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Research article

A fundamental theorem for algebroid function in k-punctured complex plane

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Abstract: The main purpose of this article is to study the value distribution of algebroid function in the k-punctured complex plane. We establish the second fundamental theorems for algebroid function concerning small algebroid functions in the k-punctured complex plane, which extend the Nevanlinna theory for algebroid functions from single connected domain to multiple connected domain.

Keywords: algebroid function; k-puncture plane; small algebroid function
Mathematics Subject Classification: 32C20, 30D45

1. Introduction

It is assumed that the readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$, etc. of Nevalinna theory, (see Hayman [4], Yang [33], Yi and Yang [34]).

It is well known that Nevanlinna value distribution theory is an important tool in studying the properties of meromorphic functions in the fields of complex analysis. The value distribution theory of meromorphic functions occupies one of the central places in Complex Analysis. In 1925, R. Nevanlinna [17] established the second fundamental theorem for meromorphic functions, which is the most important result in value distribution theory, and at the same time, he gave the question whether the second fundamental theorem can be extended to small functions. This problem attracted many mathematicians, for example, Q. T. Chuang [2] proved the second fundamental still holds for small entire functions, N. Steinmetz [20] solved this problem in 1986. Besides, numerous works are devoted to studying its connections with other areas of mathematics (topology, differential geometry, measure theory, potential theory and others); extending its inferences to wider classes of functions (meromorphic functions in arbitrary plane regions and Riemann surfaces, algebroid functions, functions of several variables, meromorphic curves). For example, G. Valiron, E. Ullrich, H. Selberg and K. L. Hiong [7, 19, 24, 25] in 1930s extended the second fundamental theorem to algebroid functions; D. C. Sun, Z. S. Gao and H. F. Liu in 2012 [21] further established the second fundamental theorem concerning small algebroid functions for v-valued algebroid functions.

Algebroid function was firstly introduced by H. Poincaré, and after that, G. Darboux pointed out that it is a very important class of functions. Let $H_k(z), \ldots, H_0(z)$ be analytic functions in a single connected domain $\mathbb{X} \subseteq \mathbb{C}$ without common zeros, then the irreducible equation

$$\Psi(z, w) = H_v(z)f^v + H_{v-1}(z)f^{v-1} + \cdots + H_0(z) = 0$$

defines a v-valued algebroid function $f(z)$ in $\mathbb{X} \subseteq \mathbb{C}$ (see [6, 21]). If $v = 1$, then $f(z)$ is a meromorphic function in $\mathbb{X}$. Nearly 90 years passed, many famous mathematicians (including G. Rémoundos, G. Valiron, E. Ullrich, H. Selberg, K. L. Hiong, Y. Z. He, etc.) had paid great attention to study the value distribution of algebroid function in some complex domains, such as: the whole complex plane $\mathbb{C}$, the unit disc $\mathbb{D}$ and the angular domain Δ, and obtained a lot of interesting and important results (see [5, 8, 12, 13, 18, 22, 25–29, 32]). As we know, the whole complex plane $\mathbb{C}$, the unit disc $\mathbb{D}$ and the angular domain Δ can all be regarded as simple connected region, in other words, they only obtained those results of algebroid functions in some simple connected regions. *Thus, a naturel question arises: what results we can obtain for algebroid function in some multiply connected regions?* In 2016, Y. Tan [23] studied the value distribution of algebroid functions on annulus, and established some basic theorems which is an analog of Nevanlinna theory of algebroid function in the whole complex plane. Indeed, annulus is only regarded as a special multiply connected region—double connected region. The Nevanlinna theory for meromorphic function on annuli were proposed by Korhonen, Khrystiyanyn and Kondratyuk (see [9–11]), after their works, Lund and Ye, Fernández, T. B. Cao, H. Y. Xu further the value distribution and uniqueness of meromorphic function on annulus(see [1, 15, 16, 31]).

However, there was very seldom paper dealing with the value distribution of algebroid function in a more general multiply connected region. In recent, the authors [30] have studied the value distribution of algebroid functions in a $k + 1$ multiply connected region—a k-punctured complex plane, and established some basic theorems of algebroid functions in the k-punctured complex plane. In this paper, we will further investigate the value distribution of algebroid functions in the k-punctured plane, and established the second fundamental theorem for algebroid functions concerning small algebroid functions in the k-punctured planes.

The structure of this paper is as follows. *In Section 2, we introduce the basic notations and fundamental theorems of algebroid functions in the k-punctured complex planes. Section 3 is devoted to prove the second fundamental theorem of algebroid functions concerning small algebroid functions in a k-punctured complex plane.*

2. Basic notations and fundamental theorems for algebroid functions in the k-punctured complex planes

Given a set of distinct points $c_s \in \mathbb{C}$, $s \in \{1, 2, \ldots, k\}, k \in \mathbb{N}_+$, $\Omega := \mathbb{C} \setminus \bigcup_{s=1}^{k}\{c_s\}$ can be called as a k-punctured complex plane. The main purpose of this article is to study meromorphic functions of those k-punctured planes, $k \geq 2$.

Denote $d = \frac{1}{2}\min\{|c_s - c_j| : j \neq s\}$ and $r_0 = \frac{1}{d} + \max\{|c_s| : s \in \{1, 2, \ldots, k\}\}$. Then $\frac{1}{r_0} < d$, $\overline{D}_{1/r_0}(c_j) \bigcap \overline{D}_{1/r_0}(c_s) = \emptyset$ for $j \neq s$ and $\overline{D}_{1/r_0}(c_s) \subset D_{r_0}(0)$ for $s \in \{1, 2, \ldots, k\}$, where $D_\delta(c) = \{z : |z - c| < \delta\}$ and $\overline{D}_\delta(c) = \{z : |z - c| \leq \delta\}$. For an arbitrary $r \geq r_0$, we define

$$\Omega_r = D_r(0) \setminus \bigcup_{s=1}^{k} \overline{D}_{1/r}(c_s).$$

Thus, it follows that $\Omega_r \supset \Omega_{r_0}$ for $r_0 < r \leq +\infty$. It is easy to see that Ω_r is a $k + 1$ connected region.

Let $A_v(z), A_{v-1}(z), \ldots, A_0(z)$ be a group of analytic functions which have no common zeros and define in the k-punctured plane Ω,

$$\psi(z, W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_1(z)W + A_0(z) = 0, \tag{2.1}$$

$$\psi_W(z, W) = vA_v(z)W^{v-1} + (v - 1)A_{v-1}(z)W^{v-2} + \cdots + A_1(z) = 0.$$

Denote $J(z)$ by

$$J(z) = (-1)^{\frac{v(v-1)}{2}} \begin{vmatrix} 1 & A_{v-1} & A_{v-2} & \cdots & A_0 & 0 & \cdots & 0 \\ 0 & A_v & A_{v-1} & \cdots & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & A_v & A_{v-1} & \cdots & A_0 \\ v & (v-1)A_{v-1} & (v-2)A_{v-2} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & vA_v & (v-1)A_{v-1} & \cdots & A_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & vA_v & \cdots & A_1 \end{vmatrix}.$$

If $A_v(z) \not\equiv 0$, and $J(z) \not\equiv 0$ in $c_s \in \mathbb{C}$, $s \in \{1, 2, \ldots, k\}$, then irreducible equation (2.1) defines a v-valued algebroid function in a k-punctured plane Ω. For an irreducible algebroid function $W(z)$, the points in the complex plane can be divided to two classes. One is a set $T_W \subseteq \Omega$ of regular points of $W(z)$, the other is a set $S_W = \Omega - T_W$ of critical points of $W(z)$. The set S_W is an isolated set (see [6, 14]). For every $a \in T_W$. there exist and only exist v number of regular function elements $\{(w_j(z), a)\}_{j=1}^{v}$. Throughout our article, we usually denote $W(z) = \{w_j(z)\}_{j=1}^{v}$.

Let $W(z)$ be a v-valued algebroid function in a k-punctured plane Ω, $r_0 \leq r < +\infty$, we use the notations

$$\begin{aligned} m_0(r, W) =& \frac{1}{2\pi}\sum_{j=1}^{v}\int_0^{2\pi} \log^+ \left|w_j(re^{i\theta})\right| d\theta + \frac{1}{2\pi}\sum_{j=1}^{v}\sum_{s=1}^{k}\int_0^{2\pi} \log^+ \left|w_j(c_s + \frac{1}{r}e^{i\theta})\right| d\theta - \\ &- \frac{1}{2\pi}\sum_{j=1}^{v}\int_0^{2\pi} \log^+ \left|w_j(r_0e^{i\theta})\right| d\theta - \frac{1}{2\pi}\sum_{j=1}^{v}\sum_{s=1}^{k}\int_0^{2\pi} \log^+ \left|w_j(c_s + \frac{1}{r_0}e^{i\theta})\right| d\theta, \end{aligned}$$

$$N_0(r,W)=\frac{1}{v}\int_{r_0}^{r}\frac{n_0(t,W)}{t}dt,\quad \overline{N}_0(r,W)=\frac{1}{v}\int_{r_0}^{r}\frac{\overline{n}_0(t,W)}{t}dt,$$

$$N_0(r,\frac{1}{W-a})=\frac{1}{v}\int_{r_0}^{r}\frac{n_0(t,\frac{1}{W-a})}{t}dt,\quad \overline{N}_0(r,\frac{1}{W-a})=\frac{1}{v}\int_{r_0}^{r}\frac{\overline{n}_0(t,\frac{1}{W-a})}{t}dt,$$

$$N_x(r,W)=\frac{1}{v}\int_{r_0}^{r}\frac{n_x(t,W)}{t}dt,$$

where $w_j(z)(j=1,2,\ldots,v)$ is a one-valued branch of $W(z)$, $n_0(t,W)$ is the counting function of poles of the function $W(z)$ in $\Omega_t := D_t(0)\setminus\bigcup_{s=1}^{k}\overline{D}_{1/t}(c_s)$, $r_0<t\le+\infty$ (counting multiplicity); $n_0(t,\frac{1}{W-a})[\overline{n}_0(t,\frac{1}{W-a})]$ is the counting function of poles of the function $\frac{1}{W-a}$ in Ω_t, counting multiplicity [ignoring multiplicity]; $n_x(r,W)$ is the counting function of branch points of the function $W(z)$ in Ω_r, and $N_x(r,W)$ is the density index of branch point of $W(z)$ in Ω_r.

Let $W(z)$ be an algebroid function in a k-punctured plane Ω, if there are λ branches of $W(z)$ which take $a(\neq\infty)$ as the value in z_0 point, then we have the fractional power series

$$W(z)=a+b_\tau(z-z_0)^{\frac{\tau}{\lambda}}+b_{\tau+1}(z-z_0)^{\frac{\tau+1}{\lambda}}+\cdots, \tag{2.2}$$

$n_0(r,a)=n_0(r,\frac{1}{W-a})=\sum\limits_{W=a}\tau$, where $n_0(r,a)$ is the counting function of zeros of $W(z)-a$ in Ω (counting multiplicity). If there are λ branches of $W(z)$ which take ∞ as the value in z_0 point, then we have the fractional power series

$$W(z)=b_{-\tau}(z-z_0)^{-\frac{\tau}{\lambda}}+b_{-\tau+1}(z-z_0)^{\frac{-\tau+1}{\lambda}}+\cdots, \tag{2.3}$$

$n_0(r,\infty)=n_0(r,W)=\sum\limits_{W=\infty}\tau$, where $n_0(r,\infty)$ is the counting function of poles of $W(z)-a$ in Ω (counting multiplicity). $z=z_0$ is a branch point of $\lambda-1$ degree of $W(z)$ on its Riemann Surface $\widetilde{\mathscr{M}}$. $n_x(r,W)=\sum(\lambda-1)$ denotes the branch points of $W(z)$ on its Riemann Surface in Ω. Obviously, for $a\in\overline{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$, we have

$$n_0(r,\frac{1}{W-a})=n_0(r,\frac{1}{\psi(z,a)}),\ \ N_0(r,\frac{1}{W-a})=N_0(r,\frac{1}{\psi(z,a)}),$$

and especially, $N_0(r,\frac{1}{W})=\frac{1}{k}N_0(r,\frac{1}{A_0})$ as $a=0$, and $N_0(r,W)=\frac{1}{k}N_0(r,\frac{1}{A_k})$ as $a=\infty$.

Definition 2.1. *Let $W(z)$ be an algebroid function in a k-punctured plane Ω, the function*

$$T_0(r,W)=m_0(r,W)+N_0(r,W),\quad r_0<r<+\infty,$$

is called the Nevanlinna characteristic of $W(z)$ in a k-punctured plane.

From the above definitions, we can obtain the following some connections with the classical characteristics of algebroid functions in $\mathbb{C}$ as follows.

$$(a)\ \ m_0(r,W)=m(r,W)+\sum_{s=1}^{v}m(\frac{1}{r},W(c_s+z))-m(r_0,W)-\sum_{s=1}^{v}m(\frac{1}{r_0},W(c_s+z)),\ \ for\ \ r>r_0,$$

$$(b)\ \ N_0(r,W)=N(r,W)+\sum_{s=1}^{v}N(\frac{1}{r},W(c_s+z))-N(r_0,W)-\sum_{s=1}^{v}N(\frac{1}{r_0},W(c_s+z)),\ \ for\ \ r>r_0,$$

$$(c)\ T_0(r,W)=T(r,W)+\sum_{s=1}^{v}T(\frac{1}{r},W(c_s+z))-T(r_0,W)-\sum_{s=1}^{v}T(\frac{1}{r_0},W(c_s+z)),\quad for\ \ r>r_0,$$

$$(d)\ T(r,W)-T(r_0,W)\le T_0(r,W)\le T(r,W)+\sum_{s=1}^{v}T(\frac{1}{r},W(c_s+z)).$$

Next, we will prove the above conclusions. Firstly, (a) is obviously. Secondly, we prove (b). Suppose $W(0)\neq\infty$. Let $n_{10}(t,W),(t>r_0)$ be the counting function of poles of $W(z)$ in $D_t(0)$ and $n_{1s}(\frac{1}{t},W(c_s+z)),(t>r_0)$ be the counting function of poles $W(c_s+z)$ in $D_s(c_s):=\{z:|z-c_s|<\frac{1}{t}\}$ for $s=1,2\dots,v$, then

$$\begin{aligned}N_0(r,W)=&\frac{1}{v}\int_{r_0}^{r}\frac{n_{10}(t,W)}{t}dt+\frac{1}{v}\sum_{s=1}^{v}\int_{r_0}^{r}\frac{n_{1s}(\frac{1}{t},W(c_s+z))}{t}dt\\
=&\frac{1}{v}\int_{r_0}^{r}\frac{n(t,W)}{t}dt-\frac{1}{v}\sum_{s=1}^{v}\int_{\frac{1}{r}}^{\frac{1}{r_0}}\frac{n_{1s}(t,W(c_s+z))}{t}dt\\
=&\frac{1}{v}\int_{0}^{r}\frac{n(t,W)}{t}dt-\frac{1}{v}\int_{0}^{r_0}\frac{n(t,W)}{t}dt+\frac{1}{v}\sum_{s=1}^{v}\int_{0}^{\frac{1}{r}}\frac{n_{1s}(t,W(c_s+z))}{t}dt\\
&-\frac{1}{v}\sum_{s=1}^{v}\int_{0}^{\frac{1}{r_0}}\frac{n_{1s}(t,W(c_s+z))}{t}dt\\
=&N(r,W)+\sum_{s=1}^{v}N(\frac{1}{r},W(c_s+z))-N(r_0,W)-\sum_{s=1}^{v}N(\frac{1}{r_0},W(c_s+z)).\end{aligned}\tag{2.4}$$

The case $W(0)=\infty$ can be proved similarly. Because $T(r,W)=m(r,W)+N(r,W)$ and from (2.4), then relation (c) follows immediately. Thus, (d) follows immediately from (c).

Similarly to Ref. [21], we give some definitions of algebroid function class, small algebroid function, etc. in a k-punctured complex plane as follows.

Definition 2.2. *Let $W(z)=\{(w_j(z),a)\}_{j=1}^{v}$ be a v-valued algebroid function in a k-punctured complex plane. The set of all algebroid mappings of $W(z)$ is denoted by Y_W. The set*

$$H_W:=\{h\circ W(z);h\in Y_W\}$$

is called the algebroid function class of $W(z)$.

Definition 2.3. *Set*

$$X_W:=\left\{f\in H_W;T_0(r,f)=o[T_0(r,W)](r\to+\infty,r\notin E_f)\right\},$$

where E_f is a real number set of finite linear measure depending on f. X_W is called the small algebroid function set of $W(z)$. The element in X_W is called the small algebroid function of $W(z)$.

Remark 2.1. *Note that the set X_W contains all the finite or infinite complex constants, all the small meromorphic functions and all the small algebroid functions.*

Definition 2.4. *For any $h_1, h_2 \in Y_W$, define*

1) Addition: $(h_1 + h_2) \circ W(z) = h_1 \circ W(z) + h_2 \circ W(z)$.

2) Subtraction: $(h_1 - h_2) \circ W(z) = h_1 \circ W(z) - h_2 \circ W(z)$.

3) Multiplication: $(h_1 \cdot h_2) \circ W(z) = (h_1 \circ W(z)) \cdot (h_2 \circ W(z))$.

4) Division: $(\frac{h_1}{h_2}) \circ W(z) = h_1 \circ W(z) \cdot \frac{1}{h_2} \circ W(z)$.

Thus, we have

Theorem 2.1. *Let $W(z) = \{(w_j(z), a)\}_{j=1}^{v}$ and $M(z) = \{(m_j(z), a)\}_{j=1}^{v} \in H_W$ be two v-valued algebroid functions. Then*

i) $T_0(r, W + M) \leq T_0(r, W) + T_0(r, M) + O(1)$;

ii) $T_0(r, W \cdot M) \leq T_0(r, W) + T_0(r, M) + O(1)$.

Proof. We assume that $\{w_j(z)\}_{j=1}^{v}$ and $\{m_j(z)\}_{j=1}^{v}$ are v simple-valued branches of $W(z)$ and $M(z)$, respectively, by cutting the k-punctured plane. Then

$$\begin{aligned}
m_0(r, W + M) =& \frac{1}{2\pi} \sum_{j=1}^{v} \int_0^{2\pi} \log^+ \left| w_j(re^{i\theta}) + m_j(re^{i\theta}) \right| d\theta \\
&+ \frac{1}{2\pi} \sum_{j=1}^{v} \sum_{s=1}^{k} \int_0^{2\pi} \log^+ \left| w_j(c_s + \frac{1}{r}e^{i\theta}) + m_j(c_s + \frac{1}{r}e^{i\theta}) \right| d\theta \\
&- \frac{1}{2\pi} \sum_{j=1}^{v} \int_0^{2\pi} \log^+ \left| w_j(r_0e^{i\theta}) + m_j(r_0e^{i\theta}) \right| d\theta \\
&- \frac{1}{2\pi} \sum_{j=1}^{v} \sum_{s=1}^{k} \int_0^{2\pi} \log^+ \left| w_j(c_s + \frac{1}{r_0}e^{i\theta}) + m_j(c_s + \frac{1}{r_0}e^{i\theta}) \right| d\theta.
\end{aligned}$$

Since for $j = 1, 2, \ldots, v$ and $s = 1, 2, \ldots, k$,

$$\int_0^{2\pi} \log^+ \left| w_j(r_0e^{i\theta}) \right| d\theta \leq O(1), \qquad \int_0^{2\pi} \log^+ \left| m_j(r_0e^{i\theta}) \right| d\theta \leq O(1),$$

and

$$\int_0^{2\pi} \log^+ \left| w_j(c_s + \frac{1}{r_0}e^{i\theta}) \right| d\theta \leq O(1), \qquad \int_0^{2\pi} \log^+ \left| m_j(c_s + \frac{1}{r_0}e^{i\theta}) \right| d\theta \leq O(1),$$

so, it follows

$$\begin{aligned}
m_0(r, W + M) =& \frac{1}{2\pi} \sum_{j=1}^{v} \left\{ \int_0^{2\pi} \log^+ \left| w_j(re^{i\theta}) \right| d\theta + \int_0^{2\pi} \log^+ \left| m_j(re^{i\theta}) \right| d\theta \right\} + O(1) \\
&+ \frac{1}{2\pi} \sum_{j=1}^{v} \sum_{s=1}^{k} \left\{ \int_0^{2\pi} \log^+ \left| w_j(c_s + \frac{1}{r}e^{i\theta}) \right| d\theta + \int_0^{2\pi} \log^+ \left| m_j(c_s + \frac{1}{r}e^{i\theta}) \right| d\theta \right\} \\
&- \frac{1}{2\pi} \sum_{j=1}^{v} \left\{ \int_0^{2\pi} \log^+ \left| w_j(r_0e^{i\theta}) \right| d\theta + \int_0^{2\pi} \log^+ \left| m_j(r_0e^{i\theta}) \right| d\theta \right\}
\end{aligned}$$

$$-\frac{1}{2\pi}\sum_{j=1}^{v}\sum_{s=1}^{k}\left\{\int_0^{2\pi}\log^+\left|w_j(c_s+\frac{1}{r_0}e^{i\theta})\right|d\theta+\int_0^{2\pi}\log^+\left|m_j(c_s+\frac{1}{r_0}e^{i\theta})\right|d\theta\right\}$$
$$\leq m_0(r,W)+m_0(r,M)+O(1), \tag{2.5}$$

and by using the argument as in (2.4), we have

$$N_0(r,W+M)=\frac{1}{v}\int_{r_0}^{r}\frac{n_{10}(t,W+M)}{t}dt+\frac{1}{v}\sum_{s=1}^{v}\int_{r_0}^{r}\frac{n_{1s}(\frac{1}{t},W(c_s+z)+M(c_s+z))}{t}dt$$
$$\leq N_0(r,W)+N_0(r,M). \tag{2.6}$$

Thus, from (2.5) and (2.6), (i) follows. By using the same argument as in (2.5) and (2.6), we can prove (ii) easily. □

In [30], the authors obtained some basic results of algebroid functions in a k-punctured plane as follows.

Theorem 2.2. *(the first fundamental theorem for algebroid function in the k-punctured planes). Let $W(z)$ be a v-valued algebroid function which is determined by* (2.1) *in a k-punctured plane Ω, and $a\in\mathbb{C}$, then for $r_0<r<+\infty$ we have*

$$m_0(r,a)+N_0(r,a)=T_0(r,W)+O(1).$$

Theorem 2.3. *Let $W(z)$ be a v-valued algebroid function which is determined by* (2.1) *in a k-punctured plane Ω, then*

$$N_x(r,W)\leq 2(v-1)T_0(r,W)+O(1).$$

Theorem 2.4. *(the second fundamental theorem for algebroid function in the k-punctured planes). Let $W(z)$ be a v-valued algebroid function which is determined by* (2.1) *in a k-punctured plane Ω, $a_t\in\overline{\mathbb{C}}(t=1,2,\ldots,p)$ are p distinct complex numbers, then we have*

$$(p-2v)T_0(r,W)\leq\sum_{t=1}^{p}N_0\left(r,\frac{1}{W-a_t}\right)-N_1(r,W)+S_0(r,W),$$

and

$$(p-2v)T_0(r,W)\leq\overline{N}_0\left(r,\frac{1}{W-a_t}\right)+S_0(r,W),$$

where $N_1(r,W)$ is the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau-1$, and

$$S_0(r,W)=m_0\left(r,\frac{W'}{W}\right)+\sum_{t=1}^{p}m_0\left(r,\frac{W'}{W-a_t}\right)+O(1).$$

Remark 2.2. *By Lemma 3 and Lemma 6 in [3] and using the same argument as in [6, 21], we can get the following conclusion about the remainder $S_0(r,W)$.*

$$S_0(r,W)=O(\log T_0(r,W))+O(\log r),$$

outside a set of finite linear measure as $r\to+\infty$.

3. The fundamental theorem for algebroid functions in the k-punctured plane concerning small algebroid functions

In this paper, the main purpose is to extend Theorem 2.4: when p distinct complex numbers $a_j(j = 1, 2, \dots, p)$ are replaced by p small algebroid functions $a_j(z)(j = 1, 2, \dots, p)$. We obtain

Theorem 3.1. *Suppose that $W(z) = \{(w_j(z), a)\}_{j=1}^{v}$ is a v-valued nonconstant algebroid function in a k-punctured plane Ω. $\{a_t(z)\}_{t=1}^{p} \subset X_W$ are $p > 2$ distinct small algebroid functions of $W(z)$. Then for any $\varepsilon \in (0, 1)$ and $r > r_0$, we have*

$$m_0(r, W) + \sum_{t=1}^{p} m_0(r, \frac{1}{W(z) - a_t(z)}) \le (2 + \varepsilon)T_0(r, W) + (2 + \varepsilon)N_x(r, W) + S_0(r, W), \tag{3.1}$$

where $S_0(r, W)$ is stated as in Remark 2.2.

Remark 3.1. *Since*

$$\begin{aligned} m_0\left(r, \frac{1}{W(z) - a_t(z)}\right) &\le T_0(r, W - a_t) - N_0\left(r, \frac{1}{W - a_t}\right) + O(1) \\ &\le T_0(r, W) - N_0\left(r, \frac{1}{W - a_t}\right) + S_0(r, W), \end{aligned}$$

and combining with Theorem 3.1, we have

$$(p - 1 - \varepsilon)T_0(r, W) \le N_0(r, W) + \sum_{t=1}^{p} N_0(r, \frac{1}{W - a_j}) + (2 + \varepsilon)N_x(r, W) + S_0(r, W). \tag{3.2}$$

Remark 3.2. *By combining with Theorem 2.3 and Theorem 3.1, we have*

$$(p - 4v + 3 - \varepsilon)T_0(r, W) \le N_0(r, W) + \sum_{t=1}^{p} N_0(r, \frac{1}{W - a_j}) + S_0(r, W). \tag{3.3}$$

To prove this theorem, we require some lemmas as follows.

Lemma 3.1. *(see [21]). Suppose that $W(z)$ is a v-valued nonconstant algebroid function and n is a positive integer. Then $\frac{W^{(n)}}{W}$ is the differential polynomial of $\frac{W'}{W}$.*

Lemma 3.2. *(see [21]). Let $f_1, f_2, \dots, f_q, g \in H_W$. Then*

$$W(f_1, f_2, \dots, f_q) := \begin{vmatrix} f_1 & f_2 & \cdots & f_q \\ f_1' & f_2' & \cdots & f_q' \\ \cdots & \cdots & & \\ f_1^{(q-1)} & f_2^{(q-1)} & \cdots & f_q^{(q-1)} \end{vmatrix} = g^q W\left(\frac{f_1}{g}, \frac{f_2}{g}, \cdots, \frac{f_q}{g}\right).$$

Lemma 3.3. *(see [21]). Suppose that $A_u = \{a_t := a_t(z)\}_{t=1}^{u} \subset X_W$ are $u \ge 1$ distinct small algebroid functions. Let $L(\chi, A_u)$ denote the vector space spanned by finitely many products $a_1^{p_1} a_2^{p_2} \cdots a_u^{p_u}$, where integer $p_t \ge 0 (t = 1, 2, \dots, u)$ and $\sum_{t=1}^{u} p_t = \chi (\ge 1)$. Let $dimL(\chi, A_u)$ denote the dimension of the vector space $L(\chi, A_u)$. Then for any $\varepsilon > 0$, there exists $\chi \ge 1$ such that*

$$\frac{dimL(\chi + 1, A_u)}{dimL(\chi, A_u)} < 1 + \varepsilon.$$

Lemma 3.4. *Suppose that $W(z) = \{(w_j(z), a)\}$ is a v-valued nonconstant algebroid function in a k-punctured plane Ω, and $\{a_t(z)\}_{t=0}^{p} \subset X_W$ are p distinct small algebroid function with respect to $W(z)$. Then for any $r > r_0$, we have*

$$\left| m_0\left(r, \sum_{t=1}^{p} \frac{1}{W(z) - a_t(z)}\right) - \sum_{t=1}^{p} m_0\left(r, \frac{1}{W(z) - a_t(z)}\right) \right| = S_0(r, W),$$

where $S_0(r, W)$ is state as in Remark 2.2, E is a positive real number set of finite linear measure.

Proof. We can cut $W(z)$ into v single-valued branch $\{w_j(z)\}_{j=1}^{v}$ in a k-punctured plane, by using the tree Y through all branch points of $W(z)$. Accordingly, we also cut every $a_t(z)$ into v single-valued branch $\{a_{t,j}(z)\}_{j=1}^{v}$ in a k-punctured plane. For any $j = 1, 2, \ldots, v$, set

$$F_j(z) := \sum_{t=1}^{p} \frac{1}{w_j(z) - a_{t,j}(z)}. \tag{3.4}$$

Since $a_t(z) \in X_W$, then we have $m_0(r, a_t) \leq T_0(r, a_t) = o(T_0(r, W))$ for $t = 1, 2, \ldots, p$ and $r > r_0$. Hence, it follows

$$\begin{aligned}
m_0\left(r, \sum_{t=1}^{p} \frac{1}{W(z) - a_t(z)}\right) &= \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \sum_{t=1}^{p} \frac{1}{w_j(re^{i\theta}) - a_t(re^{i\theta})} \right| d\theta \\
&\quad + \sum_{j=1}^{v} \frac{1}{2\pi} \sum_{s=1}^{k} \int_0^{2\pi} \log^+ \left| \sum_{t=1}^{p} \frac{1}{w_j(c_s + \frac{1}{r}e^{i\theta})) - a_t(c_s + \frac{1}{r}e^{i\theta})} \right| d\theta \\
&\quad - \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \sum_{t=1}^{p} \frac{1}{w_j(r_0e^{i\theta}) - a_t(r_0e^{i\theta})} \right| d\theta \\
&\quad - \sum_{j=1}^{v} \frac{1}{2\pi} \sum_{s=1}^{k} \int_0^{2\pi} \log^+ \left| \sum_{t=1}^{p} \frac{1}{w_j(c_s + \frac{1}{r_0}e^{i\theta}) - a_t(c_s + \frac{1}{r_0}e^{i\theta})} \right| d\theta \\
&\leq \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{w_j(re^{i\theta}) - a_t(re^{i\theta})} \right| d\theta \\
&\quad + \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \sum_{s=1}^{k} \int_0^{2\pi} \log^+ \left| \frac{1}{w_j(c_s + \frac{1}{r}e^{i\theta})) - a_t(c_s + \frac{1}{r}e^{i\theta})} \right| d\theta \\
&\quad - \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{w_j(r_0e^{i\theta}) - a_t(r_0e^{i\theta})} \right| d\theta \\
&\quad - \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \sum_{s=1}^{k} \int_0^{2\pi} \log^+ \left| \frac{1}{w_j(c_s + \frac{1}{r_0}e^{i\theta}) - a_t(c_s + \frac{1}{r_0}e^{i\theta})} \right| d\theta \\
&\quad + o(T_0(r, W)) + K \log p \\
&\leq \sum_{t=1}^{p} m_0\left(r, \frac{1}{W(z) - a_t(z)}\right) + S_0(r, W).
\end{aligned} \tag{3.5}$$

Next, we will estimate the lower bound of $\sum_{j=1}^{v} m_0(r, F_j)$, for any $z \in \Omega$, set

$$\delta_j(z) := \min_{1\le t<u\le p}\{|a_{t,j}(z) - a_{u,j}(z)|\} \ge 0.$$

Note that $\delta_j(z)$ is the function of $z \in \Omega$, by the uniqueness theorem, its zeros must be isolated. Take arbitrary $z \in \{z : \delta_j(z) \ne 0\}$.

Case 1. If for any $t \in \{1, 2, \dots, p\}$,

$$|w_j(z) - a_{t,j}(z)| \ge \frac{\delta_j(z)}{2p},$$

then we have

$$\sum_{t=1}^{p} \log^+ \frac{1}{|w_j(z) - a_{t,j}(z)|} \le p \log^+ \frac{2p}{\delta_j(z)}. \tag{3.6}$$

Case 2. If there exists some $u \in \{1, 2, \dots, p\}$ such that

$$|w_j(z) - a_{u,j}(z)| \le \frac{\delta_j(z)}{2p}. \tag{3.7}$$

Thus, for $t \ne u$, it follows

$$\begin{aligned}|w_j(z) - a_{t,j}(z)| &\ge |a_{u,j}(z) - a_{t,j}(z)| - |w_j(z) - a_{u,j}(z)|\\ &\ge \delta_j(z) - \frac{\delta_j(z)}{2p} = \frac{2p-1}{2p}\delta_j(z).\end{aligned}$$

Hence from (3.7) we have

$$\frac{1}{|w_j(z) - a_{t,j}(z)|} \le \frac{1}{2p-1}\frac{2p}{\delta_j(z)} \tag{3.8}$$

$$< \frac{1}{2p-1}\frac{1}{|w_j(z) - a_{u,j}(z)|}. \tag{3.9}$$

Thus, from (3.4) and (3.9) we have

$$\begin{aligned}|F_j(z)| &\ge \frac{1}{|w_j(z) - a_{u,j}(z)|} - \sum_{t\ne u}\frac{1}{|w_j(z) - a_{t,j}(z)|}\\ &\ge \frac{1}{|w_j(z) - a_{u,j}(z)|} - \frac{p-1}{2p-1}\frac{1}{|w_j(z) - a_{u,j}(z)|} > \frac{1}{2|w_j(z) - a_{u,j}(z)|},\end{aligned}$$

and it follows from (3.8) that

$$\begin{aligned}\log^+|F_j(z)| &> \log^+\frac{1}{|w_j(z) - a_{u,j}(z)|} - \log 2\\ &= \sum_{t=1}^{p}\log^+\frac{1}{|w_j(z) - a_{t,j}(z)|} - \sum_{t\ne u}\log^+\frac{1}{|w_j(z) - a_{t,j}(z)|} - \log 2\\ &\ge \sum_{t=1}^{p}\log^+\frac{1}{|w_j(z) - a_{t,j}(z)|} - \sum_{t\ne u}\log^+\frac{2p}{(2p-1)\delta_j(z)} - \log 2\end{aligned}$$

$$> \sum_{t=1}^{p} \log^+ \frac{1}{|w_j(z) - a_{t,j}(z)|} - p \log^+ \frac{2p}{\delta_j(z)} - \log 2. \tag{3.10}$$

Hence form (3.6) and (3.10), combining Case 1 and Case 2, we have

$$\log^+ |F_j(z)| > \sum_{t=1}^{p} \log^+ \frac{1}{|w_j(z) - a_{t,j}(z)|} - p \log^+ \frac{2p}{\delta_j(z)} - \log 2. \tag{3.11}$$

By definition, for any $z \in \{z : \delta_j(z) \neq 0\}$, there exists $t(z) \neq u(z)$ such that $\delta_j(z) = a_{t(z),j}(z) - a_{u(z),j}(z)$. Hence we have

$$\frac{1}{\delta_j(z)} = \frac{1}{|a_{t(z),j}(z) - a_{u(z),j}(z)|} \leq \sum_{1 \leq t < u \leq p} \frac{1}{|a_{t,j}(z) - a_{u,j}(z)|}.$$

Thus,

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(re^{i\theta})} &\leq \sum_{1 \leq t < u \leq p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{|a_{t,j}(re^{i\theta}) - a_{u,j}(re^{i\theta})|} + O(1) \\
&= \sum m(r, a_{t,j}(z) - a_{u,j}(z)) + O(1) \\
&\leq \sum m(r, a_{t,j}) + m(r, a_{u,j}) + O(1),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(c_s + \frac{1}{r}e^{i\theta})} &\leq \sum_{1 \leq t < u \leq p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{|a_{t,j}(c_s + \frac{1}{r}e^{i\theta}) - a_{u,j}(c_s + \frac{1}{r}e^{i\theta})|} + O(1) \\
&= \sum m(r, a_{t,j}(c_s + \frac{1}{r}e^{i\theta})) - a_{u,j}(c_s + \frac{1}{r}e^{i\theta})) + O(1) \\
&\leq \sum m(r, a_{t,j}(c_s + \frac{1}{r}e^{i\theta})) + m(r, a_{u,j}(c_s + \frac{1}{r}e^{i\theta})) + O(1),
\end{aligned} \tag{3.13}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(r_0e^{i\theta})} = O(1), \quad \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(c_s + \frac{1}{r_0}e^{i\theta})} = O(1). \tag{3.14}$$

Hence from (3.12)–(3.14), we have

$$\begin{aligned}
&\sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(re^{i\theta})} + \sum_{j=1}^{v} \sum_{s=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(c_s + \frac{1}{r}e^{i\theta})} \\
&- \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(r_0e^{i\theta})} - \sum_{j=1}^{v} \sum_{s=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{d\theta}{\delta_j(c_s + \frac{1}{r_0}e^{i\theta})} \\
\leq &\sum T_0(r, a_{t,j}) + T_0(r, a_{u,j}) + O(1) = S_0(r, W).
\end{aligned} \tag{3.15}$$

Substituting $z = re^{i\theta}$, $z = c_s + \frac{1}{r}e^{i\theta}$, $z = r_0e^{i\theta}$ and $z = c_s + \frac{1}{r_0}e^{i\theta}$ into (3.11), respectively, and integrating on θ from 0 to 2π, by (3.15) we have

$$m_0\left(r, \sum_{t=1}^{p} \frac{1}{W(z) - a_t(z)}\right)$$

$$
\begin{aligned}
&= \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F_j(re^{i\theta})| d\theta + \sum_{j=1}^{v} \frac{1}{2\pi} \sum_{s=1}^{k} \int_0^{2\pi} \log^+ |F_j(c_s + \frac{1}{r} e^{i\theta})| d\theta \\
&\quad - \sum_{j=1}^{v} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |F_j(r_0 e^{i\theta})| d\theta - \sum_{j=1}^{v} \frac{1}{2\pi} \sum_{s=1}^{k} \int_0^{2\pi} \log^+ |F_j(c_s + \frac{1}{r_0} e^{i\theta})| d\theta \\
&\geq \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|w_j(re^{i\theta}) - a_{t,j}(re^{i\theta})|} d\theta \\
&\quad + \sum_{j=1}^{v} \sum_{t=1}^{p} \sum_{s=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|w_j(c_s + \frac{1}{r} e^{i\theta}) - a_{t,j}(c_s + \frac{1}{r} e^{i\theta})|} d\theta \\
&\quad - \sum_{j=1}^{v} \sum_{t=1}^{p} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|w_j(r_0 e^{i\theta}) - a_{t,j}(r_0 e^{i\theta})|} d\theta \\
&\quad - \sum_{j=1}^{v} \sum_{t=1}^{p} \sum_{s=1}^{k} \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|w_j(c_s + \frac{1}{r_0} e^{i\theta}) - a_{t,j}(c_s + \frac{1}{r_0} e^{i\theta})|} d\theta \\
&\quad - \sum_{j=1}^{v} \frac{p}{2\pi} \int_0^{2\pi} \log^+ \frac{2p d\theta}{\delta_j(re^{i\theta})} - \sum_{j=1}^{v} \sum_{s=1}^{k} \frac{p}{2\pi} \int_0^{2\pi} \log^+ \frac{2p d\theta}{\delta_j(c_s + \frac{1}{r} e^{i\theta})} \\
&\quad + \sum_{j=1}^{v} \frac{p}{2\pi} \int_0^{2\pi} \log^+ \frac{2p d\theta}{\delta_j(r_0 e^{i\theta})} + \sum_{j=1}^{v} \sum_{s=1}^{k} \frac{p}{2\pi} \int_0^{2\pi} \log^+ \frac{2p d\theta}{\delta_j(c_s + \frac{1}{r_0} e^{i\theta})} + O(1) \\
&\geq \sum_{t=1}^{p} m_0\left(r, \frac{1}{W(z) - a_j(z)}\right) + S_0(r, W). \qquad (3.16)
\end{aligned}
$$

Therefore, this lemma is proved by (3.5) and (3.16). □

The proof of Theorem 3.1: We use the method of [21] to complete the proof of Theorem 3.1. Let $A_p = \{a_1, a_2, \ldots, a_p\}$, and let $L(\chi, A_p)$ denote the vector space spanned by finitely many products $a_1^{n_1} a_2^{n_2} \cdots a_p^{n_p}$, where $\sum_{t=1}^{p} n_t = \chi$ and $n_t \geq 0 (t = 1, 2, \ldots, p)$. Suppose that $dimL(\chi, A_p) = n$, for given χ. Thus, let $b_1, b_2, \ldots, b_n$ denote a basis of $L(\chi, A_p)$. Assume that $dimL(\chi + 1, A_p) = l$, we also assume that $B_1, B_2, \ldots, B_l$ denote a basis of $L(\chi + 1, A_p)$. Thus, by Lemma 3.3, for any $\varepsilon > 0$, there exists some χ such that

$$1 \leq \frac{dimL(\chi + 1, A_p)}{dimL(\chi, A_p)} = \frac{l}{n} < 1 + \varepsilon. \qquad (3.17)$$

Let

$$P(W) := W(B_1, B_2, \ldots, B_l, Wb_1, Wb_2, \ldots, Wb_n).$$

Because $B_1, B_2, \ldots, B_l, Wb_1, Wb_2, \ldots, Wb_n$ are linearly independent, $P(W) \not\equiv 0$. From the definition of the Wronskian determinant, it follows

$$P(W) = \sum C_q(z) \prod_{t=0}^{n+l-1} (W^{(t)})^{q_t} = W^n \sum C_q(z) \prod_{t=0}^{n+l-1} \left(\frac{W^{(t)}}{W}\right)^{q_t}. \qquad (3.18)$$

Because $m_0\left(r, \frac{W'}{W}\right) = S_0(r, W)$, we get

$$m_0(r, P(W)) \leq nm_0(r, W) + S_0(r, W). \tag{3.19}$$

From Lemma 3.2, we have

$$W(B_1, B_2, \ldots, B_l, Wb_1, \ldots, Wb_n) = P(W) = W^{n+l} \cdot W\left(\frac{B_1}{W}, \ldots, \frac{B_l}{W}, b_1, \ldots, b_n\right). \tag{3.20}$$

(i) Suppose that z_0 is a pole of $W(z)$, $(q(z), D(z_0, \delta))$ is a τ-fold pole element with $\lambda(\lambda \geq 1)$ sheet in z_0, where $D(z_0, \delta) = \{0 < |z-z_0| < \delta\}$. Let $\delta > 0$ be sufficiently small, such that $q(z)$ has no poles or zeros in $D(z_0, \delta)$. Suppose that function elements $(B_{1,q(z)}, D(z_0, \delta)), \ldots, (B_{l,q(z)}, D(z_0, \delta))$ of $B_1, \ldots, B_l, b_1, \ldots, b_n$ are corresponding to the pole function $(q(z), D(z_0, \delta))$ respectively. Thus, it follows from Lemma 3.2 that

$$\begin{aligned} P_{q(z)}(W) &:= W(B_{1,q(z)}, \ldots, B_{l,q(z)}, q(z)b_{1,q(z)}, \ldots, q(z)b_{n,q(z)}) \\ &= q^{n+l}(z)W\left(\frac{B_{1,q(z)}}{q(z)}, \ldots, \frac{B_{l,q(z)}}{q(z)}, b_{1,q(z)}, \ldots, b_{n,q(z)}\right). \end{aligned}$$

By observing the right hand side of the above equality, if z_0 is a τ-fold pole of $q(z)$, it can be seen that outside the poles of the small algebroid functions $\{B_i\}, \{b_t\}$, the order of pole of $P(W)$ at $(q(z), z_0)$ is $(n + l)\tau$. If z_0 is a zero of $q(z)$, by the left of the above equality, it can be seen that outside the poles of the small algebroid functions $\{B_i\}, \{b_t\}$, $(q(z), z_0)$ is not the pole of $P(W)$.

(ii) Suppose that z_0 is a branch point of $W(z)$, $(p(z), D(z_0, \delta))$ is a λ sheet algebraic function element in z_0, but not a pole element. Let $\delta > 0$ be sufficiently small, such that $p(z)$ has no poles or zeros in $D(z_0, \delta)$. If its corresponding derivative function element $(p'(z), D(z_0, \delta))$ is the pole element, then its order is at most $\lambda - 1$. Suppose that function elements $(B_{1,p(z)}, D(z_0, \delta))$,..., $(B_{l,p(z)}, D(z_0, \delta))$, $(b_{1,p(z)}, D(z_0, \delta)), \ldots, (b_{n,p(z)}, D(z_0, \delta))$ of $B_1, \ldots, B_l, b_1, \ldots, b_n$ are corresponding to the pole function $(p(z), D(z_0, \delta))$ respectively. Denote

$$W_t(B'_{1,p(z)}, \ldots, B'_{l,p(z)}, (p(z)b_{1,p(z)})', \ldots, (p(z)b_{n,p(z)})')$$

to be the algebraic cofactor that generate by moving the first line and the t column of the $n + l$ order determinant

$$P_{p(z)}(W) := W_t(B_{1,p(z)}, \ldots, B_{l,p(z)}, (p(z)b_{1,p(z)}), \ldots, (p(z)b_{n,p(z)})).$$

In view of Lemma 3.2, it follows

$$\begin{aligned} P_{p(z)}(W) &= \sum_{t=1}^{l}[B_{t,p(z)}W_t(B'_{1,p(z)}, \ldots, B'_{l,p(z)}, (p(z)b_{1,p(z)})', \ldots, (p(z)b_{n,p(z)})')] \\ &\quad + \sum_{t=l+1}^{n+l}[p(z)b_{t,p(z)} \cdot W_t(B'_{1,p(z)}, \ldots, B'_{l,p(z)}, (p(z)b_{1,p(z)})', \ldots, (p(z)b_{n,p(z)})')] \\ &= \sum_{t=1}^{l} B_{t,p}(p'b_{t,p} + pb'_{t,p})^{n+l-1} \cdot W_t\left(\frac{B'_{1,p}}{(pb_{t,p})'}, \ldots, \frac{B'_{l,p}}{(pb_{t,p})'}, \frac{(pb_{1,p})'}{(pb_{t,p})'}, \ldots, \frac{(pb_{n,p})'}{(pb_{t,p})'}\right) \end{aligned}$$

$$+\sum_{t=l+1}^{n+l} pb_{t,p}((p'b_{t,p}+pb'_{t,p})^{n+l-1}\cdot W_t\left(\frac{B'_{1,p}}{(pb_{t,p})'},\ldots,\frac{B'_{l,p}}{(pb_{t,p})'},\frac{(pb_{1,p})'}{(pb_{t,p})'},\ldots,\frac{(pb_{n,p})'}{(pb_{t,p})'}\right).$$

Thus outside the poles of the small algebroid functions $\{B_i\},\{b_t\}$, the order of pole of $P(W)$ at $(q(z),D(z_0,\delta))$ is at most $(\lambda-1)(n+l-1)$.

(iii) Suppose that z_0 is a branch point of $W(z)$, $(q(z),D(z_0,\delta))$ is a λ sheet algebraic function element in z_0, but not a pole element, and also not the pole of the $t-1$ order derivative $q^{(t-1)}(z)$. If z_0 is a pole of the t order derivative $q^{(t)}(z)$, then its order is at most $\lambda-1$. Similar to the above argument, outside of the poles of $\{B_i\},\{b_t\}$, the order of pole of $P(W)$ at $(q(z),D(z_0,\delta))$ at most $(\lambda-1)(n+l-1)$.

Thus, in view of (i)-(iii), it yields

$$N_0(r,P(W))\le(n+l)N_0(r,W)+(n+l-1)N_x(r,W)+S_0(r,W).$$

From (3.20) and Theorem 2.1, we have

$$T_0(r,P(W))\le nT_0(r,W)+lN_0(r,W)+(n+l-1)N_x(r,W)+S_0(r,W). \tag{3.21}$$

Let a be a linear combination of $\{a_t\}$, then

$$\begin{aligned}P(W-a)&=W(B_1,B_2,\ldots,B_l,Wb_1-ab_1,Wb_2-ab_2,\ldots,Wb_n-ab_n)\\&=W(B_1,B_2,\ldots,B_l,Wb_1,Wb_2,\ldots,Wb_n)\pm\sum W(B_1,B_2,\ldots,B_l,\ldots),\end{aligned}$$

where the element "$\ldots$" behind B_l in $\sum W(B_1,B_2,\ldots,B_l,\ldots)$ consists of ab_t. However ab_t and $B_1,B_2,\ldots,B_l$ are linearly dependent, thus we have $\sum W(B_1,B_2,\ldots,B_l,\ldots)=0$. So, we obtain

$$P(W-a)=P(W). \tag{3.22}$$

Thus, it follows from Lemma 3.1 and (3.18) that

$$P(W)=W^n\cdot Q\left(\frac{W'}{W}\right), \tag{3.23}$$

where $Q\left(\frac{W'}{W}\right)$ is the differential polynomial of $\frac{W'}{W}$. Let

$$V_t:=W-a_t,\ \ Q_t:=Q\left(\frac{V'_t}{V_t}\right),\quad t=1,2,\ldots,p.$$

From (3.22) and (3.23), it follows $P(W)=P(V_t)=V_t^nQ_t$, that is,

$$\frac{1}{(W-a_t)^n}=\frac{Q_t}{P(W)}.$$

Therefore, we have

$$\frac{1}{|W-a_t|}=\frac{|Q_t|^{n^{-1}}}{|P(W)|^{n^{-1}}}. \tag{3.24}$$

Set

$$F(z):=\sum_{t=1}^{p}\frac{1}{W(z)-a_j(z)},$$

thus by Lemma 3.4 it follows

$$m_0(r,F) = m_0\left(r, \sum_{t=1}^{p} \frac{1}{W(z)-a_t(z)}\right) = \sum_{t=1}^{p} m_0\left(r, \frac{1}{W(z)-a_t(z)}\right) + S_0(r,W), \tag{3.25}$$

and from (30) we have

$$|F(z)| \leq \sum_{t=1}^{p} \frac{1}{|W(z)-a_t(z)|} \leq \frac{1}{|P(W)|^{n^{-1}}} \sum_{t=1}^{p} |Q_t|^{n^{-1}}.$$

Hence from the above inequality and Theorem 2.1, we have

$$\begin{aligned}
m_0(r,F) &\leq \frac{1}{n} m_0\left(r, \frac{1}{P(W)}\right) + \frac{1}{n}\sum_{t=1}^{p} m_0(r,Q_t) + O(1) \\
&\leq \frac{1}{n} T_0(r,P(W)) - \frac{1}{n} N_0\left(r, \frac{1}{P(W)}\right) + S_0(r,W) \\
&\leq T_0(r,W) + \frac{l}{n} N_0(r,W) + \frac{n+l-1}{n} N_x(r,W) - \frac{1}{n} N_0\left(r, \frac{1}{P(W)}\right) + S_0(r,W) \\
&\leq T_0(r,W) + \frac{l}{n} N_0(r,W)(2+\varepsilon) N_x(r,W) - \frac{1}{n} N_0\left(r, \frac{1}{P(W)}\right) + S_0(r,W).
\end{aligned} \tag{3.26}$$

Hence from (23), (31) and (32), we have

$$\begin{aligned}
& m_0(r,W) + \sum_{t=1}^{p} m_0\left(r, \frac{1}{W(z)-a_t(z)}\right) \\
\leq & \frac{l}{n} m_0(r,W) + m_0(r,F) \\
\leq & (1+\frac{l}{n}) T_0(r,W) + (2+\varepsilon) N_x(r,W) + S_0(r,W) \\
< & (2+\varepsilon) T_0(r,W) + (2+\varepsilon) N_x(r,W) + S_0(r,W),
\end{aligned} \tag{3.27}$$

that is, (7) holds.

Therefore, this completes the proof of Theorem 3.1.

4. Conclusions

Theorem 3.1 can be called as the fundamental theorem for algebroid functions in the k-punctured plane concerning small algebroid functions, which is important in the study on the uniqueness and the value distribution of algebroid functions in the k-punctured plane. From the conclusion of Theorem 3.1, a very natural question is raised: can the constant ε be removed?

Conflict of interest

The authors declare that they have no competing interests.

Author's contributions

Conceptualization, H. Y. Xu; writing-original draft preparation, H.Y. Xu; writing-review and editing, H. Y. Xu, Y. X. Chen and Z. J. Wu; funding acquisition, H. Y. Xu, Z. J. Wu and J. Liu.

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