dB^H(s),$$

$$\int_0^{t+} \int_Y \int_{\mathbb{R}^d} \Psi(t-s, x-z) h(s,z,u(s,z);y) dz \tilde{\eta}(dy, ds)$$

are well defined stochastic processes in $\mathcal{D}_T$.

Proof. We divide the proof into two parts.

(i) First, we prove that $\int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-z)g(s,z)dzdB^H(s) \in L_2^{\mathcal{F}_t}([0,T], L^2(\mathbb{R}^d))$. By assumption (G), inequality (2.1), the Young inequality, the Hölder inequality and Lemma 1, we get that

$$\begin{aligned}&\mathbb{E}\|\int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-z)g(s,z)dzdB^H(s)\|_2^2\\ =&\mathbb{E}\|\int_0^t \Psi(t-s) * g(s) dB^H(s)\|_2^2\\ \le&2HT^{2H-1}\int_0^t \|\Psi(t-s,\cdot)\|_1^2 \|g(s,\cdot)\|_{\mathcal{L}_2^0}^2 ds\\ \le&2HT^{2H-1+\frac{r(2\beta-1)-1}{r}}\left(\int_0^T \|g(s,\cdot)\|_{\mathcal{L}_2^0}^{2r} ds\right)^{1/r}\\ \le&C(T,H,\beta,r).\end{aligned}$$

Next we show that $\int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-z)g(s,z)dzdB^H(s) \in C([0,T]; L^2(\mathbb{R}^d))$. For any $\varepsilon > 0$, we consider

$$\begin{aligned}
&\mathbb{E}\{\|\int_0^{t+\varepsilon}\int_{\mathbb{R}^d}\Psi(t-s+\varepsilon, x-z)g(s,z)dzdB^H(s) - \int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)g(s,z)dzdB^H(s)\|_2^2\}\\
\leq &2\mathbb{E}\{\|\int_0^t\int_{\mathbb{R}^d}[\Psi(t-s+\varepsilon, x-z) - \Psi(t-s,x-z)]g(s,z)dzdB^H(s)\|_2^2\}\\
&+ 2\mathbb{E}\{\|\int_t^{t+\varepsilon}\int_{\mathbb{R}^d}\Psi(t-s+\varepsilon,x-z)g(s,z)dzdB^H(s)\|_2^2\}\\
\triangleq &G_1 + G_2.
\end{aligned}$$

By inequality (2.1) and the Young inequality, for any sufficient small $\delta > 0$, which is independent on ε, we get that

$$\begin{aligned}
G_1 =&2\mathbb{E}\{\|\int_0^t[\Psi(t-s+\varepsilon) - \Psi(t-s)] * g(s)dB^H(s)\|_2^2\}\\
\leq&4Ht^{2H-1}\int_0^t\|\Psi(t-s+\varepsilon,\cdot) - \Psi(t-s,\cdot)\|_1^2\|g(s,\cdot)\|_{\mathcal{L}_2^0}^2ds\\
\leq&4Ht^{2H-1}\int_0^{t-\delta}\|\Psi(t-s+\varepsilon,\cdot) - \Psi(t-s,\cdot)\|_1^2\|g(s,\cdot)\|_{\mathcal{L}_2^0}^2ds\\
&+ 4Ht^{2H-1}\int_{t-\delta}^t\|\Psi(t-s+\varepsilon,\cdot) - \Psi(t-s,\cdot)\|_1^2\|g(s,\cdot)\|_{\mathcal{L}_2^0}^2ds\\
\triangleq&G_{1,1} + G_{1,2}.
\end{aligned}$$

One can check that

$$G_{1,1} \leq 4Ht^{2H-1+\frac{r(2\beta-1)-1}{r}}\{\int_0^{t-\delta}\|g(s,\cdot)\|_{\mathcal{L}_2^0}^{2r}ds\}^{1/r} < +\infty.$$

Since $\Psi(t,x) \in C((0,\infty), L^1(\mathbb{R}^d))$, it is uniformly continuous on $[\delta, T]$. Hence $G_{1,1} \to 0$ as $\varepsilon \to 0$. On the other hand, it is obvious that

$$G_{1,2} \leq 4Ht^{2H-1+\frac{r(2\beta-1)-1}{r}}\{\int_{t-\delta}^t\|g(s,\cdot)\|_{\mathcal{L}_2^0}^{2r}ds\}^{1/r} \to 0 \text{ as } \delta \to 0.$$

By the arbitrariness of δ, we get that

$$G_1 \to 0 \text{ as } \varepsilon \to 0.$$

By similar computations, we obtain

$$G_2 \leq 4H\varepsilon^{2H-1}\int_t^{t+\varepsilon}\|\Psi(t-s+\varepsilon,\cdot)\|_1^2\|g(s,\cdot)\|_{\mathcal{L}_2^0}^2ds \to 0 \text{ as } \varepsilon \to 0.$$

Hence $\int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)g(s,z)dzdB^H(s)$ is a well defined stochastic process in $\mathcal{D}_T$.

(ii) Denote $X_t \triangleq \int_0^{t+}\int_Y\int_{\mathbb{R}^d}\Psi(t-s,x-z)h(s,z,u(s,z);y)dz\tilde{\eta}(dy,ds)$. We state that X_t is a well defined stochastic process which is right continuous with left limit. Let

$$H_{t,x}(s,y) = \int_{\mathbb{R}^d}\Psi(t-s,x-z)h(s,z,u(s,z);y)dz.$$

We assert that $H_{t,x}(s,y) \in S^{\pi;2}_{\mathcal{F}}([0,T]\times Y, L^2)$. By assumption (H), the Young inequality and Lemma 1, we arrive at

$$\mathbb{E}\int_0^t \int_Y \|H_{t,x}(s,y)\|_2^2 \pi(dy)ds \leq \mathbb{E}\int_0^t \int_Y \|\Psi(t-s)\|_1^2 \|h(s;y)\|_2^2 \pi(dy)ds < \infty.$$

The assertion is proved.

We now claim that $X_t \in L_2^{\mathcal{F}_t}([0,T], L^2(\mathbb{R}^d))$. Following identity (2.2), the Schwartz inequality, the Fubini Theorem and assumption (A1), we deduce that

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d}[\int_0^{t+}\int_Y\int_{\mathbb{R}^d}\Psi(t-s,x-z)h(s,z,u(s,z);y)dz\tilde{\eta}(dy,ds)]^2dx\\
=&\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\int_Y|\int_{\mathbb{R}^d}\Psi(t-s,x-z)h(s,z,u(s,z);y)dz|^2\pi(dy)dsdx\\
\leq&\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\int_Y\{\int_{\mathbb{R}^d}\Psi(t-s,x-z)dz\}\{\int_{\mathbb{R}^d}\Psi(t-s,x-z)[h(s,z,u(s,z);y)]^2dz\}\pi(dy)dsdx\\
\leq&c(T,d)\mathbb{E}\int_{\mathbb{R}^d}\int_0^t[\int_{\mathbb{R}^d}\Psi(t-s,x-z)dx]\Lambda(z)[1+|u(s,z)|]dzds\\
\leq&c(T,d)\int_0^T\{\|\Lambda\|_1+\|\Lambda\|_2\mathbb{E}[\|u(t,\cdot)\|_2]\}ds\\
\leq&c(T,d).
\end{aligned}$$

We remain to discuss the regularity of X_t. Since X_t is defined as a right limit, it is right continuous on t. Furthermore, by the continuity theorem in [22, Theorem 6.9], there exists a square integrable $\mathcal{F}_t$-adapted martingale $\hat{X}_t$ which is right continuous with left-hand limit and satisfies

$$\mathbb{P}\{\hat{X}_t = X_t\} = 1, \ \forall t \in [0,T].$$

Therefore, X_t is right continuous with left limit for all $x \in \mathbb{R}^d$ and a.s. $\omega \in \Omega$ in this sense. Hence we conclude that $X_t \in \mathcal{D}_T$. □

We introduce our main result of this section as follows.

Theorem 1. *Let $\beta \in (1/2,1)$, $\alpha \in (1,2)$ and $H \in (1/2,1)$. If (A1), (A2), (G) and (H) hold, then problem (1.1) has a unique mild solution u, subject to $u_0 \in L_2(\Omega, L^2(\mathbb{R}^d))$ and $v \in V$.*

Proof. We use the iterative technique and the Gronwall inequality to obtain the existence and uniqueness of mild solution. There are three steps to prove the theorem.

Step 1. The sequence of iterations.

Let

$$
\begin{aligned}
u_1(t,x) &= R^{\beta}_{\alpha}(t)u_0 = \int_{\mathbb{R}^d} \Phi(t, x-z)u_0(z)dz,\\
u_{n+1}(t,x) &= u_1(t,x) + \int_0^t \int_{\mathbb{R}^d} \Psi(t-s, x-z) f(s,z,u_n(s,z))dzds\\
&\quad + \int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-z)(\mathcal{A}v)(z)dzds + \int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-z)g(s,z)dzdB^H(s)\\
&\quad + \int_0^{t+} \int_Y \int_{\mathbb{R}^d} \Psi(t-s,x-z)h(s,z,u_n(s,z);y)dz\tilde{\eta}(dy,ds).\\
&\triangleq u_1(t,x) + \sum_{i=1}^{4} J_i(t,x),
\end{aligned}
$$

where $n = 1, 2, \cdots$. Next, we show $u_n \in \mathcal{D}_T$, for any $n \in \mathbb{N}$. From Lemma 2, $R^{\beta}_{\alpha}(t)u_0 \in C([0,T], L^2(\mathbb{R}^d))$ and

$$
\|R^{\beta}_{\alpha}(t)u_0\|_{C([0,T],L^2(\mathbb{R}^d))} \le \|u_0\|_2
$$

and further $u_1 \in \mathcal{D}_T$. Let $u_n \in \mathcal{D}_T$ for some $n > 1$. We need to prove $u_{n+1} \in \mathcal{D}_T$. By Lemma 4, we get that $J_3(t,x)$, $J_4(t,x) \in \mathcal{D}_T$. Consequently, we are left with $J_1(t,x)$ and $J_2(t,x)$. By the Schwartz inequality, the Fubini Theorem, (A1) and Lemma 1, we obtain

$$
\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d} |\int_0^t \int_{\mathbb{R}^d} \Psi(t-s,x-z) f(s,z,u_n(s,z))dzds|^2 dx\\
\le& t\mathbb{E}\int_{\mathbb{R}^d}\int_0^t |\int_{\mathbb{R}^d} \Psi(t-s,x-z) f(s,z,u_n(s,z))dz|^2 dsdx\\
\le& t\mathbb{E}\int_{\mathbb{R}^d}\int_0^t \{\int_{\mathbb{R}^d} \Psi(t-s,x-z)dz\}\{\int_{\mathbb{R}^d} \Psi(t-s,x-z)|f(s,z,u_n(s,z))|^2 dz\}dsdx\\
\le& C(T,d)\mathbb{E}\int_{\mathbb{R}^d}\int_0^t [\int_{\mathbb{R}^d} \Psi(t-s,x-z)dx]\Lambda(z)(1+|u_n(s,z)|)dzds\\
\le& C(T,d)\mathbb{E}\int_0^T \{\|\Lambda\|_1 + \|\Lambda\|_2\|u_n(s,\cdot)\|_2\}ds\\
\le& C(T,d).
\end{aligned}
$$

Hence $J_1 \in L^{\mathcal{F}_t}_2([0,T], L^2(\mathbb{R}^d))$. On the other hand, for arbitrarily $\varepsilon > 0$, we consider

$$
\begin{aligned}
&\mathbb{E}\|J_1(t+\varepsilon,x) - J_1(t,x)\|_2^2\\
\le& 2\mathbb{E}\int_{\mathbb{R}^d} |\int_0^t \int_{\mathbb{R}^d} [\Psi(t-s+\varepsilon,x-z) - \Psi(t-s,x-z)] f(s,z,u_n)dzds|^2 dx\\
&+ 2\mathbb{E}\int_{\mathbb{R}^d} |\int_t^{t+\varepsilon} \int_{\mathbb{R}^d} \Psi(t-s+\varepsilon,x-z) f(s,z,u_n)dzds|^2 dx\\
\triangleq& \Psi_1 + \Psi_2.
\end{aligned}
$$

By the Schwartz inequality, the Fubini Theorem, (A1) and Lemma 1 we get that for any sufficient small

$\delta > 0$,

$$
\begin{aligned}
\Psi_1 \leq & C(T,d)\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}[\Psi(t-s+\varepsilon,x-z)-\Psi(t-s,x-z)]|f(s,z,u_n(s,z))|^2dzdsdx \\
\leq & C(T,d)\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}[\Psi(t-s+\varepsilon,x-z)-\Psi(t-s,x-z)]\Lambda(z)(1+|u_n(s,z)|)dzdsdx \\
\leq & C(T,d)\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\{\int_{\mathbb{R}^d}[\Psi(t-s+\varepsilon,x-z)-\Psi(t-s,x-z)]dx\}|\Lambda(z)|^2dzds \\
& + C(T,d)\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\{\int_{\mathbb{R}^d}[\Psi(t-s+\varepsilon,x-z)-\Psi(t-s,x-z)]dx\}(1+|u_n(s,z)|^2)dzds \\
\leq & C(T,d)\mathbb{E}\int_0^t\|\Psi(t-s+\varepsilon,\cdot)-\Psi(t-s,\cdot)\|_1ds \\
\leq & C(T,d)\mathbb{E}\{\int_0^{t-\delta}+\int_{t-\delta}^t\}\|\Psi(t-s+\varepsilon,\cdot)-\Psi(t-s,\cdot)\|_1ds.
\end{aligned}
$$

Similar to the discussion for G_1, we get that $\Psi_1 \to 0$ as $\varepsilon \to 0$ and

$$\Psi_2 \leq C(T)\int_t^{t+\varepsilon}(t-s+\varepsilon)^{\beta-1}ds \leq C(T)\varepsilon^\beta \to 0 \text{ as } \varepsilon \to 0.$$

Therefore, $J_1 \in \mathcal{D}_T$. Since control $v \in V$ and operator $\mathcal{A} \in \mathcal{L}(V; L^2(\mathbb{R}^d))$, we deduce that

$$\mathbb{E}\int_{\mathbb{R}^d}|\int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)(\mathcal{A}v)(z)dzds|^2dx \leq C(T)\int_0^t\|\Psi(t-s,\cdot)\|_1^2\|\mathcal{A}v\|_2^2ds \leq C(T),$$

by the Schwartz inequality and the Young inequality. So it is easily to see that $J_2(t,x) \in \mathcal{D}_T$. We conclude that $u_{n+1} \in \mathcal{D}_T$. By induction, we get $\{u_n\}_{n=1}^\infty \subset \mathcal{D}_T$.

Step 2. The existence

(i) We show that the sequence of iterations is a Cauchy sequence in $\mathcal{D}_T$.

Let $W^n(t) = \mathbb{E}\int_{\mathbb{R}^d}[u_{n+1}(t,x)-u_n(t,x)]^2dx$, for all $0 \leq t \leq T$. Note that

$$
\begin{aligned}
W^1(t) \leq & 2C\mathbb{E}\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)[|f(s,z,u_1(s,z))|^2 \\
& +|\int_Y h(s,z,u_1(s,z);y)\pi(dy)|^2]dzdsdx \\
& + 2CHT^{2H-1}\int_0^t\|\Psi(t-s,\cdot)\|_1^2[\|\mathcal{A}v\|_2^2+\|g(s,\cdot)\|_{\mathcal{L}_2^0}^2]ds \\
\leq & C(T,d)\mathbb{E}\int_0^T\{\|\Lambda\|_1+\|\Lambda\|_2\|u_1(s,\cdot)\|_2\}ds + C(T,H)\int_0^t[\|\mathcal{A}v\|_2^2+\|g(s,\cdot)\|_{\mathcal{L}_2^0}^2]ds \\
\leq & C(T,H,d),
\end{aligned}
$$

by (A1) and (G). For all $n \geq 2$, we assert that

$$0 \leq W^n(t) \leq \frac{|C(T)|^{n-1}}{(n-1)!}W^1(T), \tag{3.3}$$

where $0 \le t \le T$. Suppose that (3.3) is true for n, we consider the case of $n+1$.

$$
\begin{aligned}
W^{n+1}(t) \le & 2\mathbb{E}\int_{\mathbb{R}^d}\{\int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)|f(s,z,u_{n+1}(s,z))-f(s,z,u_n(s,z))|dzds\}^2dx \\
& + 2\mathbb{E}\int_{\mathbb{R}^d}\{\int_0^{t+}\int_Y\int_{\mathbb{R}^d}\Psi(t-s,x-z)|h(s,z,u_{n+1}(s,z);y)-h(s,z,u_n(s,z);y)|dz\pi(dy)ds\}^2dx \\
\le & C\mathbb{E}\int_0^t\int_{\mathbb{R}^d}\{\|\Psi_{t-s,\cdot}\|_1\}|u_{n+1}(s,z)-u_n(s,z)|^2dzds \\
\le & C\int_0^t\mathbb{E}\int_{\mathbb{R}^d}|u_{n+1}(s,z)-u_n(s,z)|^2dzds \\
\le & C\int_0^t W^n(s)ds \le \frac{|C(T)|^n}{n!}W^1(T).
\end{aligned}
$$

Hence the assertion holds. Furthermore, formula (3.3) implies that $\{u_n(t,\cdot)\}$ converges uniformly in $L^2(\mathbb{R}^d)$ for any $0 \le t \le T$ and a.s. $\omega \in \Omega$. The limit is denoted by u, i.e.

$$
u(t,\cdot) = \lim_{n\to\infty} u_n(t,\cdot) \text{ in } L^2(\mathbb{R}^d).
$$

In addition, $u \in \mathcal{D}_T$.

(ii) u is a mild solution.

By assumption (A2), we get

$$
\begin{aligned}
& \mathbb{E}\int_{\mathbb{R}^d}\{u_n(s,z)-u_1(s,z)-\int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)f(s,z,u(s,z))dzds \\
& -\int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)(\mathcal{A}v)(z)dzds-\int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)g(s,z)dzdB^H(s) \\
& -\int_0^t\int_Y\int_{\mathbb{R}^d}\Psi(t-s,x-z)h(s,z,u(s,z);y)dz\pi(dy)ds\}^2dx \\
\le & \mathbb{E}\int_{\mathbb{R}^d}\{\int_0^t\int_{\mathbb{R}^d}\Psi(t-s,x-z)|f(s,z,u_{n-1}(s,z))-f(s,z,u(s,z))|dzds \\
& +\int_0^t\int_Y\int_{\mathbb{R}^d}\Psi(t-s,x-z)|h(s,z,u_{n-1}(s,z);y)-h(s,z,u(s,z);y)|dz\pi(dy)ds\}^2dx \\
\le & c(T,L,d)\int_0^t\mathbb{E}\int_{\mathbb{R}^d}[u_{n-1}(s,z)-u(s,z)]^2dzds \to 0 \text{ as } n\to\infty.
\end{aligned}
$$

We obtain that u satisfies (3.2), a.s.. Therefore, u is a mild solution of problem (1.1).

Step 3. The uniqueness

Let $\bar{u}$, $\bar{\bar{u}}$ be two different mild solutions of (1.1) with the same initial state u_0 and control v. Let

$$
\zeta(t) = \mathbb{E}\int_{\mathbb{R}^d}[\bar{u}(t,x)-\bar{\bar{u}}(t,x)]^2dx.
$$

First, we can check that $\sup_{t\in[0,T]}\zeta(t) < \infty$. Second, by similar arguments to that for (3.3), we get that

$$
\zeta(t) \le C\int_0^t \zeta(s)ds.
$$

Hence we conclude that $\bar{u}(t,x) = \bar{\bar{u}}(t,x)$, for all $t \in [0,T]$, a.e. $x \in \mathbb{R}^d$ and a.s. $\omega \in \Omega$ by the Gronwall inequality. □

4. Optimal control

In this section, we consider the existence of optimal control for problem (1.1) with a non-convex cost function. From Theorem 1, there exists a solution map: $v \mapsto u$, for any $v \in V$. We call (u, v) a solution pair. We consider the cost function J has the following abstract form

$$J(v) = \mathbb{E}\{\int_0^T \mathcal{P}(t, u; v)dt + Q(u(T))\}. \tag{4.1}$$

We assume that

(J1) $\mathcal{P} : [0, T] \times L^2(\mathbb{R}^d) \times V \to [0, +\infty)$. Functional $\mathcal{P}$ is measurable in $t \in [0, T]$. $\mathcal{P}(t, \cdot; \eta)$ is continuous, a.e. $t \in [0, T]$ and uniformly for $\eta \in V$. $\mathcal{P}(t, \xi; \cdot)$ is continuous and bounded, for a.e. $t \in [0, T]$ and all $\xi \in L^2(\mathbb{R}^d)$. Further there exists a positive function $a \in L^1([0, T])$ and a positive constant $b > 0$ such that

$$\mathcal{P}(t, \xi; \eta) \leq a(t) + b\|\xi\|_2^2, \ \forall \eta \in V.$$

(J2) $Q : L^2(\mathbb{R}^d) \to [0, +\infty)$. Q is lower semi-continuous on $L^2(\mathbb{R}^d)$ and there exist $b_1 \in [0, \infty)$ and $b_2 \in (0, \infty)$ such that

$$Q(\xi) \leq b_1 + b_2\|\xi\|_2^2.$$

Let the admissible set V_{ad} be a compact subset of V. If there exists an element $v_0 \in V_{ad}$ such that

$$J(v_0) = \inf_{v \in V_{ad}} J(v),$$

then v_0 is called an optimal control of J.

Theorem 2. *Let the conditions in Theorem 1 and (J1), (J2) hold. Then problem (1.1) with cost function (4.1) has at least one optimal control in V_{ad}.*

Proof. Let $J_0 \triangleq \inf_{v \in V_{ad}} J(v)$. Then $J_0 \geq 0$ and there exists $\{v^n\}$ such that $J(v^n) \to J_0$ as $n \to \infty$. By the compactness of V_{ad}, there exists a convergent subsequence of $\{v^n\}$, still denoted by $\{v^n\}$. Suppose $v^n \to \tilde{v}$ in V_{ad}.

Let $\{(u^n, v^n)\}_{n=1}^{\infty}$ and $(\tilde{u}, \tilde{v})$ be solution pairs. Next we prove $u^n \to \tilde{u}$ in $L_2^{\mathcal{F}_t}([0, T]; L^2(\mathbb{R}^d))$. Since $\{u^n\}_{n=1}^{\infty}$ and $\tilde{u}$ are mild solutions, by similar arguments to the proof of (3.3), we obtain

$$\begin{aligned}\mathbb{E}\|u^n(t, \cdot) - \tilde{u}(t, \cdot)\|_2^2 \leq & C(T) \int_0^T \mathbb{E}\{\|u^n(s, \cdot) - \tilde{u}(s, \cdot)\|_2^2\}ds \\ & + C(T) \int_0^T \mathbb{E}\|\Psi(t - s, \cdot)\|_1^2 \|\mathcal{A}(v^n - \tilde{v})\|_2^2 ds.\end{aligned}$$

From the Gronwall inequality that

$$\begin{aligned}\mathbb{E}\|u^n(t, \cdot) - \tilde{u}(t, \cdot)\|_2^2 & \leq C(T) \int_0^t \int_0^T \mathbb{E}\|\Psi(\tau - s, , \cdot)\|_1^2 \|\mathcal{A}(v^n - \tilde{v})\|_2^2 ds d\tau \\ & \leq C(T, \beta) \int_0^T \mathbb{E}\|\mathcal{A}(v^n - \tilde{v})\|_2^2 ds \to 0\end{aligned} \tag{4.2}$$

as $n \to \infty$, uniformly in $t \in [0, T]$. We arrive at $u^n \to \tilde{u}$ as $n \to \infty$ in $L_2^{\mathcal{F}_t}([0, T]; L^2(\mathbb{R}^d))$. Hence $\{u^n\}$ is bounded in $L_2^{\mathcal{F}_t}([0, T]; L^2(\mathbb{R}^d))$, i.e., there exists $M > 0$ such that

$$\|u^n\|^2_{L_2^{\mathcal{F}_t}} = \mathbb{E}[\sup_{t\in[0,T]} \|u^n(t, \cdot)\|_2^2] < M, \ \forall n \in \mathbb{N}. \tag{4.3}$$

Especially, we get

$$\mathbb{E}[\|u^n(T, \cdot)\|_2^2] < M, \ \forall n \in \mathbb{N}. \tag{4.4}$$

We show that J is lower semi-continuous. By assumption (J1) and (4.3), we get that

$$\mathbb{E} \int_0^T \mathcal{P}(t, u^n; v^n)dt \le \int_0^T a(t)dt + b\mathbb{E} \int_0^T \|u^n(t, \cdot)\|_2^2 dt < \infty.$$

Consider

$$\begin{aligned} \mathbb{E} \int_0^T \mathcal{P}(t, \tilde{u}; \tilde{v})dt \le& \mathbb{E} \int_0^T [\mathcal{P}(t, \tilde{u}; \tilde{v}) - \mathcal{P}(t, \tilde{u}; v^n)]dt \\ &+ \mathbb{E} \int_0^T [\mathcal{P}(t, \tilde{u}; v^n) - \mathcal{P}(t, u^n; v^n)]dt + \mathbb{E} \int_0^T \mathcal{P}(t, u^n; v^n)dt. \end{aligned} \tag{4.5}$$

Since $\mathcal{P}(t, \xi; \cdot)$ is continuous, by (J1) and the Lebesgue Dominated Convergence Theorem we obtain

$$\mathbb{E} \int_0^T [\mathcal{P}(t, \tilde{u}; \tilde{v}) - \mathcal{P}(t, \tilde{u}; v^n)]dt \to 0 \text{ as } n \to \infty. \tag{4.6}$$

As $\mathcal{P}(t, \cdot; \eta)$ is continuous, a.e. $t \in [0, T]$ and uniformly in $\eta \in V$,

$$\mathbb{E} \int_0^T [\mathcal{P}(t, \tilde{u}; v^n) - \mathcal{P}(t, u^n; v^n)]dt \to 0 \text{ as } n \to \infty. \tag{4.7}$$

By (4.5)–(4.7), letting $n \to \infty$ we get

$$\mathbb{E} \int_0^T \mathcal{P}(t, \tilde{u}; \tilde{v})dt \le \varliminf_{n\to\infty} \mathbb{E} \int_0^T \mathcal{P}(t, u^n; v^n)dt. \tag{4.8}$$

On the other hand, by (4.2) and the lower semi-continuity of Q we get that

$$Q(\tilde{u}(T)) \le \varliminf_{n\to\infty} Q(u^n(T)), \text{ a.s.}$$

and by (J2)

$$Q(u^n(T)) \le b_1 + b_2\|u^n(T)\|_2^2, \text{ a.s.}$$

and further by (4.4) we have

$$\mathbb{E}Q(u^n(T)) < M.$$

Hence by the Fatou Lemma and (4.2), we conclude that

$$\mathbb{E}Q(\tilde{u}(T)) \le \varliminf_{n\to\infty} \mathbb{E}Q(u^n(T)). \tag{4.9}$$

It is clearly that J is lower semi-continuous by (4.8) and (4.9). Therefore, there exists $v_0 \in V_{ad}$ such that

$$J(v_0) = J_0 = \inf_{v\in V_{ad}} J(v).$$

We get that v_0 is an optimal control of problem (1.1) with cost function (4.1). □

5. Examples

In this section, two examples are given to illustrate the results. One is a simple example in 1-dimensional space, another is an abstract example in $\mathbb{R}^3$.

Example 1. *Consider the following fractional stochastic control system*

$$\begin{cases} {}_0^c D_t^{0.75} u + (-\Delta)^{\frac{\alpha}{2}} u = u^2 + x^2 + t^2 \frac{dB^H(t)}{dt}, \ t \in [0,1], \ x \in [0,1], \\ u = 0, \ x \in \mathbb{R} \setminus [0,1], \\ u_0 = e^{-x}, \ x \in [0,1], \end{cases} \tag{5.1}$$

where $\alpha \in (1,2)$ and $H \in (1/2, 1)$.

The order of Caputo fractional derivative is $\beta = 0.75$ and $t \in [0,1]$, then $g = t^2$ satisfies assumption (G) for $r \in (2, +\infty)$. Functions $f = u^2$ and $h \equiv 0$ satisfy assumptions (A1), (A2) and (H). Hence, by Theorem 1, fractional stochastic control system (5.1) has an unique mild solution. Let $V_{ad} = [0,1]$, $\mathcal{A}v = v^2$, $v = x$ and

$$J = \mathbb{E} \int_0^T \|u\|_2^2 dt. \tag{5.2}$$

It implies that $\mathcal{P} = \|u\|_2^2$ and Q is null operator. One can check that operators $\mathcal{P}$ and Q satisfy assumptions (J1) and (J2). Therefore, by Theorem 2, system (5.1) with cost function (5.2) has at least one optimal control.

Example 2. *We consider the following fractional stochastic control system*

$$\begin{cases} {}_0^c D_t^{0.6} u + (-\Delta)^{0.8} u = a_1 u + \mathcal{A}v + a_2 t \frac{dB^{0.75}(t)}{dt} + \int_Y a_3 u y \tilde{\eta}(dy), \ t \in (0,1], \ x \in \mathbb{R}^3, \\ u_0 = e^{-|x|}, \ x \in \mathbb{R}^3, \end{cases} \tag{5.3}$$

where a_1, a_2 and a_3 are positive constants.

Note that $\beta = 0.6$, $\alpha = 1.6$ and $H = 0.75$. Then $g = a_2 t$ satisfies (G) for $r \in (5, +\infty)$. Further assume that

$$\int_Y y^2 \tilde{\eta}(dy) < \infty.$$

Thus $f = a_1 u$ and $h = a_3 u y$ satisfy (A1), (A2) and (H). Therefore, we deduce that there exists an unique mild solution of system (5.3). Let

$$J(v) = \mathbb{E}\{\int_0^T e^{-\lambda t} u^\theta v^\theta dt + a_4 u^\theta(T)\}, \tag{5.4}$$

where $\lambda \in (0, +\infty)$ and $\theta \in (0,1)$. So $\mathcal{P} = e^{-\lambda t} u^\theta v^\theta$ and $Q = a_4 u^\theta(T)$. It is easily to see that function J satisfies (J1) and (J2). By Theorem 2, there exists an optimal control of system (5.3) with cost function (5.4).

6. Conclusions

In this paper, by iterative technique and energy estimates, we obtain the existence and uniqueness of mild solution to problem (1.1) in suitable framework and under Lipschitz type conditions. We prove the existence of optimal control for problem (1.1) with a non-convex cost function by nonlinear analysis method. At last, two examples are given to demonstrate applications of the theorems. It is well known that a weak solution is also a mild solution. However, the converse is not true. We will keep on discussing the properties of weak solutions to fully nonlocal stochastic problems.

Acknowledgments

The authors are supported by the National Natural Science Foundation of China (Grant No. 11771107). We would like to thank referees for their valuable suggestions to improve the manuscript.

Conflict of interest

The authors declare that there is no conflict of interest.

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