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Research article

Square-mean asymptotically almost periodic solutions of second order nonautonomous stochastic evolution equations

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Abstract: In this paper, we study the existence of square-mean asymptotically almost periodic mild solutions for a class of second order nonautonomous stochastic evolution equations in Hilbert spaces. By using the principle of Banach contractive mapping principle, the existence and uniqueness of square-mean asymptotically almost periodic mild solutions of the equation are obtained. To illustrate the abstract result, a concrete example is given.

Keywords: square-mean asymptotically almost periodic function; mild solution; second order nonautonomous stochastic evolution equations
Mathematics Subject Classification: 60H15, 47D99

1. Introduction

The concept of asymptotically almost periodicity was introduced by Fréchet [1] in the early 1940s. The study of almost periodic solutions and asymptotically almost periodic solutions of differential equations have become a hot spot in the qualitative theory of differential equations [2–13]. Huang [13] established the asymptotically almost periodic solutions of the delayed Nicholson-type system involving patch structure

$$x_i'(t) = -a_{ii}(t)x_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{m}\beta_{ij}(t)x_i(t-\tau_{ij}(t))e^{-\gamma_{ij}(t)x_i(t-\tau_{ij}(t))}.$$

with weaker conditions.

In recent years, some scholars have established the asymptotic almost periodic theories in probability to study stochastic processes. These theories have good applications prospect in statistics, mathematical physics, mechanics and mathematical biology. Cao [14] studied the asymptotically almost periodic solutions of first order stochastic functional differential equation

$$dx(t) = (Ax(t) + F(t, x(t), x_t))dt + G(t, x(t), x_t)dW(t),\ \ t \in R$$

Where $A : D(A) \subset L^2(P, H) \to L^2(P, H)$ generates strongly continuous semigroups $\{T(t)\}_{t\geq 0}$. $W(t)$ is a Q-Wiener process with covariance operator Q whose value is taken on $L^2(P, H)$.

Liu [15] studied the asymptotically almost periodic mild solutions for the class of stochastic functional differential equations

$$dx(t) = (A(t)x(t) + F(t, x(t), x_t))dt + G(t, x(t), x_t)dW(t), \ t \in R$$

where $A(t) : D(A) \subset L^2(P, H) \to L^2(P, H)$ can display the center flow. $W(t)$ is a certain Q-Wiener process with covariance operator Q whose value is taken on $L^2(P, H)$.

On the other hand, the second order stochastic differential equation is the correct model of continuous time, which can be used to explain the synthesis process of making it into continuous time. McKibben [16] first established the second order damped functional stochastic evolution equation. In addition, McKibben [17] studied the existence and uniqueness of mild solutions for a class of second order neutral stochastic evolution equations with finite delay. Since then, it has attracted people's attention in many literatures, such as [18–22]. The existence of solutions for the second order abstract Cauchy problem is closely related to the concept of cosine function. Research on abstract second order differential equations controlled by evolutionary operators $\{U(t, s) : t, s \in J\}$ was developed by Kozak. Kozak [23] has proved that homogeneous equation

$$u''(t) = A(t)u(t), \ t \in J$$

with

$$u(s) = x, u'(s) = y$$

exists a mild solution $u(t) = -\frac{\partial}{\partial s}U(t, s)x + U(t, s)y + \int_s^t U(t, \xi) f(\xi)d\xi$.

Various methods for determining the existence of evolution operators generated by the family of $\{A(t) : t \in J\}$ can be found in references [24,25]. It is a better way to study the second order differential system directly instead of transforming it into the first order system.

Recently, Ren [26] established the existence and uniqueness of mild solutions to the following second order nonautonomous neutral stochastic evolution equations with infinite delay, which are driven by standard cylindrical Wiener process and independent cylindrical fractional Brownian motion.

$$d[y'(t) - f(t, y_t)] = [A(t)y(t)dt + g(t, y_t)]dt + h(t, y_t)dW(t) + \sigma(t)dB_Q^H(t), \ t \in I = [0, T]$$

and

$$y_0 = \phi \in \mathcal{B}, y'(0) = \xi.$$

The existence of asymptotically almost periodic solutions for second order nonautonomous stochastic evolution equations is an untreated topic. Under the stimulation of these works and certain conditions, and by using the Banach contraction mapping principle and the evolution operator theory, this paper established the existence and uniqueness of square-mean asymptotically almost periodic mild solutions to the following second order nonautonomous stochastic evolution equations

$$dx'(t) = A(t)x(t)dt + F(t, x(t))dt + G(t, x(t))dW(t), \ t \in R^+ = [0, +\infty) \tag{1.1}$$

with

$$x(0) = x_0, x'(0) = x_1 \tag{1.2}$$

in a real separable Hilbert space, where $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that generate an evolution operators $\{U(t,s)\}_{t,s\geq 0}$, and $\{W(t)\}_{t\geq 0}$ is a Q-Wiener process. Here F, G are appropriate functions specified later.

The structure of this paper is as follows. In Section 2, we introduce the concepts of evolution operator, square mean asymptotically almost periodic stochastic process, and give some properties and Lemmas of them. In Section 3, we obtain the existence and uniqueness of the square-mean asymptotically almost periodic mild solution for the second order nonautonomous stochastic evolution equation. In Section 4, we give an example to illustrate our main results.

2. Preliminaries

In this section, we give some definitions, basic properties and Lemmas, which will be used in the sequel. As in [5–10, 27–29], two real separable Hilbert spaces are represented by $(H, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(K, \|\cdot\|_K, \langle\cdot,\cdot\rangle)$. Denote the complete probability space by (Ω, F, P) . The symbol $L^2(P,H)$ denotes the spatial variable x of all random variables with the value of H, such that

$$E\|x\|^2 = \int_\Omega \|x\|^2 dP < \infty.$$

For $x \in L^2(P,H)$, let

$$\|x\|_2 = \left(\int_\Omega \|x\|^2 dP\right)^{\frac{1}{2}}.$$

Then it is a Banach space equipped with the norm $\|\cdot\|_2$.

Definition 2.1 (see [5]) A stochastic process $x : R \to L^2(P,H)$ is said to be continuous in the square-mean sense if

$$\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0, \text{ for all s} \in R.$$

Definition 2.2 (see [5]) Let $x : R \to L^2(P,H)$ be continuous in the square-mean sense. x is said to be square-mean almost periodic if for each $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains at least a number τ for which

$$\sup_{t\in R} E\|x(t+\tau) - x(t)\|^2 < \varepsilon.$$

The collection of all such functions will be denoted by $AP(L^2(P,H))$. $AP(L^2(P,H))$ is a Banach space when it is equipped with the norm $\|x\|_\infty = \sup_{t\in R}(E\|x(t)\|^2)^{\frac{1}{2}}$.

Definition 2.3 (see [5]) A continuous function $f : R \times L^2(P,H) \to L^2(P,H)$, $(t,x) \to f(t,x)$ which is jointly continuous, is said to be square-mean almost periodic in $t \in R$ uniformly for all $x \in K$, where K is compact subset of $L^2(P,H)$, if for any $\varepsilon > 0$, there exists $l(\varepsilon, K) > 0$ such that any interval of length $l(\varepsilon, K)$ contains at least a number τ for which

$$\sup_{t\in R} E\|f(t+\tau, x) - f(t,x)\|^2 < \varepsilon$$

for each stochastic process $x : R \to K$.

The set of all these functions is represented by $AP(R \times L^2(P,H), L^2(P,H))$.

The notation $C_0(R^+, L^2(P,H))$ denotes the set of all continuous stochastic processes φ from R^+ into $L^2(P,H)$, such that $\lim\limits_{t\to+\infty} E\|\varphi(t)\|^2 = 0$. Similarly, we use $C_0(R^+ \times L^2(P,H), L^2(P,H))$ to denote the space of all continuous functions $\phi : R^+ \times L^2(P,H) \to L^2(P,H)$ such that $\lim\limits_{t\to+\infty} E\|\phi(t,x)\|^2 = 0$, uniformly for x in any compact subset of $L^2(P,H)$.

Definition 2.4 (see [14]) A stochastic process $f : R^+ \to L^2(P,H)$ is said to be square-mean asymptotically almost periodic if it can be decomposed as $f = g + h$, where g is square-mean almost periodic function and $h \in C_0(R^+, L^2(P,H))$.

By $AAP(R^+, L^2(P,H))$ we denote the collection of all such functions.

Definition 2.5 (see [14]) A stochastic process $f : R^+ \times L^2(P,H) \to L^2(P,H)$ is said to be square-mean asymptotically almost periodic in t, uniformly for x in compact subset K of $L^2(P,H)$, if it can be decomposed as $f = g + h$, where g is square-mean almost periodic function and $h \in C_0(R^+ \times L^2(P,H), L^2(P,H))$.

Denote by $AAP(R^+ \times L^2(P,H), L^2(P,H))$ the collection of all such functions.

The following Lemma generalizes Theorem 5 of [2]. It can be proved in an analogous way.

Lemma 2.6 A continuous function $f : R^+ \to L^2(P,H)$ is square-mean asymptotically almost periodic if and only if, for every $\varepsilon > 0$, there exists $L(\varepsilon, E\|f\|^2, L^2(P,H)) > 0$ and a relatively dense subset of R^+, denoted by $\mathrm{T}(\varepsilon, E\|f\|^2, L^2(P,H))$, such that $E\|f(t+\tau) - f(t)\|^2 < \varepsilon$ for every $t \geq L(\varepsilon, E\|f\|^2, L^2(P,H))$ and every $\tau \in \mathrm{T}(\varepsilon, E\|f\|^2, L^2(P,H))$.

The following Lemmas can be obtained directly from [14].

Lemma 2.7 $(AAP(R^+, L^2(P,H)), \|\cdot\|_\infty)$ is a Banach space with the norm given by

$$\|x\|_\infty = \sup_{t\in R^+} \|x(t)\|_2 = \sup_{t\in R^+} (E\|x(t)\|^2)^{\frac{1}{2}}.$$

Let $K \subset L^2(P,H)$. We denote by $C_K(R^+ \times L^2(P,H), L^2(P,H))$ the set of all the functions $f : R^+ \times L^2(P,H) \to L^2(P,H)$ satisfying $f(t,\cdot)$ is uniformly continuous on $L^2(P,H)$ uniformly for $t \in R^+$.

Lemma 2.8 Let $x \in AAP(R^+, L^2(P,H))$ and $f \in AAP(R^+ \times L^2(P,H), L^2(P,H)) \cap C_K(R^+ \times L^2(P,H), L^2(P,H))$ with $K = \overline{\{x(t), t \in R^+\}}$. Then $f(t, x(t)) \in AAP(R^+, L^2(P,H))$.

This concept of evolution operator has been developed by Kozak [23], recentley used by Henríquez et al. [24, 25] and Ren [26].

Definition 2.9 The familly $\{U(t,s)\}_{t,s\geq 0}$ is said to be an evolution operator generated by the $\{A(t)\}_{t\geq 0}$ if the following conditions hold:

(A1) for each $x \in X$ the map $(t,s) \to U(t,s)x$ is continuously differentiable and

(a) for each $t \in R^+$, $U(t,t) = 0$;

(b) for all $t, s \in R^+$, $\frac{\partial}{\partial t}U(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}U(t,s)x|_{t=s} = -x$.

(A2) for all $t, s \in R^+$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, the map $(t,s) \to U(t,s)x$ is of class C^2 and

(a) $\frac{\partial^2}{\partial t^2}U(t,s)x = A(t)U(t,s)x$;

(b) $\frac{\partial^2}{\partial s^2}U(t,s)x = U(t,s)A(s)x$;

(c) $\frac{\partial^2}{\partial s \partial t}U(t,s)x|_{t=s} = 0$.

(A3) for all $t, s \in R^+$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}U(t,s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}U(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}U(t,s)x$ and

(a) $\frac{\partial^3}{\partial t^2 \partial s} U(t,s)x = A(t)\frac{\partial}{\partial s} U(t,s)x$. Moreover, the map $(t,s) \to A(t)\frac{\partial}{\partial s} U(t,s)x$ is continuous;
(b) $\frac{\partial^3}{\partial s^2 \partial t} U(t,s)x = \frac{\partial}{\partial t} U(t,s)A(s)x$.

3. Results

In this section, we suppose that the following assumptions hold:
(H_1) The evolution operator $\{U(t,s)\}_{t,s\geq 0}$ generated by $A(t)$ satisfies the following conditions:
(1) There exists constants $M_0, M_1 > 0$ such that

$$\|U(t,s)\| \leq M_0 e^{-\delta(t-s)}, \qquad \left\|\frac{\partial}{\partial s} U(t,s)\right\| \leq M_1 e^{-\delta(t-s)}$$

for all $t \geq s \geq 0$ and $\delta > 0$.
(2) For each $\varepsilon_1 > 0$, there exists constant $l(\varepsilon_1) > 0$, such that every interval of length $l(\varepsilon_1)$ contains a constant τ with the property that

$$\|U(t+\tau, s+\tau) - U(t,s)\| \leq \varepsilon_1 e^{-\delta(t-s)}.$$

for all $t, s \in R^+$, where $\delta > 0$ is the constant required in (1).
(H_2) The functions $F, G : R^+ \times L^2(P,H) \to L^2(P,H)$ satisfy the following conditions:
(1) $F, G \in AAP(R^+ \times L^2(P,H), L^2(P,H))$ and $F(t,\cdot)$, $G(t,\cdot)$ are uniformly continuous in every bounded subset $K \subset L^2(P,H)$ uniformly for $t \in R^+$;
(2) there exist constants $L_F, L_G > 0$ such that

$$E\|F(t,x) - F(t,y)\|^2 \leq L_F E\|x-y\|^2,$$

$$E\|G(t,x) - G(t,y)\|^2 \leq L_G E\|x-y\|^2,$$

for all $x, y \in K$ and $t \in R^+$.
Definition 3.1 An F_t-adapted continuous stochastic process $x(t)$ is called a mild solution to problems (1.1) and (1.2) if the following hold:
(1) x_0, x_1 satisfying $\|x_0\|^2 < \infty$, $\|x_1\|^2 < \infty$;
(2) the stochastic integral equation satisfied

$$x(t) = -\frac{\partial}{\partial s} U(t,0)x_0 + U(t,0)x_1 + \int_0^t U(t,s)F(s,x(s))ds + \int_0^t U(t,s)G(s,x(s))dW(s) \tag{3.1}$$

for all $t \in R^+$.
Lemma 3.2 Assume that (H_1) is satisfied. If $v : R^+ \to L^2(P,H)$ is square-mean asymptotically almost periodic, then the function

$$u(t) = \int_0^t U(t,s)v(s)ds, \quad t \in R^+$$

is square-mean asymptotically almost periodic.

Proof Let $\varepsilon > 0$ be given and $T(\frac{\delta^2}{4}\varepsilon, E\|v\|^2, L^2(P,H))$, $L = L(\frac{\delta^2}{4}\varepsilon, E\|v\|^2, L^2(P,H))$ be as in Lemma 2.6. Let $L_1 > 0$ and $\frac{16M_0^2}{\delta^2}e^{-2\delta(L_1-L)}E\|v\|^2 < \frac{\varepsilon}{4}$. For $t \geq L + L_1$ and $\tau \in T(\frac{\delta^2}{4}\varepsilon, E\|v\|^2, L^2(P,H))$, by using the Cauchy-Schwarz inequality, one has

$$\begin{aligned}
&E\|u(t+\tau)-u(t)\|^2\\
&= E\left\|\int_0^{t+\tau} U(t+\tau,s)v(s)ds - \int_0^t U(t,s)v(s)ds\right\|^2\\
&= E\Big\|\int_0^{\tau} U(t+\tau,s)v(s)ds + \int_0^L U(t+\tau,s+\tau)(v(s+\tau)-v(s))ds\\
&\quad + \int_L^t U(t+\tau,s+\tau)(v(s+\tau)-v(s))ds + \int_0^t (U(t+\tau,s+\tau)-U(t,s))v(s)ds\Big\|^2\\
&\leq 4E\left(\int_0^{\tau} \|U(t+\tau,s)v(s)\|ds\right)^2 + 4E\left(\int_0^L \|U(t+\tau,s+\tau)(v(s+\tau)-v(s))\|ds\right)^2\\
&\quad + 4E\left(\int_L^t \|U(t+\tau,s+\tau)(v(s+\tau)-v(s))\|ds\right)^2 + 4E\left(\int_0^t \|(U(t+\tau,s+\tau)-U(t,s))v(s)\|ds\right)^2\\
&\leq 4M_0^2E\left(\int_0^{\tau} e^{-\delta(t+\tau-s)}\|v(s)\|ds\right)^2 + 4M_0^2E\left(\int_0^L e^{-\delta(t-s)}\|v(s+\tau)-v(s)\|ds\right)^2\\
&\quad + 4M_0^2E\left(\int_L^t e^{-\delta(t-s)}\|v(s+\tau)-v(s)\|ds\right)^2 + 4\varepsilon_1^2E\left(\int_0^t e^{-\delta(t-s)}\|v(s)\|ds\right)^2\\
&\leq 4M_0^2\left(\int_0^{\tau} e^{-\delta(t+\tau-s)}ds\right)^2 E\|v\|^2 + 16M_0^2\left(\int_0^L e^{-\delta(t-s)}ds\right)^2 E\|v\|^2\\
&\quad + 4M_0^2\left(\int_L^t e^{-\delta(t-s)}ds\right)^2 E\|v(t+\tau)-v(t)\|^2 + 4\varepsilon_1^2\left(\int_0^t e^{-\delta(t-s)}ds\right)^2 E\|v\|^2\\
&\leq \frac{4M_0^2}{\delta^2}e^{-2\delta t}E\|v\|^2 + \frac{16M_0^2}{\delta^2}e^{-2\delta(t-L)}E\|v\|^2 + \frac{4M_0^2}{\delta^2}\varepsilon + \frac{4\varepsilon_1^2}{\delta^2}E\|v\|^2
\end{aligned}$$

and hence

$$E\|u(t+\tau)-u(t)\|^2 < \varepsilon.$$

Therefore, by Lemma 2.6, $u(t) \in AAP(R^+, L^2(P,H))$. This completes the proof.

Lemma 3.3 Assume that (H_1) is satisfied. If $v : R^+ \to L^2(P,H)$ is square-mean asymptotically almost periodic, then the function

$$w(t) = \int_0^t U(t,s)v(s)dW(s), \quad t \in R^+$$

is square-mean asymptotically almost periodic.

Proof Let $\varepsilon > 0$ be given and $T(\frac{\delta}{4}\varepsilon, E\|v\|^2, L^2(P,H))$, $L = L(\frac{\delta}{4}\varepsilon, E\|v\|^2, L^2(P,H))$ be as in Lemma 2.6. Let $L_1 > 0$ and $\frac{8M_0^2}{\delta}e^{-2\delta(L_1-L)}E\|v\|^2 < \frac{\varepsilon}{4}$. Let $\widetilde{W}(s) = W(s+\tau) - W(\tau)$ for each $s \geq 0$. Note that $\widetilde{W}$ is also a Brownian motion and has the same distribution as W. By using Itô's isometry identity [27] and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&E\|w(t+\tau)-w(t)\|^2\\
&= E\left\|\int_0^{t+\tau} U(t+\tau,s)v(s)dW(s) - \int_0^t U(t,s)v(s)dW(s)\right\|^2\\
&= E\|\int_0^{\tau} U(t+\tau,s)v(s)d\widetilde{W}(s) + \int_0^{L} U(t+\tau,s+\tau)(v(s+\tau)-v(s))d\widetilde{W}(s)\\
&\quad + \int_L^t U(t+\tau,s+\tau)(v(s+\tau)-v(s))d\widetilde{W}(s) + \int_0^t (U(t+\tau,s+\tau)-U(t,s))v(s)d\widetilde{W}(s)\|^2\\
&\le 4E\int_0^{\tau} \|U(t+\tau,s)\|\|v(s)\|^2\,ds + 4E\int_0^{L} \|U(t+\tau,s+\tau)\|\|(v(s+\tau)-v(s))\|^2\,ds\\
&\quad + 4E\int_L^t \|U(t+\tau,s+\tau)\|\|(v(s+\tau)-v(s))\|^2\,ds + 4E\int_0^t \|(U(t+\tau,s+\tau)-U(t,s))\|\|v(s)\|^2\,ds\\
&\le 4M_0^2\int_0^{\tau} e^{-2\delta(t+\tau-s)}E\|v(s)\|^2ds + 4M_0^2\int_0^{L} e^{-2\delta(t-s)}E\|v(s+\tau)-v(s))\|^2ds\\
&\quad + 4M_0^2\int_L^t e^{-2\delta(t-s)}E\|v(s+\tau)-v(s))\|^2ds + 4\varepsilon_1^2\int_0^t e^{-2\delta(t-s)}E\|v(s)\|^2ds\\
&\le \frac{2M_0^2}{\delta}e^{-2\delta t}E\|v\|^2 + \frac{8M_0^2}{\delta}e^{-2\delta(t-L)}E\|v\|^2 + \frac{2M_0^2}{\delta}\varepsilon + \frac{2\varepsilon_1^2}{\delta}E\|v\|^2.
\end{aligned}
$$

For $t \ge L(\frac{\delta}{4}\varepsilon, E\|v\|^2, L^2(P,H)) + L_1$, $\tau \in \mathrm{T}(\frac{\delta}{4}\varepsilon, E\|v\|^2, L^2(P,H))$, we obtain

$$E\|w(t+\tau)-w(t)\|^2 < \varepsilon.$$

Therefore, by Lemma 2.6, $w(t) \in AAP(R^+, L^2(P,H))$. This completes the proof.

Theorem 3.4 Assume that assumptions (H_1)-(H_3) hold. If $M_0\sqrt{\left(\frac{2L_F}{\delta^2}+\frac{L_G}{\delta}\right)} < 1$, the stochastic differential equations (1.1) and (1.2) have a unique square-mean asymptotically almost periodic mild solution.

Proof Define the operator $\Gamma : AAP(R^+, L^2(P,H)) \to AAP(R^+, L^2(P,H))$ by

$$
\begin{aligned}
(\Gamma x)(t) &= -\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)x_1 + \int_0^t U(t,s)F(s,x(s))ds + \int_0^t U(t,s)G(s,x(s))dW(s)\\
&= -\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)x_1 + (\Gamma_1 x)(t) + (\Gamma_2 x)(t),
\end{aligned}
$$

where $(\Gamma_1 x)(t) = \int_0^t U(t,s)F(s,x(s))ds$, $(\Gamma_2 x)(t) = \int_0^t U(t,s)G(s,x(s))dW(s)$.

We need to prove that Γ is well defined that is $\Gamma(AAP(R^+, L^2(P,H))) \subset AAP(R^+, L^2(P,H))$.

From previous assumptions of $\{U(t,s)\}_{t,s\ge 0}$, one can easily see that

$$
\begin{aligned}
E\|-\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)x_1\|^2 &\le 2E\|-\frac{\partial}{\partial s}U(t,0)x_0\|^2 + 2E\|U(t,0)x_1\|^2\\
&\le 2M_1^2e^{-2\delta t}E\|x_0\|^2 + 2M_0^2e^{-2\delta t}E\|x_1\|^2.
\end{aligned}
$$

then we get

$$\lim_{t\to+\infty} E\| - \frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)x_1\|^2 = 0,$$

that is $-\frac{\partial}{\partial s}U(t,0)x_0 + U(t,0)x_1 \in C_0(R^+, L^2(P,H))$.

Let $x \in AAP(R^+, L^2(P,H))$. By (H_2) and Lemma 2.8, the function $F(t,x(t))$ and $G(t,x(t))$ belongs to $AAP(R^+, L^2(P,H))$.

By Lemma 3.2 and 3.3, Γ maps $AAP(R^+, L^2(P,H))$ into itself. To complete the proof, it suffices to prove that Γ has a fixed point. Clearly, we get

$$\begin{aligned}
E\|(\Gamma x)(t) - (\Gamma y)(t)\|^2 &= E\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t) + (\Gamma_2 x)(t) - (\Gamma_2 y)(t)\|^2 \\
&\le 2E\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\|^2 + 2E\|(\Gamma_2 x)(t) - (\Gamma_2 y)(t)\|^2 \\
&= 2E\left\|\int_0^t U(t,s)[F(s,x(s)) - F(s,y(s))]dW(s)\right\|^2 \\
&\quad + 2E\left\|\int_0^t U(t,s)[G(s,x(s)) - G(s,y(s))]dW(s)\right\|^2 \\
&\le 2M_0^2 E\left(\int_0^t e^{-\delta(t-s)}\|F(s,x(s)) - F(s,y(s))\|ds\right)^2 \\
&\quad + 2E\left(\left\|\int_0^t U(t,s)[G(s,x(s)) - G(s,y(s))]dW(s)\right\|\right)^2.
\end{aligned}$$

We evaluate the first term of the right-hand side as follows:

$$\begin{aligned}
&E\left(\int_0^t e^{-\delta(t-s)}\|F(s,x(s)) - F(s,y(s))\|ds\right)^2 \\
&\le E\left[\left(\int_0^t e^{-\delta(t-s)}ds\right)\left(\int_0^t e^{-\delta(t-s)}\|F(s,x(s)) - F(s,y(s))\|^2 ds\right)\right] \\
&\le \left(\int_0^t e^{-\delta(t-s)}ds\right)\left(\int_0^t e^{-\delta(t-s)}E\|F(s,x(s)) - F(s,y(s))\|^2 ds\right) \\
&\le L_F\left(\int_0^t e^{-\delta(t-s)}ds\right)\left(\int_0^t e^{-\delta(t-s)}E\|x(s) - y(s)\|^2 ds\right) \\
&\le L_F\left(\int_0^t e^{-\delta(t-s)}ds\right)^2 \sup_{t\ge0} E\|x(t) - y(t)\|^2 \\
&\le L_F\left(\int_0^\infty e^{-\delta(t-s)}ds\right)^2 \sup_{t\ge0} E\|x(t) - y(t)\|^2 \\
&\le \frac{L_F}{\delta^2}\sup_{t\ge0} E\|x(t) - y(t)\|^2.
\end{aligned}$$

As to the second term, we use again an estimate on the Itô's integral established in [27] to obtain:

$$
\begin{aligned}
&E(\|\int_0^t U(t,s)[G(s,x(s)) - G(s,y(s))]dW(s)\|)^2 \\
&\qquad \le E\left(\int_0^t \|U(t,s)\|\|G(s,x(s)) - G(s,y(s))\|^2 ds\right) \\
&\qquad \le M_0^2 \int_0^t e^{-2\delta(t-s)} E\|G(s,x(s)) - G(s,y(s))\|^2 ds \\
&\qquad \le M_0^2 L_G \left(\int_0^t e^{-2\delta(t-s)} ds\right) \sup_{t\ge 0} E\|x(t) - y(t)\|^2 \\
&\qquad \le M_0^2 L_G \left(\int_0^\infty e^{-2\delta(t-s)} ds\right) \sup_{t\ge 0} E\|x(t) - y(t)\|^2 \\
&\qquad \le \frac{M_0^2 L_G}{2\delta} \sup_{t\ge 0} E\|x(t) - y(t)\|^2.
\end{aligned}
$$

So, we have

$$E\|(\Gamma x)(t) - (\Gamma y)(t)\|^2 \le M_0^2\left(\frac{2L_F}{\delta^2} + \frac{L_G}{\delta}\right) \sup_{t\ge 0} E\|x(t) - y(t)\|^2,$$

that is

$$\|(\Gamma x)(t) - (\Gamma y)(t)\|_2^2 \le M_0^2\left(\frac{2L_F}{\delta^2} + \frac{L_G}{\delta}\right) \sup_{t\ge 0} \|x(t) - y(t)\|_2^2. \tag{3.2}$$

Note that

$$\sup_{t\ge 0} \|x(t) - y(t)\|_2^2 \le \left(\sup_{t\ge 0} \|x(t) - y(t)\|_2\right)^2. \tag{3.3}$$

Hence, by (3.2) and (3.3), for $t \ge 0$, we obtain

$$\|(\Gamma x)(t) - (\Gamma y)(t)\|_2 \le M_0 \sqrt{\left(\frac{2L_F}{\delta^2} + \frac{L_G}{\delta}\right)} \|x(t) - y(t)\|_\infty.$$

Therefore, we get

$$\|(\Gamma x)(t) - (\Gamma y)(t)\|_\infty \le M_0 \sqrt{\left(\frac{2L_F}{\delta^2} + \frac{L_G}{\delta}\right)} \|x(t) - y(t)\|_\infty$$

which implies that Γ is a contraction mapping by $M_0 \sqrt{\left(\frac{2L_F}{\delta^2} + \frac{L_G}{\delta}\right)} < 1$. So by the Banach contraction mapping principle, we conclude that there exists a unique fixed point $x(\cdot)$ for $\Gamma \in AAP(R^+, L^2(P,H))$, such that $\Gamma x = x$, that is

$$(\Gamma x)(t) = -\frac{\partial}{\partial s} U(t,0)x_0 + U(t,0)x_1 + \int_0^t U(t,s)F(s,x(s))ds + \int_0^t U(t,s)G(s,x(s))dW(s),$$

for $t \ge 0$. This completes the proof.

4. Example

To complete this work, we apply the previous results to consider the following second-order stochastic partial equation

$$\frac{\partial^2 z(t,\xi)}{\partial t^2} = \left(\frac{\partial^2 z(t,\xi)}{\partial \xi^2} + a(t)\frac{\partial z(t,\xi)}{\partial \xi}\right)\partial t + f(t, z(t,\xi))\partial t + g(t, z(t,\xi))dW(t), \quad t \geq 0, \ \xi \in [0,\pi] \tag{4.1}$$

with

$$z(t,0) = z(t,\pi) = 0, \ t \geq 0 \tag{4.2}$$

and

$$z(0,\xi) = z_0(\xi), \frac{\partial}{\partial t}z(0,\xi) = z_1(\xi), \ \xi \in [0,\pi], \tag{4.3}$$

where W is a Q-Wiener process with $TrQ < \infty$ and f, g are appropriate functions.

Take $H = L^2([0,\pi])$ equipped with its natural topology. The operator $A(t) = A + B(t)$, where A is defined by $Az = \frac{d^2 z(\xi)}{d\xi^2}$, with $D(A) = \{z \in H : z(0) = z(\pi)\}$ and $B(t)z = a(t)\frac{dz(\xi)}{d\xi}$. The spectrum of A consists of the eigenvalues $-n^2$ for $n \in \mathbb{N}$, with associated eigenvectors $e_n(\xi) = \frac{1}{\sqrt{2\pi}}e^{in\xi}, n \in \mathbb{N}$. Furthermore, the set $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis of H. In particular, $Ax = \sum_{n=1}^{\infty} -n^2\langle x, e_n\rangle e_n$, $x \in D(A)$. It is well known that A generates a cosine function $C(t)$ on H, defined by

$$C(t)x = \sum_{n=1}^{\infty} cos(nt)\langle x, e_n\rangle e_n, \quad t \in R,$$

with associated sine function

$$S(t)x = t\langle x, e_0\rangle e_0 + \sum_{n=1}^{\infty} \frac{sin(nt)}{n}\langle x, e_n\rangle e_n, \quad t \in R.$$

It is clear that $\|C(t)\| \leq 1$. It is easy to see that $A(t) = A + B(t)$ is a closed linear operator, and $U(t,s) : H \to H$ is well defined and satisfies the condition of Definition 2.9. We refer to [24] for more details.

Let $z(t)(\xi) = z(t,\xi)$. Define $F : [0,\pi] \times H \to H$, $G : [0,\pi] \times H \to L_2(H)$ by $F(t,z)(\cdot) = f(t, z(t,\cdot))$ and $G(t,z)(\cdot) = g(t, z(t,\cdot))$. Therefore, the above system can be be written in the following abstract form:

$$dz'(t) = A(t)z(t)dt + F(t, z(t))dt + G(t, z(t))dW(t), \ t \in R^+ = [0, +\infty) \tag{4.4}$$

with

$$z(0) = z_0, z'(0) = z_1. \tag{4.5}$$

Assume that $U(t,s)$, F and G satisfy the conditions of Theorem 3.4. Then the above system has a unique square-mean asymptotically almost periodic solutions.

5. Conclusions

This paper established the existence and uniqueness of square-mean asymptotically almost periodic mild solutions for a class of second order nonautonomous stochastic evolution equations in Hilbert spaces. The results are based on the properties of evolution operators and the Lipschitz condition. However, if we generalize the results to the second order nonautonomous neutral stochastic evolution equations with infinite delay or not, can we get similar results? This is an interesting and meaningful work. In the future, we will study these problems. Also, we will study the asymptotically almost periodic mild solutions of other types of second order nonautonomous stochastic differential equations.

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Conflict of interest

We confirm that we have no conflict of interest.

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