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Research article

Maximum H-index of bipartite network with some given parameters

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Abstract: A network is an abstract structure that consists of nodes that are connected by links. A bipartite network is a type of networks where the set of nodes can be divided into two disjoint sets in a way that each link connects a node from one partition with a node from the other partition. In this paper, we first determine the maximum H-index of networks in the class of all n-node connected bipartite network with matching number t. We obtain that the maximum H-index of a bipartite network with a given matching number is $K_{t,n-t}$. Secondly, we characterize the network with the maximum H-index in the class of all the n-vertex connected bipartite network of given diameter. Based on our obtain results, we establish the unique bipartite network with maximum H-index among bipartite networks with a given independence number and cover of a network.

Keywords: H-index; bipartite network; matching number; independence number; cover of a network; diameter
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1. Introduction

In this paper we consider simple and finite network. Undefined notation and terminology can be found in [1, 2]. The distance between any two nodes is an important quantity in network theory. Generally, the *distance* between two nodes u, v in $\mathbb{N}$ is the length of a shortest u-v path of $\mathbb{N}$, which is denoted by $d_{\mathbb{N}}(u, v)$ (or $d(u, v)$ for short). The maximum distance between any two nodes of $\mathbb{N}$ is called a diameter. Let $D_{\mathbb{N}}(v)$ is the overall sum of distances from any node v in $\mathbb{N}$. Similarly, $\mathbb{D}_{\mathbb{N}}(v)$ denotes the sum of all reciprocals of distances from v in $\mathbb{N}$. A well-known distance-based invariant is the Wiener index, which was defined as

$$W(\mathbb{N}) = \sum_{\{u,v\}\subseteq V_{\mathbb{N}}} d(u,v) = \frac{1}{2}\sum_{v\subseteq V_{\mathbb{N}}} D_{\mathbb{N}}(v). \tag{1.1}$$

Due to the interesting and successful physio-chemical properties of Wiener index, many other distance based topological indices of networks have been flourished. Fortunately, one of such measuring invariants known as the Harary index was proposed by Plavšić et al. [3] and by Ivanciuc et al. [4] independently in 1993, which is defined as

$$H(\mathbb{N}) = \sum_{\{u,v\}\subseteq V_{\mathbb{N}}} \frac{1}{d_{\mathbb{N}}(u,v)} = \frac{1}{2}\sum_{v\subseteq V_{\mathbb{N}}} \mathbb{D}_{\mathbb{N}}(v), \tag{1.2}$$

where $d_{\mathbb{N}}(u,v)$ is the distance between the nodes u and v. This is a "reciprocal analogue" of the Wiener index. More-formally the Wiener index $W(\mathbb{N})$ is half-sum of the distance matrix of $\mathbb{N}$, and it is obvious to develop a matrix $H(\mathbb{N})$, which is the half-sum of reciprocal analogue of the distance matrix. Such matrix is so-called reciprocal distance matrix or the Harary matrix [5].

The upper (resp. lower) bound and the corresponding extremal graphs of topological indices are very important. Gutman [14] showed that the path and the star are respectively the graphs with minimal and maximal Harary index among all trees. In [15–18], the authors presented several upper and lower bounds for the Harary index of connected graphs, triangle-free, quadrangle-free graphs, graphs with given diameter, matching number. Ilić et al. [19] investigated the Harary index of trees with various parameters. There are many results concerning the Harary index of graph classes with several constraints, like connectivity [11], trees with given degree sequence [20], unicyclic graphs [21], bicyclic graphs [22], the ordering [23]. Other results related to distance and its invariants, one can see [24]. Recently, Feng et al. [25] investigated the minimal Harary index of trees with small diameters.

The main motivation of establishing most of the results of this paper came from the references [6–9]. Li et al. [6] studied on the maximal connective eccentricity index of bipartite graphs with given parameters. Li and Song [7] determined on the sum of all distances in bipartite graphs. In [9], Wang et al. characterized the connective eccentricity index of networks and its applications to octane isomers and benzenoid hydrocarbons. To study similar extremal property for the H-index is natural and interesting for us.

2. Notation and terminology

Let $\mathbb{N} = (V_{\mathbb{N}}, E_{\mathbb{N}})$ be a network with node set $V_{\mathbb{N}}$ and link set $E_{\mathbb{N}}$. The set of neighbors of a node v in $\mathbb{N}$ is denoted by $N_{\mathbb{N}}(v)$ or simply $N(v)$. The network obtained from $\mathbb{N}$ by deleting an link $uv \in E_{\mathbb{N}}$ is denoted by $\mathbb{N} - uv$. Similarly, $\mathbb{N} + uv$ is obtained from $\mathbb{N}$ by adding an link $uv \notin E_{\mathbb{N}}$.

The *union* of two networks H_1 and H_2 is denoted by $H_1 \cup H_2$ with $V_{H_1\cup H_2} = V_{H_1} \cup V_{H_2}$ and $E_{H_1\cup H_2} = E_{H_1} \cup E_{H_2}$. If H_1 and H_2 are node disjoint, then we let $H_1 \uplus H_2$ denote the *join* of H_1 and H_2, which is the network obtained from $H_1 \cup H_2$ by adding all the links between the nodes $x \in V_{H_1}$ and $y \in V_{H_2}$. For disjoint networks $H_1, H_2, \ldots, H_k$ with $k \geq 3$, the sequential join $H_1 \uplus H_2 \uplus \cdots \uplus H_k$ is the network $(H_1 \uplus H_2) \cup (H_2 \uplus H_3) \cup \cdots \cup (H_{k-1} \uplus H_k)$. For short, denote by $k\mathbb{N}$ and $[k]\mathbb{N}$ the union and the sequential join of k disjoint copies of $\mathbb{N}$, respectively. For example, $kK_1 \cong \overline{K}_k$ which is the k isolated nodes and $[p]H_1 \uplus H_2 \uplus [q]H_3$ denotes the sequential join $\underbrace{H_1 \uplus H_1 \uplus \ldots \uplus H_1}_{p} \uplus H_2 \uplus \underbrace{H_3 \uplus H_3 \uplus \ldots \uplus H_3}_{q}$.

A bipartite network $\mathbb{N}$ is denoted with bipartition (X, Y) by $\mathbb{N}[X, Y]$, and defined as every link has one end in X and the other end in Y. Moreover, if every node of X is connected to every node of Y in $\mathbb{N}[X, Y]$, then $\mathbb{N}$ is said to be a complete bipartite network. Denote $K_{m,n}$ a unique complete bipartite network with parts of sizes m and n.

Assume that, the set of all n-node connected bipartite networks with matching number "t" is denoted by $\mathfrak{M}_{n,t}$. Whereas, the set of all n-node connected bipartite networks with diameter "d" is denoted by $\mathfrak{B}_{n,d}$.

The set of pairwise non-adjacent links in a network $\mathbb{N}$ is called a matching. Without loss of generality, assume that if M is a matching, then the two ends of each link of M are said to be matched under M, and each node incident with an link of M is said to be covered by M. If M covers as many nodes as possible then M is called a maximum matching. The number of links in a maximum matching of a network $\mathbb{N}$ is called the matching number of $\mathbb{N}$.

A node (resp. link) independent set of a network $\mathbb{N}$ is a set of nodes (resp. links) such that any two distinct nodes (resp. links) of the set are not adjacent (resp. incident on a common node). A node (resp. link) cover of a network $\mathbb{N}$ is a set of nodes (resp. links) such that each link (resp. node) of $\mathbb{N}$ is incident with at least one node (resp. link) of the set.

Further on, we need the following lemmas. Note that Lemma 2.2 is the extension of Lemma 2.1, introduced by Feng et al. [10].

Lemma 2.1. [11] *Let $\mathbb{N}$ be a network and for any link $e \notin E_{\mathbb{N}}$, then one has $H(\mathbb{N} + e) > H(\mathbb{N})$.*

Lemma 2.2. [10] *If $\mathbb{N}' = \mathbb{N} + uv$ for a connected network $\mathbb{N}$ and $uv \notin E_{\mathbb{N}}$, then it holds that*

$$H(\mathbb{N}') \geqslant H(\mathbb{N}) + \frac{1}{2},$$

where the equality holds if and only if u and v are pendent nodes sharing the same neighbor.

Lemma 2.3. (The König-Egerváry Theorem). (See [12, 13]). *In any bipartite network, the number of links in a maximum matching is equal to the number of nodes in a minimum covering and denoted by $\eta(\mathbb{N})$.*

Let $\mathbb{N} = \mathbb{N}[X, Y]$ be a bipartite network such that $\mathbb{N} \in \mathfrak{M}_{n,t}$. Based on Lemma 2.3, it is obvious to see $\eta(\mathbb{N}) = t$. Let S be a minimum covering of $\mathbb{N}$ and $X_M = S \cap X$, $Y_M = S \cap Y$. Without loss of generality, suppose that $|X_M| \geq |Y_M|$. Since, S is a covering of $\mathbb{N}$, obviously $E(X \setminus X_M, Y \setminus Y_M) = \emptyset$.

3. Maximum H-index of bipartite networks with a given matching number

This section deals the sharp upper bound on H-index of n-node bipartite networks with matching number t, and all the corresponding extremal bipartite networks. A covering of a network $\mathbb{N}$ is a node subset $K \subseteq V_{\mathbb{N}}$ such that each link of $\mathbb{N}$ has at least one end in the set K. The number of nodes in a minimum covering of a network $\mathbb{N}$ is called the covering number of $\mathbb{N}$.

Lemma 3.1. *Let $\mathbb{N}_1[X,Y]$ be a bipartite network with the same node set as $\mathbb{N}$, where $\mathbb{N} \in \mathfrak{M}_{n,t}$ such that $E(\mathbb{N}_1) = \{xy : x \in X_M, y \in Y\} \cup \{xy : x \in X \setminus X_M, y \in Y_M\}$. Then, $H(\mathbb{N}) \leqslant H(\mathbb{N}_1)$ with equality if and only if $\mathbb{N} \cong \mathbb{N}_1$.*

Proof. It is easy to check that $\mathbb{N}$ is a subnetwork of $\mathbb{N}_1$. From Lemma 2.1, the result is obvious. □

Based on network $\mathbb{N}_1$, we define a new network $\mathbb{N}_2$ as: $\mathbb{N}_2 = \mathbb{N}_1 - \{uv : u \in X \setminus X_M, v \in Y_M\} + \{uw : u \in X \setminus X_M, w \in X_M\}$, which is depicted in Figure 1.

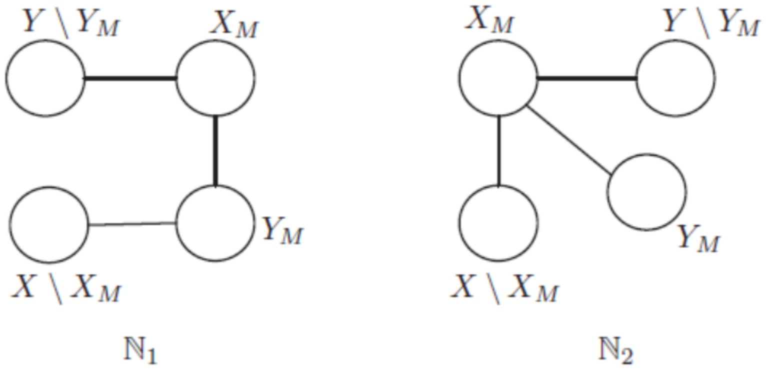

Figure 1. Networks $\mathbb{N}_1$ and $\mathbb{N}_2$.

Lemma 3.2. *Let $\mathbb{N}_1$ and $\mathbb{N}_2$ be the networks defined above (see Figure 1). Then one has*

$$H(\mathbb{N}_1) < H(\mathbb{N}_2).$$

Proof. Based on $\mathbb{N}_1$, we construct a new network, say $\mathbb{N}_2$, which is obtained from $\mathbb{N}_1$ by deleting all the links between $X \setminus X_M$ and Y_M, and adding all the links between $X \setminus X_M$ and X_M, see Figure 1. It is routine to check that $\mathbb{N}_2 \in \mathfrak{M}_{n,t}$ with $\mathbb{N} \ncong \mathbb{N}_2 \cong K_{t,n-t}$.

Let $|X \backslash X_M| = m_1$, $|Y \backslash Y_M| = m_2$ suppose $m_2 \geqslant m_1 \geqslant t$. We partition $V_{\mathbb{N}_1} = V_{\mathbb{N}_2}$ into $X_M \cup Y_M \cup (X \setminus X_M) \cup (Y \setminus Y_M)$ as shown in Figure 1. For the sake of simplicity, assume that, for all $a \in Y \backslash Y_M,\ b \in X_M,\ c \in Y_M$ and $d \in X \backslash X_M$, then one has

$$\mathbb{D}_{\mathbb{N}_1}(a) = \sum_{b \in X_M} \frac{1}{d_{\mathbb{N}_1}(a,b)} + \sum_{c \in Y_M} \frac{1}{d_{\mathbb{N}_1}(a,c)} + \sum_{d \in X \backslash X_M} \frac{1}{d_{\mathbb{N}_1}(a,d)} + \sum_{\bar{a} \in Y \backslash Y_M} \frac{1}{d_{\mathbb{N}_1}(a,\bar{a})} = t + \frac{t}{2} + \frac{m_1}{3} + \frac{m_2 - 1}{2},$$

$$\mathbb{D}_{\mathbb{N}_1}(b) = \sum_{a \in Y \backslash Y_M} \frac{1}{d_{\mathbb{N}_1}(b,a)} + \sum_{c \in Y_M} \frac{1}{d_{\mathbb{N}_1}(b,c)} + \sum_{d \in X \backslash X_M} \frac{1}{d_{\mathbb{N}_1}(b,d)} + \sum_{\bar{b} \in X_M} \frac{1}{d_{\mathbb{N}_1}(b,\bar{b})} = m_2 + t + \frac{m_1}{2} + \frac{t-1}{2},$$

$$\mathbb{D}_{\mathbb{N}_1}(c) = \sum_{a \in Y \backslash Y_M} \frac{1}{d_{\mathbb{N}_1}(c,a)} + \sum_{b \in X_M} \frac{1}{d_{\mathbb{N}_1}(c,b)} + \sum_{d \in X \backslash X_M} \frac{1}{d_{\mathbb{N}_1}(c,d)} + \sum_{\bar{c} \in Y_M} \frac{1}{d_{\mathbb{N}_1}(c,\bar{c})} = \frac{m_2}{2} + t + m_1 + \frac{t-1}{2},$$

$$\mathbb{D}_{\mathbb{N}_1}(d) = \sum_{a \in Y \backslash Y_M} \frac{1}{d_{\mathbb{N}_1}(d,a)} + \sum_{b \in X_M} \frac{1}{d_{\mathbb{N}_1}(d,b)} + \sum_{c \in Y_M} \frac{1}{d_{\mathbb{N}_1}(d,c)} + \sum_{\bar{d} \in X \backslash X_M} \frac{1}{d_{\mathbb{N}_1}(d,\bar{d})} = \frac{m_2}{3} + \frac{t}{2} + t + \frac{m_1 - 1}{2},$$

$$\mathbb{D}_{\mathbb{N}_2}(a) = \sum_{b \in X_M} \frac{1}{d_{\mathbb{N}_2}(a,b)} + \sum_{c \in Y_M} \frac{1}{d_{\mathbb{N}_2}(a,c)} + \sum_{d \in X \backslash X_M} \frac{1}{d_{\mathbb{N}_2}(a,d)} + \sum_{\bar{a} \in Y \backslash Y_M} \frac{1}{d_{\mathbb{N}_2}(a,\bar{a})} = t + \frac{t}{2} + \frac{m_1}{2} + \frac{m_2 - 1}{2},$$

$$\mathbb{D}_{\mathbb{N}_2}(b)=\sum_{a\in Y\setminus Y_M}\frac{1}{d_{\mathbb{N}_2}(b,a)}+\sum_{c\in Y_M}\frac{1}{d_{\mathbb{N}_2}(b,c)}+\sum_{d\in X\setminus X_M}\frac{1}{d_{\mathbb{N}_2}(b,d)}+\sum_{\bar{b}\in X_M}\frac{1}{d_{\mathbb{N}_2}(b,\bar{b})}= \quad m_2+t+m_1+\frac{t-1}{2},$$

$$\mathbb{D}_{\mathbb{N}_2}(c)=\sum_{a\in Y\setminus Y_M}\frac{1}{d_{\mathbb{N}_2}(c,a)}+\sum_{b\in X_M}\frac{1}{d_{\mathbb{N}_2}(c,b)}+\sum_{d\in X\setminus X_M}\frac{1}{d_{\mathbb{N}_2}(c,d)}+\sum_{\bar{c}\in Y_M}\frac{1}{d_{\mathbb{N}_2}(c,\bar{c})}= \quad \frac{m_2}{2}+t+\frac{m_1}{2}+\frac{t-1}{2},$$

$$\mathbb{D}_{\mathbb{N}_2}(d)=\sum_{a\in Y\setminus Y_M}\frac{1}{d_{\mathbb{N}_2}(d,a)}+\sum_{b\in X_M}\frac{1}{d_{\mathbb{N}_2}(d,b)}+\sum_{c\in Y_M}\frac{1}{d_{\mathbb{N}_2}(d,c)}+\sum_{\bar{d}\in X\setminus X_M}\frac{1}{d_{\mathbb{N}_2}(d,\bar{d})}= \quad \frac{m_2}{2}+t+\frac{t}{2}+\frac{m_1-1}{2}.$$

This gives

$$\begin{aligned}
H(\mathbb{N}_1)-H(\mathbb{N}_2) &= \frac{1}{2}\Big(\sum_{u\in V_{\mathbb{N}_1}}\mathbb{D}_{\mathbb{N}_1}(u)-\sum_{u\in V_{\mathbb{N}_2}}\mathbb{D}_{\mathbb{N}_2}(u)\Big)\\
&= \frac{1}{2}\Big(\sum_{a\in Y\setminus Y_M}\mathbb{D}_{\mathbb{N}_1}(a)-\sum_{a\in Y\setminus Y_M}\mathbb{D}_{\mathbb{N}_2}(a)+\sum_{b\in X_M}\mathbb{D}_{\mathbb{N}_1}(b)-\sum_{b\in X_M}\mathbb{D}_{\mathbb{N}_2}(b)\\
&+\sum_{c\in Y_M}\mathbb{D}_{\mathbb{N}_1}(c)-\sum_{c\in Y_M}\mathbb{D}_{\mathbb{N}_2}(c)+\sum_{d\in X\setminus X_M}\mathbb{D}_{\mathbb{N}_1}(d)-\sum_{d\in X\setminus X_M}\mathbb{D}_{\mathbb{N}_2}(d)\Big)\\
&= \frac{1}{2}\Big(m_2(\frac{m_1}{3}-\frac{m_1}{2})+t(\frac{m_1}{2}-m_1)+t(m_1-\frac{m_1}{2})+m_1(\frac{m_2}{3}-\frac{m_2}{2})\Big)\\
&= \frac{1}{2}\Big(m_2(\frac{m_1}{3}-\frac{m_1}{2})+m_1(\frac{m_2}{3}-\frac{m_2}{2})\Big)\\
&= \frac{-m_1m_2}{6}\\
&< 0.
\end{aligned}$$

Hence, we obtain that $H(\mathbb{N}_2)>H(\mathbb{N}_1)$. □

Lemma 3.3. *Let $\mathbb{N}$ be a connected bipartite network with $V_{\mathbb{N}}=(X,Y)$ with $|X|=m_1\geqslant|Y|=m_2$.*

1. *If $m_1=1$, then $H(\mathbb{N})=1$. Hence, and $\mathbb{N}=K_2$.*
2. *If $m_1>1$ and $m_2=1$, then $H(\mathbb{N})=\frac{1}{4}(m_1^2+3m_1)$ and $\mathbb{N}$ is just the network K_{1,m_1}.*
3. *If $m_2>1$, then $H(\mathbb{N})\leqslant\frac{1}{4}[m_2(2m_1+m_2-1)+m_1(2m_2+m_1-1)]$ with equality if and only if $\mathbb{N}\cong K_{m_1,m_2}$.*

Hence, due to Lemma 3.3, the considered bipartite network is of order $n>2$.

Theorem 3.1. *Let $\mathbb{N}\in\mathfrak{M}_{n,t}$, then $H(\mathbb{N})\leqslant\frac{1}{4}(n^2-2t^2+2nt-n)$. The equality holds if and only if $\mathbb{N}\cong K_{t,n-t}$.*

Proof. It is obvious to obtain that

$$H(K_{t,n-t})=\frac{1}{4}(n^2-2t^2+2nt-n).$$

Hence, we only need to show that among $\mathfrak{M}_{n,t}$ with maximum H-index is a unique network $K_{t,n-t}$.

Choose $\mathbb{N}$, in $\mathfrak{M}_{n,t}$ such that its H-index is maximum. For $t=\lfloor\frac{n}{2}\rfloor$, due to Lemma 2.1 the extremal network is just $K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$ as desired. Therefore, we only consider the case $t<\lfloor\frac{n}{2}\rfloor$.

Without loss of generality, assume that the bipartition node set of $\mathbb{N}$ is denoted by (X, Y), such that $|Y| \geqslant |X| \geqslant t$. Let M be a maximal matching of $\mathbb{N}$, then due to Lemma 2.1, the addition of new link(s) increases the H-index of a network. In what follows, if $|X| = t$, then the extremal network is $\mathbb{N} = K_{t,n-t}$. Hence, we consider the case $|X| > t$.

Assume that M is a matching set and X_M(resp. Y_M) be the set of nodes of X(resp. Y) which are incident to the links of M. Therefore, $|X_M| = |Y_M| = t$. Keeping in mind that $\mathbb{N}$ does not contains links between the nodes of $X \backslash X_M$ and the nodes of $Y \backslash Y_M$. Otherwise, any such link together with M producing a matching of cardinality greater than that of M, which is a contradiction to the maximality of M.

By Lemma 3.1 adding all possible links between the nodes of X_M and Y_M, X_M and $Y \backslash Y_M$, $X \backslash X_M$ and Y_M we get a network $\mathbb{N}_1$ as depicted in Figure 1. Together with Lemma 2.1 we have $H(\mathbb{N}_1) > H(\mathbb{N})$. Note that the matching number of $\mathbb{N}_1$ is at least $t + 1$. Hence, $\mathbb{N}_1 \notin \mathfrak{M}_{n,t}$ and $\mathbb{N} \ncong \mathbb{N}_1$. Based on $\mathbb{N}_1$, we construct a new network, say $\mathbb{N}_2$, which is obtained from $\mathbb{N}_1$ by deleting all the links between $X \setminus X_M$ and Y_M, and adding all the links between $X \setminus X_M$ and X_M, see Figure 1. It is routine to check that $\mathbb{N}_2 \cong K_{t,n-t}$. By Lemma 3.2, $H(\mathbb{N}_2) > H(\mathbb{N}_1)$. Hence, we obtained our desire result. □

Remark 3.1. *The maximum cardinalities of all node (resp. link) independent set is called node (resp. link) independence number of* $\mathbb{N}$*, and is denoted by* $\gamma(\mathbb{N})$ *(resp.* $\gamma'(\mathbb{N})$*). The minimum cardinalities of all node (link) covers are said to be a node (resp. link) cover number of* $\mathbb{N}$*, and is denoted by* $\eta(\mathbb{N})$ *(resp.* $\eta'(\mathbb{N})$*).*

Together Lemma 2.3, and Remark 3.1 with Theorem 3.1 the following useful result is obvious.

Corollary 3.1. *The network* $K_{\sigma,n-\sigma}$ *is a unique network having maximum* H*-index, among all connected bipartite networks of order* n *with node cover number or node independence number or link cover number* σ*.*

4. Maximum H-index of bipartite networks with a given diameter

In this section, we characterize the networks in $\mathfrak{B}_{n,d}$ attaining the maximum H-index. Without loss of generality, assume that $P = v_0 v_1 \ldots v_d$ is a diametric path in $\mathfrak{B}_{n,d}$. Thereby, any $\mathbb{N} = (V_{\mathbb{N}}, E_{\mathbb{N}})$ in $\mathfrak{B}_{n,d}$, there is a partition $R_0, R_1, \ldots R_d$ of $V_{\mathbb{N}}$ with $d(v_0, v) = i$ such that $v \in R_i (i = 0, 1, 2, \ldots, d)$. Thus, we assume R_i a distance layer of $V_{\mathbb{N}}$, and R_i, R_j of $V_{\mathbb{N}}$ are adjacent if $|i - j| = 1$. Suppose that $|R_i| = l_i$ throughout this section.

For $d \geqslant 3$, if d is odd, then assume $\mathfrak{Q}(n, d) := [\frac{d-1}{2}]K_1 + \lfloor\frac{n-d-1}{2}\rfloor K_1 + \lceil\frac{n-d+1}{2}\rceil K_1 + [\frac{d-1}{2}]K_1$.

For $d \geqslant 4$, if d is even, then assume $\hat{\mathfrak{Q}}(n, d) := \{Q(n, d) = [\frac{d}{2} - 1]K_1 + a_1 K_1 + \lfloor\frac{n-d+2}{2}\rfloor K_1 + a_2 K_1 + [\frac{d}{2} - 1]K_1 : a_1 + a_2 = \lceil\frac{n-d+2}{2}\rceil\}$. The following is our main result of this section.

Theorem 4.1. *Let* $\mathbb{N}$ *be a network in* $\mathfrak{B}_{n,d}$ *with the maximum* H*-index.*

1. *If* $d = 2$*, then* $\mathbb{N} \cong K_{\lfloor\frac{n}{2}\rfloor,\lceil\frac{n}{2}\rceil}$*.*
2. *If* $d \geqslant 3$*, then* $\mathbb{N} \cong \mathfrak{Q}(n, d)$ *for odd* d*, and* $\mathbb{N}$ *is an arbitrary network in* $Q(n, d)$ *otherwise.*

Proof. Choose a network $\mathbb{N}$ in $\mathfrak{B}_{n,d}$ which maximizes the H-index.

(i) In view of Lemma 2.1, we have $\mathbb{N} \cong K_{n-q,q}$ for $d = 2$, where $q, n - q \geqslant 2$. Assume $|X| = n - q$ and $|Y| = q$, then it is routine to check that, for all x (resp. y) in X (resp. Y), one has

$\mathbb{D}_{\mathbb{N}}(x) = q + \frac{1}{2}(n-q-1) = \frac{1}{2}(n+q-1)$, $\mathbb{D}_{\mathbb{N}}(y) = (n-q) + \frac{1}{2}(q-1) = n - \frac{1}{2}q - \frac{1}{2}$. This gives

$$\begin{aligned} H(K_{n-q,q}) &= \frac{1}{2}\Big(\sum_{x\in X}\mathbb{D}_{\mathbb{N}}(x) + \sum_{y\in Y}\mathbb{D}_{\mathbb{N}}(y)\Big) \\ &= \frac{1}{2}\Big(\frac{1}{2}(n-q)(n+q-1) + q(n - \frac{1}{2}q - \frac{1}{2})\Big) \\ &= \frac{1}{4}\Big(n^2 - 2q^2 - n + 2nq\Big). \end{aligned}$$

If n is odd, then $H(K_{n-q,q}) \leqslant \frac{1}{8}(3n^2-2n-1)$ with equality if and only if $q = \frac{n-1}{2}$, or $q = \frac{n+1}{2}$, i.e., $\mathbb{N} \cong K_{\frac{n+1}{2},\frac{n-1}{2}}$; and if n is even, then $H(K_{n-q,q}) \leqslant \frac{1}{8}n(3n-2)$ with equality if and only if $q = \frac{n}{2}$ i.e., $\mathbb{N} \cong K_{\frac{n}{2},\frac{n}{2}}$ as desired. □

(ii) In order to prove this part, we use the following structural properties.

Proposition 4.1. $\mathbb{N}[R_i] \cong |R_i|K_1$, *i.e., the induced subnetwork* $\mathbb{N}[R_i]$ *contains no link for* $i = 1, 2, \ldots, d$, *and* $|R_d| = 1$ *for* $d \geqslant 3$.

Proof of Property 4.1. By a contradiction, we assume that there exist two nodes z^+, z^- in some R_i such that $z^+z^- \in E_{\mathbb{N}[R_i]} \subseteq E_{\mathbb{N}}$. Since both z^+ and z^- are in R_i, there exists two distinct paths, we say U_1 and U_2, such that U_1 (resp. U_2) connects the nodes z^0, z^+ (resp. z^0, z^-). Clearly, the paths $U_1 \cup U_2 \cup z^+z^-$ contains an odd cycle in $\mathbb{N}$. In fact, if U_1 and U_2 contain no common internal node, then $U_1 \cup U_2 + z^+z^-$ is an odd cycle. Otherwise, suppose that w_0 is the last common internal node of U_1, U_2, then $U_1(w_0, z^+) \cup U_2(w_0, z^-) + z^+z^-$ is an odd cycle. This is impossible since $\mathbb{N}$ is bipartite.

In what follows we prove the second part. In fact, if $|R_d| \geq 2$, then we may choose $r \in R_d \setminus \{x_d\}$ and put $\breve{\mathbb{N}} = \mathbb{N} + \{rz^+ : z^+ \in R_{d-3}\}$. It is easy to check that $\breve{\mathbb{N}} \in \mathfrak{B}_{n,d}$ with its node partition

$$R_0 \cup R_1 \cup R_2 \cup \ldots \cup R_{d-3} \cup (R_{d-2} \cup \{r\}) \cup R_{d-1} \cup (R_d \setminus \{r\}).$$

In view of Lemma 2.1, one obtains $\xi^{ee}(\breve{\mathbb{N}}) > \xi^{ee}(\mathbb{N})$, which contradicts to the choice of $\mathbb{N}$. Thus, $|R_d| = 1$. □

Proposition 4.2. $\mathbb{N}[R_{j-1} \cup R_j] \cong K_{|R_{j-1}|,|R_j|}$, *i.e.,* $\mathbb{N}[R_{j-1} \cup R_j]$ *induces a complete bipartite network for each* $j = 1, 2, \ldots, d$.

Proof of Property 4.2. Without loss of generality, assume that $\mathbb{N}[R_{j-1} \cup R_j]$ is not a complete bipartite network for some j. By Property 4.1, we get $\mathbb{N}[R_{j-1}] \cong |R_{j-1}|K_1$ and $\mathbb{N}[R_j] \cong |R_j|K_1$. Thus, there exists v_i in R_{j-1} and v_j in R_j, such that v_i, v_j are not adjacent. Construct $\mathbb{N}' = \mathbb{N} + v_iv_j$. Obviously, $\mathbb{N}' \in \mathfrak{B}_{n,d}$ and we have $H(\mathbb{N}') > H(\mathbb{N})$ by Lemma 2.1. Hence, this contradicts to the choice of $\mathbb{N}$, so we get our desired result. □

Bear in mind the same notations as above, we have the following structural property.

Proposition 4.3. *For* $d \geq 3$, *each of the following holds.*

1. *For odd d, we have*

$$\begin{aligned} |R_0| = |R_1| = |R_2| = \cdots = |R_{\frac{d-3}{2}}| = |R_{\frac{d+3}{2}}| = \cdots = |R_{d-1}| = |R_d| = 1, \\ \text{and} \Big||R_{\frac{d-1}{2}}| - |R_{\frac{d+1}{2}}|\Big| \leqslant 1. \end{aligned} \tag{4.1}$$

2. *For even d, one has*

$$|R_0| = |R_1| = |R_2| = \cdots = |R_{\frac{d}{2}-2}| = |R_{\frac{d}{2}+2}| = \cdots = |R_{d-1}| = |R_d| = 1, \\ and \left||R_{\frac{d}{2}-1}| + |R_{\frac{d}{2}+1}| - |R_{\frac{d}{2}}|\right| \leqslant 1. \tag{4.2}$$

Proof of Property 4.3. (i) Note that $|R_0| = |R_d| = 1$, here we only show that $|R_1| = 1$ holds. Similarly, we can show that $|R_2| = \cdots = |R_{\frac{d-3}{2}}| = |R_{\frac{d+3}{2}}| = \cdots = |R_{d-1}| = 1$. We omit the procedure here.

If $d = 3$, then the result is obvious. In what follows, we consider that $d \geqslant 5$. If $|R_1| \geqslant 2$, then choose $u \in R_1$ and let $\mathbb{N}' = \mathbb{N} - u_0v + \{ux : x \in R_4\}$. In fact, the node partition of $\mathbb{N}'$ is $R_0 \cup (R_1 \setminus \{u\}) \cup R_2 \cup (R_3 \cup \{u\}) \cup R_4 \cup \ldots \cup R_d$; in view of Property 4.1 and the choice of $\mathbb{N}$, any two of adjacent blocks of $R_{\mathbb{N}'}$ induce a complete bipartite subnetwork and $|R_d| = 1$ for $d \geqslant 5$. Note that, $\mathbb{D}_{\mathbb{N}}(u) = \mathbb{D}_{\mathbb{N}'}(u) + \frac{2}{3} - \sum_{i=4}^{d} \frac{2l_i}{(i-1)(i-3)}$, $\mathbb{D}_{\mathbb{N}}(v) = \mathbb{D}_{\mathbb{N}'}(v) + \frac{2}{3}$ for all $v \in R_0$, $\mathbb{D}_{\mathbb{N}}(v) = \mathbb{D}_{\mathbb{N}'}(v)$ for all $v \in (R_1 \setminus \{u\}) \cup R_2 \cup R_3$, $\mathbb{D}_{\mathbb{N}}(v) = \mathbb{D}_{\mathbb{N}'}(v) - \frac{2}{(i-1)(i-3)}$ for all $v \in R_4 \cup R_5 \cup \ldots \cup R_d$.

$$\begin{aligned}
H(\mathbb{N}) - H(\mathbb{N}') &= \frac{1}{2}\Big(\sum_{v \in V_{\mathbb{N}}} \mathbb{D}_{\mathbb{N}}(v) - \sum_{v \in V_{\mathbb{N}'}} \mathbb{D}_{\mathbb{N}'}(v)\Big) \\
&= \frac{1}{2}\Big[\sum_{v \in R_0} \Big(\mathbb{D}_{\mathbb{N}}(v) - \mathbb{D}_{\mathbb{N}'}(v)\Big) + \Big(\mathbb{D}_{\mathbb{N}}(u) - \mathbb{D}_{\mathbb{N}'}(u)\Big) \\
&+ \sum_{j=4}^{d} \sum_{v \in R_j} \Big(\mathbb{D}_{\mathbb{N}}(v) - \mathbb{D}_{\mathbb{N}'}(v)\Big)\Big] \\
&= \frac{1}{2}\Big(\frac{2}{3} + \sum_{j=4}^{d} \frac{-2l_i}{(i-1)(i-3)} + \sum_{j=4}^{d} \frac{-2l_i}{(i-1)(i-3)} + \frac{2}{3}\Big) \\
&= \frac{1}{2}\Big(\frac{4}{3} - 4\sum_{j=4}^{d} \frac{l_i}{(i-1)(i-3)}\Big) \\
&= -2\Big(\sum_{j=4}^{d} \frac{l_i}{(i-1)(i-3)} - \frac{1}{3}\Big) \\
&= -2\Big(\frac{l_4}{(4-1)(4-3)} + \sum_{j=5}^{d} \frac{l_i}{(i-1)(i-3)} - \frac{1}{3}\Big) \\
&< 0.
\end{aligned}$$

The last inequality follows that $l_4 > 0$ and $\sum_{j=5}^{d} \frac{l_i}{(i-1)(i-3)} > 0$. i.e $H(\mathbb{N}') > H(\mathbb{N})$, a contradiction to the choice of $\mathbb{N}$. Hence, $|R_1| = 1$.

Next we show that if d is odd, then $\left||R_{\frac{d-1}{2}}| - |R_{\frac{d-1}{2}+1}|\right| \leqslant 1$. Without loss of generality, we assume that $|R_{\frac{d-1}{2}}| \geqslant |R_{\frac{d-1}{2}+1}|$. Then it suffices to show that $|R_{\frac{d-1}{2}}| - |R_{\frac{d-1}{2}+1}| \leqslant 1$. If this is not true, then $|R_{\frac{d-1}{2}}| - |R_{\frac{d-1}{2}+1}| \geqslant 2$. Choose $w \in R_{\frac{d-1}{2}}$, let $\mathbb{N}' = \mathbb{N} - \Big\{wx : x \in R_{\frac{d-3}{2}} \cup R_{\frac{d+1}{2}}\Big\} + \Big\{wy : y \in R_{\frac{d-1}{2}} \cup R_{\frac{d+3}{2}}\Big\}$.

Then the node partition of $\mathbb{N}'$ is $R_0 \cup R_1 \ldots \cup R_{\frac{d-3}{2}} \cup (R_{\frac{d-1}{2}} \setminus \{w\}) \cup (R_{\frac{d+1}{2}} \cup \{u\}) \cup R_{\frac{d+3}{2}} \cup \ldots \cup R_d$ and each of the two adjacent blocks of $R_{\mathbb{N}'}$ induces a complete bipartite network. By direct calculation, we have

$$H(\mathbb{N}') - H(\mathbb{N}) = \Big((|R_{\frac{d-1}{2}}| - 1) + \frac{1}{2}|R_{\frac{d+1}{2}}|\Big) - \Big(\frac{1}{2}(|R_{\frac{d-1}{2}}| - 1) + |R_{\frac{d+1}{2}}|\Big)$$
$$= \frac{1}{2}\Big(|R_{\frac{d-1}{2}}| - |R_{\frac{d+1}{2}}| - 1\Big)$$
$$> 0,$$

a contradiction to the choice of $\mathbb{N}$. This completes the proof of Property 4.3(i).

Together Property 4.1 and Property 4.2 with (4.1), we obtain that $\mathbb{N} \cong \mathfrak{Q}(n, d)$.

(ii) By the same discussion as the proof of the first part of (i) as above, we can show that $|R_0| = |R_1| = |R_2| = \cdots = |R_{\frac{d}{2}-2}| = |R_{\frac{d}{2}+2}| = \cdots = |R_{d-1}| = |R_d| = 1$, we omit the procedure here.

Next we show that if d is even, then $\Big||R_{\frac{d}{2}}| - (|R_{\frac{d}{2}-1}| + |R_{\frac{d}{2}+1}|)\Big| \leqslant 1$. Without loss of generality, we assume that $|R_{\frac{d}{2}}| < |R_{\frac{d}{2}-1}| + |R_{\frac{d}{2}+1}|$. Then it suffices to show that $|R_{\frac{d}{2}+1}| + |R_{\frac{d}{2}-1}| - |R_{\frac{d}{2}}| \leqslant 1$. If this is not true, then $|R_{\frac{d}{2}+1}| + |R_{\frac{d}{2}-1}| - |R_{\frac{d}{2}}| \geqslant 2$. It is routine to check that at least one of $R_{\frac{d}{2}-1}$ and $R_{\frac{d}{2}+1}$ contains at least two nodes. Hence, we assume without loss of generality that $|R_{\frac{d}{2}-1}| \geqslant 2$. Choose $w \in R_{\frac{d}{2}-1}$ and let

$$\mathbb{N}^* = \mathbb{N} - \Big\{wx : x \in R_{\frac{d}{2}-2} \cup R_{\frac{d}{2}}\Big\} + \Big\{wy : y \in R_{\frac{d}{2}-1} \cup R_{\frac{d}{2}+1}\Big\}.$$

Then the node partition of $\mathbb{N}^*$ is $R_0 \cup R_1 \ldots \cup (R_{\frac{d}{2}-1} \setminus \{w\}) \cup (R_{\frac{d}{2}} \cup \{w\}) \cup R_{\frac{d}{2}+1} \cup \ldots \cup R_d$ and each of the two adjacent blocks of $R_{\mathbb{N}^*}$ induces a complete bipartite network. By direct calculation, we have

$$H(\mathbb{N}^*) - H(\mathbb{N}) = \Big((|R_{\frac{d}{2}-1}| - 1) + \frac{1}{2}|R_{\frac{d}{2}}| + |R_{\frac{d}{2}+1}|\Big) - \Big(\frac{1}{2}(|R_{\frac{d}{2}-1}| - 1) + |R_{\frac{d}{2}}| + \frac{1}{2}|R_{\frac{d}{2}+1}|\Big)$$
$$= \frac{1}{2}\Big(|R_{\frac{d}{2}-1}| + |R_{\frac{d}{2}+1}| - (|R_{\frac{d}{2}}| + 1)\Big)$$
$$\geqslant \frac{1}{2}(1) > 0,$$

a contradiction to the choice of $\mathbb{N}$. This completes the proof of Property 4.3(ii).

For even d, by Property 4.1 and Property 4.3(ii), we obtain that $|R_0| = |R_1| = |R_2| = \cdots = |R_{\frac{d}{2}-2}| = |R_{\frac{d}{2}+2}| = \cdots = |R_{d-1}| = |R_d| = 1$, and $\Big||R_{\frac{d}{2}-1}| + |R_{\frac{d}{2}+1}| - |R_{\frac{d}{2}}|\Big| \leqslant 1$, and it is easy to check that if $\mathbb{N}, \mathbb{N}^* \in \hat{\mathfrak{Q}}(n, d)$, then $H(\mathbb{N}) = H(\mathbb{N}^*)$, which implies that $\mathbb{N} \cong Q(n, d) \in \hat{\mathfrak{Q}}(n, d)$, as desired. □

5. Conclusions

In this contribution, we determined the unique bipartite network with maximum H-index among all bipartite networks with given matching number, independence number, cover of a network and diameter.

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