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Research article

Ground state sign-changing solutions for fractional Laplacian equations with critical nonlinearity

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Abstract: In this paper, we investigate the existence of the least energy sign-changing solutions for nonlinear elliptic equations driven by nonlocal integro-differential operators with critical nonlinearity. By using constrained minimization method and topological degree theory, we obtain a least energy sign-changing solution for them under much weaker conditions. As a particular case, we drive an existence theorem of sign-changing solutions for the fractional Laplacian equations with critical growth.

Keywords: fractional Laplacian equation; sign-changing solution; Ekeland's variational principle; Brouwer's degree theory; variational methods
Mathematics Subject Classification: 35J65, 47J05, 47J30

1. Introduction

This paper is mainly concerned with the existence of the least energy sign-changing solutions for the following nonlocal elliptic equations

$$\begin{cases} -\mathcal{L}_K u = |u|^{2^*-2}u + \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $\mathcal{L}_K$ is the integro-differential operator defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \qquad x \in \mathbb{R}^N,$$

here $K : \mathbb{R}^N \backslash \{0\} \longrightarrow (0, +\infty)$ is a function with the properties that
(K_1) $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$;
(K_2) there exist $\gamma > 0$ and $s \in (0, 1)$ such that $K(x) \geq \gamma |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

And λ is a positive real parameter, the nonlinear term f satisfies the following conditions:
(f_1) $f: \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathédory function, and there exist $C > 0$ and $q \in (2, 2^*)$ such that $|f(x,t)| \leq C(1 + |t|^{q-1})$, $\quad a.e.\ x \in \Omega,\ t \in \mathbb{R}$;
(f_2) $\lim\limits_{t\to 0} \frac{f(x,t)}{t} = 0$ uniformly in $x \in \Omega$;
(f_3) $\frac{f(x,t)}{t}$ is strictly increasing in $|t| > 0$ for $a.e.\ x \in \Omega$.
As an example for f we can take the function $f(x,t) = e^{|x|}|t|^{q-2}q$, with $x \in \Omega,\ t \in \mathbb{R}$ and $q \in (2, 2^*)$.

A typical model for K is given by the singular kernel $K(x) = |x|^{-(N+2s)}$ which coincides with the fractional Laplace operator $-(-\triangle)^s$ of the following fractional Laplacian equations

$$\begin{cases} (-\triangle)^s u = |u|^{2^*-2}u + \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.2}$$

where

$$-(-\triangle)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \qquad x \in \mathbb{R}^N,$$

where λ is a positive real parameter.

In problem (1.1) and problem (1.2), the set $\Omega \subset \mathbb{R}^N (N > 2s)$ is an open bounded with Lipschitz boundary and $2^* := \frac{2N}{N-2s}$ is the fractional critical Sobolev exponent.

The operator $(-\triangle)^s$ can be seen as the infinitesimal generators of Lévy stable diffusion Processes, see [1] for example. It is easy to see that the integro-differential operator $\mathcal{L}_K$ is a generalization of the fractional Laplace operator $-(-\triangle)^s$. Elliptic equations involving nonlocal integro-differential operators appear frequently in many different areas of research and find many applications in engineering and finance, including statistical mechanics, fluid flow, pricing of financial instruments, and portfolio optimization, see [2–4]. In the past few years, a great deal of attention has been devoted to nonlocal operators of elliptic type, both for their interesting theoretical structure and in view of concrete applications, see [5–12] and the references therein. By minimax method, invariant sets of descending flow method or constrained minimization method, many authors obtained the existence results of sign-changing solutions of some nonlinear elliptic equations, see [13–20]. To show their results, the authors always assumed the nonlinearity $f(x,t)$ is subcritical and/or $f(x,t)$ satisfies (AR) condition and/or $f(x,t)$ is differentiable with respect to t. The existence of nontrivial solutions, positive solutions, negative solutions and sign-changing solutions, for nonlocal elliptic problem (1.1) has been investigated by using variational method, fixed point index theory and critical point theorems, see [2–4,12,20].

Motivated by the papers mentioned above, the main purpose of this paper is to establish the existence of sign-changing solution for problem (1.1) and problem (1.2) under much weaker conditions.

We define the sets X and E as

$$X = \{u \mid u:\ \mathbb{R}^N \longrightarrow\ \mathbb{R},\ u\,|_{\Omega} \in L^2(\Omega) \text{ and } (u(x) - u(y))\sqrt{K(x-y)} \in L^2(\mathbb{R}^{2N}\setminus \mathcal{O})\ \},$$

and

$$E = \{g \mid g \in X \text{ and } g = 0\ a.e. \text{ in } \mathbb{R}^N \setminus \Omega\},$$

where $u\mid_\Omega$ represents the restriction to Ω of function $u(x)$, $\mathcal{O} = (\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)$.

We note that E is non-empty and

$$\|g\| := \Big(\int_{\mathbb{R}^{2N}} |g(x) - g(y)|^2 K(x-y)dxdy\Big)^{\frac{1}{2}} \tag{1.3}$$

is a norm on E, equivalent to the standard one (see [2,3]). Also, $(E,\ \|\cdot\|)$ is a Hilbert space and

$$(u,\ v) := \int_{\mathbb{R}^{2N}} (u(x) - u(y))(v(x) - v(y))K(x-y)dxdy, \qquad \forall\ u,\ v \in E. \tag{1.4}$$

For any $\lambda > 0$ fixed, we define the the energy functional $\mathcal{J}_\lambda :\ E \longrightarrow \mathbb{R}$ by

$$\mathcal{J}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x-y)dxdy - \frac{1}{2^*}\int_\Omega |u(x)|^{2^*}dx - \lambda\int_\Omega F(x, u(x))dx,\ \ u \in E. \tag{1.5}$$

Under the condition (f_1), by standard argument, it is easy to obtain that $\mathcal{J}_\lambda \in C^1(E, \mathbb{R})$ and

$$\begin{aligned}\langle \mathcal{J}_\lambda'(u), v\rangle &= \int_{\mathbb{R}^{2N}} (u(x) - u(y))(v(x) - v(y))K(x-y)dxdy \\ &- \int_\Omega |u(x)|^{2^*-2}u(x)v(x)dx - \lambda\int_\Omega f(x, u(x))v(x)dx,\ \ u, v \in E.\end{aligned} \tag{1.6}$$

It is easy to see that for $u \in E$

$$\|u\|^2 = \|u^+\|^2 + \|u^-\|^2 - \int_{\mathbb{R}^{2N}} (u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy \geq \|u^+\|^2 + \|u^-\|^2, \tag{1.7}$$

$$\mathcal{J}_\lambda(u) = \mathcal{J}_\lambda(u^+) + \mathcal{J}_\lambda(u^-) - \int_{\mathbb{R}^{2N}} (u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy,$$

and

$$\langle \mathcal{J}_\lambda'(u), u^+\rangle = \langle \mathcal{J}_\lambda'(u^+), u^+\rangle - \int_{\mathbb{R}^{2N}} (u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy.$$

Obviously, the critical points of $\mathcal{J}_\lambda$ are equivalent to the weak solutions of problem (1.1). Furthermore, if $u \in E$ is a solutions of (1.1) and $u^\pm \neq 0$ in $\mathbb{R}^N$, then u is a sign-changing solution of (1.1), where $u^+(x) := \max\{u(x), 0\}$, $u^-(x) := \min\{u(x), 0\}$.

Lemma 1.1 [2] The embedding $E \hookrightarrow L^\nu(\mathbb{R}^N)$ is continuous if $\nu \in [1, 2^*]$ and compact if $\nu \in [1, 2^*)$, where $u \in L^\nu(R^N)$ means $u = 0$ $a.e.$ in $\mathbb{R}^N\backslash\Omega$.

The main results of this paper are the following theorems:

Theorem 1.1 Let $K :\ \mathbb{R}^N\backslash\{0\} \longrightarrow (0, +\infty)$ be a function satisfying $(K_1) - (K_2)$ and assume that f satisfies $(f_1) - (f_3)$. Then, there exists $\lambda_* > 0$ such that for each $\lambda > \lambda_*$, problem (1.1) admits a least energy sign-changing solution.

Theorem 1.2 Assume that f satisfies $(f_1) - (f_3)$. Then, there exists $\lambda^* > 0$ such that for each $\lambda > \lambda^*$, problem (1.2) admits a least energy sign-changing solution.

2. Preliminaries

In this section we collect some preliminary lemmas which will be used in the next section to prove the existence of sign-changing solutions of problem (1.1) and problem (1.2).

Let

$$\mathcal{M} := \{u \in E : \ u^{\pm} \neq 0 \ and \ \langle \mathcal{J}'_\lambda(u), u^+\rangle = \langle \mathcal{J}'_\lambda(u), u^-\rangle = 0\}.$$

Lemma 2.1 If $u \in E$ with $u^{\pm} \neq 0$, then there exists a pair (t_λ, s_λ) of positive numbers such that $t_\lambda u^+ + s_\lambda u^- \in \mathcal{M}$.

Proof. For given $u \in E$ with $u^{\pm} \neq 0$, let $g_u,\ h_u : (0, +\infty) \times (0, +\infty) \longrightarrow \mathbb{R}$ be two functions defined by

$$\begin{aligned} g_u(t,s) \ &:= t^2\|u^+\|^2 - st\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy \\ &-t^{2^*}\int_\Omega |u^+|^{2^*}dx - \lambda\int_\Omega f(x, tu^+)tu^+dx, \end{aligned}$$

$$\begin{aligned} h_u(t,s) \ &:= s^2\|u^-\|^2 - st\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy \\ &-s^{2^*}\int_\Omega |u^-|^{2^*}dx - \lambda\int_\Omega f(x, su^-)su^-dx. \end{aligned}$$

It is obvious that $g_u(t,s) = \langle \mathcal{J}'_\lambda(tu^+ + su^-), tu^+\rangle,\ h_u(t,s) = \langle \mathcal{J}'_\lambda(tu^+ + su^-), su^-\rangle$ and $g_u,\ h_u$ are both continuous in $(0, +\infty) \times (0, +\infty)$.

From (f_1) and (f_2), for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for $a.e.\ x \in \Omega$ and any $t \in \mathbb{R}$

$$|f(x,t)| \leq \varepsilon|t| + \delta(\varepsilon)|t|^{q-1}, \quad |F(x,t)| \leq \varepsilon|t|^2 + \delta(\varepsilon)|t|^q. \tag{2.1}$$

For $\varepsilon = \frac{\lambda_1}{4\lambda}$ in (2.1), there exists $c_1 > 0$ such that

$$|f(x,t)| \leq \frac{\lambda_1}{4\lambda}|t| + c_1|t|^{q-1} \qquad a.e. x \in \Omega,\ t \in \mathbb{R}, \tag{2.2}$$

where $\lambda_1 = \min\limits_{u \in E\backslash\{0\}} \frac{\int_{R^N}|u(x)-u(y)|^2K(x-y)dxdy}{\int_\Omega |u(x)|^2dx}$ is the first eigenvalue of the operator $-\mathcal{L}_K$ with homogeneous Dirichlet boundary data. By using lemma 1.1 and (2.2), we have

$$g_u(t,s) \geq t^2\|u^+\|^2 - t^{2^*}\int_\Omega |u^+|^{2^*}dx - \lambda\int_\Omega f(x, tu^+)tu^+dx \geq \frac{t^2}{2}\|u^+\|^2 - c_2t^{2^*}\|u^+\|^{2^*} - \lambda c_3t^q\|u^+\|^q.$$

Since $2 < q < 2^*$, we can obtain that there exists $r_1 > 0$ small enough such that $g_u(t,s) > 0$ for all $s > 0,\ t \in (0, r_1]$. Similarly there exists $r_2 > 0$ small enough such that $h_u(t,s) > 0$ for all $t > 0,\ s \in (0, r_2]$.

By (f_1)–(f_3) and the second integral mean value theorem, we easily deduce that

$$\frac{1}{2}f(x,t)t - F(x,t) \ is\ strictly\ increasing\ in\ |t| > 0\ for\ a.e.\ x \in \Omega. \tag{2.3}$$

and

$$\frac{1}{2}f(x,t)t - F(x,t) > 0, \qquad F(x,t) > 0, \qquad a.e.\ x \in \Omega, t \in \mathbb{R}\backslash\{0\}. \tag{2.4}$$

So we are

$$\begin{aligned} g_u(t,s) &\le t^2\|u^+\|^2 - st\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y)+u^-(x)u^+(y))K(x-y)dxdy \\ &-t^{2^*}\int_\Omega |u^+|^{2^*}dx \to -\infty,\ as\ t\to+\infty, \end{aligned}$$

since $2^* > 2$. Therefore, there exists $R_1 > 0$ sufficiently large such that $g_u(t,s) < 0$ for all $s > 0,\ t \in [R_1,+\infty)$, and Similarly we can find $R_2 > 0$ such that $h_u(t,s) < 0$ for all $t > 0,\ s \in [R_2,+\infty)$. By applying Miranda's Theorem, there exist $t_\lambda > 0, s_\lambda > 0$ such that $g_u(t_\lambda,s_\lambda) = 0 = h_u(t_\lambda,s_\lambda),\ i.e., t_\lambda u^+ + s_\lambda u^- \in \mathcal{M}$.

Lemma 2.2 If $u \in \mathcal{M}$, then

$$\phi_u(t,s) < \phi_u(1,1) = \mathcal{J}_\lambda(u), \qquad for\ all\ t,s\ge 0\ with\ (t,s)\neq(1,1),$$

where $\phi_u(t,s) := \mathcal{J}_\lambda(tu^+ + su^-),\ (t,s)\in\mathbb{R}^2_+ := [0,+\infty)\times[0,+\infty)$.

Proof. Since $u \in \mathcal{M}$, then $\langle \mathcal{J}'_\lambda(u), u^\pm\rangle = 0$, that is

$$\|u^+\|^2 - \int_{\mathbb{R}^{2N}}(u^+(x)u^-(y)+u^-(x)u^+(y))K(x-y)dxdy = \int_\Omega |u^+|^{2^*}dx + \lambda\int_\Omega f(x,u^+)u^+dx,$$

$$\|u^-\|^2 - \int_{\mathbb{R}^{2N}}(u^+(x)u^-(y)+u^-(x)u^+(y))K(x-y)dxdy = \int_\Omega |u^-|^{2^*}dx + \lambda\int_\Omega f(x,u^-)u^-dx.$$

From the definition of $\phi_u(t,s)$, it follows that $(\bar t,\bar s)$ is a critical point of $\phi_u(t,s)$ if and only if $\bar t u^+ + \bar s u^-$ is a weak solution of (1.1).

Let $t,s\ge 0$, then, by (1.5) and (2.4) we have $\phi_u(0,0) = 0$ and

$$\begin{aligned} \phi_u(t,s) &= \tfrac{1}{2}\|tu^+ + su^-\|^2 - \tfrac{1}{2^*}\int_\Omega |tu^+ + su^-|^{2^*}dx - \lambda\int_\Omega F(x,tu^+ + su^-)dx \\ &\le \tfrac{t^2}{2}\|u^+\|^2 + \tfrac{s^2}{2}\|u^-\|^2 - st\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y)+u^-(x)u^+(y))K(x-y)dxdy \\ &-\tfrac{t^{2^*}}{2^*}\int_\Omega |u^+|^{2^*}dx - \tfrac{s^{2^*}}{2^*}\int_\Omega |u^-|^{2^*}dx. \end{aligned}$$

Without loss of generality, we can suppose that $t \ge s > 0$. Thus we can get that

$$\phi_u(t,s) \le \frac{t^2+s^2}{2}[\|u^+\|^2 + \|u^-\|^2 - \int_{\mathbb{R}^{2N}}(u^+(x)u^-(y)+u^-(x)u^+(y))K(x-y)dxdy] - \frac{t^{2^*}}{2^*}\int_\Omega |u^+|^{2^*}dx,$$

that is,

$$\frac{\phi_u(t,s)}{t^2+s^2} \le \frac{1}{2}[\|u^+\|^2 + \|u^-\|^2 - \int_{\mathbb{R}^{2N}}(u^+(x)u^-(y)+u^-(x)u^+(y))K(x-y)dxdy] - \frac{t^{2^*-2}}{2\cdot 2^*}\int_\Omega |u^+|^{2^*}dx.$$

Since $2^* > 2$, we can infer that $\lim\limits_{|(t,s)|\to+\infty}\phi_u(t,s) = -\infty$. By using the continuity of ϕ_u we can deduce the existence of $(\hat t,\hat s)\in\mathbb{R}^2_+$ that is a global maximum point of ϕ_u, $i.e.,\ \phi_u(\hat t,\hat s) = \max\limits_{(t,s)\in\mathbb{R}^2_+}\phi_u(t,s)$.

Now we prove that $\hat t,\hat s > 0$. Suppose by contradiction that $\hat s = 0$. But it is obvious that

$$\begin{aligned} \phi_u(\hat t,s) &= \tfrac{\hat t^2}{2}\|u^+\|^2 - \tfrac{\hat t^{2^*}}{2^*}\int_\Omega |u^+|^{2^*}dx - \lambda\int_\Omega F(x,\hat t u^+)dx \\ &-s\hat t\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y)+u^-(x)u^+(y))K(x-y)dxdy \\ &+\tfrac{s^2}{2}\|u^-\|^2 - \tfrac{s^{2^*}}{2^*}\int_\Omega |u^-|^{2^*}dx - \lambda\int_\Omega F(x,su^-)dx \end{aligned}$$

is an increasing function with respect to s if s is small enough, then the pair $(\hat{t}, 0)$ is not a global maximum point of ϕ_u in $\mathbb{R}^2_+$, so $\hat{s} > 0$. Similarly we can prove that $\hat{t} > 0$.

Next, we prove that $\hat{t}, \hat{s} \le 1$. Since $(\phi_u)'_t(\hat{t}, \hat{s}) = (\phi_u)'_s(\hat{t}, \hat{s}) = 0$, we have

$$\hat{t}^2\|u^+\|^2 - \hat{t}\hat{s}\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy = \hat{t}^{2^*}\int_\Omega |u^+|^{2^*}dx + \lambda\int_\Omega f(x, \hat{t}u^+)\hat{t}u^+dx,$$

$$\hat{s}^2\|u^-\|^2 - \hat{t}\hat{s}\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy = \hat{s}^{2^*}\int_\Omega |u^-|^{2^*}dx + \lambda\int_\Omega f(x, \hat{s}u^-)\hat{s}u^-dx.$$

Assume that $\hat{t} \ge \hat{s}$, we have

$$\begin{aligned} &\hat{t}^2\|u^+\|^2 - \hat{t}^2\textstyle\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy \\ &\ge \hat{t}^{2^*}\textstyle\int_\Omega |u^+|^{2^*}dx + \lambda\int_\Omega f(x, \hat{t}u^+)\hat{t}u^+dx. \end{aligned} \tag{2.5}$$

Since $\langle \mathcal{J}'_\lambda(u), u^+\rangle = 0$, that is

$$\|u^+\|^2 - \int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy = \int_\Omega |u^+|^{2^*}dx + \lambda\int_\Omega f(x, u^+)u^+dx,$$

which together with (2.5) gives

$$0 \ge (\hat{t}^{2^*-2} - 1)\int_\Omega |u^+|^{2^*}dx + \lambda\int_\Omega\{\frac{f(x, \hat{t}u^+)}{\hat{t}u^+} - \frac{f(x, u^+)}{u^+}\}(u^+)^2dx.$$

By (f_3) we can infer that $\hat{t} \le 1$, and $\hat{s} \le \hat{t} \le 1$.

Finally we prove that $\phi_u(\hat{t}, \hat{s}) < \phi_u(1, 1)$ if $(\hat{t}, \hat{s}) \in [0, 1] \times [0, 1] \setminus \{(1, 1)\}$. By the definition of ϕ_u and the above conclusions we have

$$\begin{aligned} \phi_u(\hat{t}, \hat{s}) &= \mathcal{J}_\lambda(\hat{t}u^+ + \hat{s}u^-) = \mathcal{J}_\lambda(\hat{t}u^+ + \hat{s}u^-) - \tfrac{1}{2}\langle \mathcal{J}'_\lambda(\hat{t}u^+ + \hat{s}u^-), \hat{t}u^+ + \hat{s}u^-\rangle \\ &= (\tfrac{1}{2} - \tfrac{1}{2^*})\hat{t}^{2^*}\textstyle\int_\Omega |u^+|^{2^*}dx + (\tfrac{1}{2} - \tfrac{1}{2^*})\hat{s}^{2^*}\int_\Omega |u^-|^{2^*}dx \\ &\quad +\lambda\textstyle\int_\Omega \tfrac{1}{2}f(x, \hat{t}u^+)\hat{t}u^+ - F(x, \hat{t}u^+)dx + \lambda\int_\Omega \tfrac{1}{2}f(x, \hat{s}u^-)\hat{s}u^- - F(x, \hat{s}u^-)dx \\ &< (\tfrac{1}{2} - \tfrac{1}{2^*})\textstyle\int_\Omega |u^+|^{2^*}dx + (\tfrac{1}{2} - \tfrac{1}{2^*})\int_\Omega |u^-|^{2^*}dx \\ &\quad +\lambda\textstyle\int_\Omega \tfrac{1}{2}f(x, u^+)u^+ - F(x, u^+)dx + \lambda\int_\Omega \tfrac{1}{2}f(x, u^-)u^- - F(x, u^-)dx \\ &= (\tfrac{1}{2} - \tfrac{1}{2^*})\textstyle\int_\Omega |u|^{2^*}dx + \lambda\int_\Omega \tfrac{1}{2}f(x, u)u - F(x, u)dx \\ &= \mathcal{J}_\lambda(u) - \tfrac{1}{2}\langle \mathcal{J}'_\lambda(u), u\rangle = \mathcal{J}_\lambda(u) = \phi_u(1, 1). \end{aligned}$$

For fixed $\lambda > 0$, let $C_\lambda := \inf_{u\in\mathcal{M}} \mathcal{J}_\lambda(u)$, then we have the following results.

Lemma 2.3 $C_\lambda > 0$ and there exists $D > 0$ such that $\|u^\pm\| \ge D$ for all $u \in \mathcal{M}$.

Proof. (1) For $\omega \in E\backslash\{0\}$, by using (2.2) and Lemma1.1, we have

$$\mathcal{J}_\lambda(\omega) \ge \frac{1}{4}\|\omega\|^2 - c_4\|\omega\|^{2^*} - \lambda c_5\|\omega\|^q.$$

It follows from $2 < q < 2^*$ that $\inf_{\omega\in S_\rho} \mathcal{J}_\lambda(\omega) > 0$ for sufficiently small $\rho > 0$, where $S_\rho := \{u \in E : \|u\| = \rho\}$.

For each $u \in \mathcal{M}$, there is $t_1 > 0$ such that $t_1 u \in S_\rho$. By Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}\mathcal{J}_\lambda(u) &= \mathcal{J}_\lambda(u^+ + u^-) = \max_{(t,s)\in\mathbb{R}^2_+} \mathcal{J}_\lambda(tu^+ + su^-)\\ &\geq \mathcal{J}_\lambda(t_1u^+ + t_1u^-) = \mathcal{J}_\lambda(t_1u) \geq \inf_{\omega\in S_\rho} \mathcal{J}_\lambda(\omega).\end{aligned}$$

Hence, $C_\lambda = \inf_{u\in\mathcal{M}} \mathcal{J}_\lambda(u) \geq \inf_{\omega\in S_\rho} \mathcal{J}_\lambda(\omega) > 0$.

(2) For each $u \in \mathcal{M}$, we have $\langle \mathcal{J}'_\lambda(u), u^\pm\rangle = 0$, so

$$\langle \mathcal{J}'_\lambda(u^\pm), u^\pm\rangle = \int_{\mathbb{R}^{2N}} (u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy \leq 0,$$

which together with (2.2) and Lemma 1.1 gives

$$\|u^\pm\|^2 \leq \int_\Omega |u^\pm|^{2^*}dx + \lambda \int_\Omega F(x, u^\pm)dx \leq c_6\|u^\pm\|^{2^*} + \tfrac{1}{4}\|u^\pm\|^2 + c_7\|u^\pm\|^q.$$

Then, $\|u^\pm\|^2 \leq c_8\|u^\pm\|^r$, where $r = q$ if $\|u^\pm\| < 1$ or $r = 2^*$ if $\|u^\pm\| \geq 1$. Since $r > 2$ we have what we need.

Lemma 2.4 There exists $\lambda_* > 0$ such that for $\lambda > \lambda_*$,

$$0 < C_\lambda < \frac{s}{N}(S_K)^{\frac{N}{2s}},$$

where S_K is the best fractional critical Sobolev constant, namely

$$S_K := \inf_{u\in E\backslash\{0\}} \frac{\|u\|^2}{\|u\|^2_{2^*}}. \tag{2.6}$$

Proof. By Lemma 2.1, for each $\lambda > 0$ and $u \in E$ with $u^\pm \neq 0$, there exists a pair (t_λ, s_λ) of positive numbers such that $t_\lambda u^+ + s_\lambda u^- \in \mathcal{M}$. Namely

$$\begin{aligned}&t_\lambda^2\|u^+\|^2 - t_\lambda s_\lambda \int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy\\ &= t_\lambda^{2^*} \int_\Omega |u^+|^{2^*}dx + \lambda \int_\Omega f(x, t_\lambda u^+)t_\lambda u^+dx,\end{aligned} \tag{2.7}$$

$$\begin{aligned}&s_\lambda^2\|u^-\|^2 - t_\lambda s_\lambda \int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy\\ &= s_\lambda^{2^*} \int_\Omega |u^-|^{2^*}dx + \lambda \int_\Omega f(x, s_\lambda u^-)s_\lambda u^-dx.\end{aligned} \tag{2.8}$$

By using (2.4), we have

$$\|u^+\|^2 - \frac{s_\lambda}{t_\lambda}\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy \geq t_\lambda^{2^*-2}\int_\Omega |u^+|^{2^*}dx,$$

$$\|u^-\|^2 - \frac{t_\lambda}{s_\lambda}\int_{\mathbb{R}^{2N}}(u^+(x)u^-(y) + u^-(x)u^+(y))K(x-y)dxdy \geq s_\lambda^{2^*-2}\int_\Omega |u^-|^{2^*}dx,$$

which imply that $\{t_\lambda\}$ and $\{s_\lambda\}$ are bounded. Then, for the sequence $\lambda_n \to \infty$ as $n \to \infty$, there exist $t_0 \geq 0,\ s_0 \geq 0$ such that $t_{\lambda_n} \to t_0$ and $s_{\lambda_n} \to s_0$. If we assume that $t_0 > 0$, it follows from (2.4) that

$$\lim_{n\to\infty} t_{\lambda_n}^{2^*} \int_\Omega |u^+|^{2^*} dx + \lambda_n \int_\Omega f(x, t_{\lambda_n} u^+) t_{\lambda_n} u^+ dx = +\infty,$$

which leads to a contradiction with (2.7). Thus, $t_0 = 0$. Similarly we can deduce that $s_0 = 0$. Namely,

$$\lim_{\lambda\to+\infty} t_\lambda = \lim_{\lambda\to+\infty} s_\lambda = 0. \tag{2.9}$$

By (1.5), (2.4) and Lemma 2.3, for C_λ we have that choosing $u \in E$ with $u^\pm \neq 0$,

$$0 < C_\lambda \leq \mathcal{J}_\lambda(t_\lambda u^+ + s_\lambda u^-) \leq \frac{1}{2}\|t_\lambda u^+ + s_\lambda u^-\|^2 \leq \frac{1}{2}(t_\lambda\|u^+\| + s_\lambda\|u^-\|)^2.$$

By (2.9), it is easy to see that there exists $\lambda_* > 0$ such that for $\lambda > \lambda_*$, $\quad 0 < C_\lambda < \frac{s}{N}(S_K)^{\frac{N}{2s}}$.

3. Proofs of the main results

Proof of Theorem 1.1 For any fixed $\lambda > \lambda_*$, by Lemma 2.4, $\mathcal{J}_\lambda$ is bounded below over $\mathcal{M}$. By Ekeland's variational principle, there exists $\{w_n\} \subset \mathcal{M}$ such that

$$\mathcal{J}_\lambda(w_n) \to C_\lambda \in (0, \frac{s}{N}(S_K)^{\frac{N}{2s}}), \quad \mathcal{J}'_\lambda(w_n) \to 0. \tag{3.1}$$

Similar to the proof of Proposition 2 in [4], one can prove that $\{w_n\}$ is bounded in E. By (1.7), $\{w_n^+\}$ and $\{w_n^-\}$ are both bounded in E. we can assume that

$$w_n^\pm \rightharpoonup w^\pm \ \text{ in } E,\ w_n^\pm \to w^\pm \text{ in } L^\nu(\mathbb{R}^N) \quad (2 \leq \nu < 2^*),\ w_n^\pm(x) \to w^\pm(x) \ \text{ a.e. in } \mathbb{R}^N. \tag{3.2}$$

We can claim that $w_n^\pm \to w^\pm$ in E. In the following, by contradiction, we assume that $w_n^+ \nrightarrow w^+$ or $w_n^- \nrightarrow w^-$ in E. Let $v_n^\pm := w_n^\pm - w^\pm$, then it follows from the Brezis-lieb theorem that

$$Q(v_n) \to C_\lambda - \mathcal{J}_\lambda(w), \qquad Q'(v_n^\pm) \to 0,$$

where $v_n = v_n^+ + v_n^-$, $w = w^+ + w^-$ and $Q(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx$. Since $Q'(v_n^\pm) \to 0$, we can get that $\|v_n^\pm\|^2 + o(1) = \int_\Omega |v_n^\pm|^{2^*} dx = \|v_n^\pm\|_{2^*}^{2^*}$. On the other hand, it follows from (2.6) that

$$S_K\|v_n^+\|_{2^*}^2 \leq \|v_n^+\|^2 + o(1) = \|v_n^+\|_{2^*}^{2^*} + o(1) \quad or \quad S_K\|v_n^-\|_{2^*}^2 \leq \|v_n^-\|^2 + o(1) = \|v_n^-\|_{2^*}^{2^*} + o(1),$$

which further implies

$$\|v_n^+\|_{2^*}^{2^*} \geq (S_K)^{\frac{N}{2s}} + o(1) \quad or \quad \|v_n^-\|_{2^*}^{2^*} \geq (S_K)^{\frac{N}{2s}} + o(1).$$

Similar to the proof of Proposition 2 in [4], we can deduce that $\mathcal{J}_\lambda(w) \geq 0$. Thus,

$$\begin{aligned} C_\lambda + o(1) &\geq C_\lambda + o(1) - \mathcal{J}_\lambda(w) = Q(v_n) \geq Q(v_n^+) + Q(v_n^-) \\ &= (\tfrac{1}{2} - \tfrac{1}{2^*})\textstyle\int_\Omega |v_n^+|^{2^*} dx + (\tfrac{1}{2} - \tfrac{1}{2^*})\int_\Omega |v_n^-|^{2^*} dx \geq \tfrac{s}{N}(S_K)^{\frac{N}{2s}} + o(1), \end{aligned}$$

which contradicts with $C_\lambda < \frac{s}{N}(S_K)^{\frac{N}{2s}}$. Hence we have $w_n^\pm \to w^\pm$ in E. Therefore, by Lemma 1.1, we have

$$\|w_n^\pm\| \to \|w^\pm\|, \quad \int_\Omega |w_n^\pm|^{2^*} dx \to \int_\Omega |w^\pm|^{2^*} dx, \quad \int_\Omega f(x, w_n^\pm) w_n^\pm dx \to \int_\Omega f(x, w^\pm) w^\pm dx. \tag{3.3}$$

By Lemma 2.3, $\|w_n^\pm\| \geq D > 0$, so $\|w^\pm\| \geq D$ and $w^\pm \neq 0$. Thus $w = w^+ + w^-$ is sign-changing. From Lemma 2.1, there exist $t, s > 0$ such that

$$\langle \mathcal{J}_\lambda'(tw^+ + sw^-), tw^+ \rangle = 0, \quad \langle \mathcal{J}_\lambda'(tw^+ + sw^-), sw^- \rangle = 0 \tag{3.4}$$

and $tw^+ + sw^- \in \mathcal{M}$. Since $\{w_n\} \subset \mathcal{M}$, we have $\langle \mathcal{J}_\lambda'(w_n), w_n^\pm \rangle = 0$, by (3.2) and (3.3), we have

$$\langle \mathcal{J}_\lambda'(w), w^+ \rangle = 0, \quad \langle \mathcal{J}_\lambda'(w), w^- \rangle = 0. \tag{3.5}$$

By putting together (3.4) and (3.5), and arguing as in the proof of Lemma 3.2 we can deduce that $t, s \leq 1$.

Since $tw^+ + sw^- \in \mathcal{M}$, $w_n \in \mathcal{M}$, from (2.3), (3.1), (3.4) and $t, s \in (0, 1]$ we can obtain that

$$\begin{aligned}
C_\lambda & \leq \mathcal{J}_\lambda(tw^+ + sw^-) = \mathcal{J}_\lambda(tw^+ + sw^-) - \tfrac{1}{2}\langle \mathcal{J}_\lambda'(tw^+ + sw^-), tw^+ + sw^- \rangle \\
& = (\tfrac{1}{2} - \tfrac{1}{2^*})t^{2^*} \textstyle\int_\Omega |w^+|^{2^*} dx + (\tfrac{1}{2} - \tfrac{1}{2^*})s^{2^*} \int_\Omega |w^-|^{2^*} dx \\
& \quad + \lambda \textstyle\int_\Omega \tfrac{1}{2} f(x, tw^+)tw^+ - F(x, tw^+) dx + \lambda \int_\Omega \tfrac{1}{2} f(x, sw^-)sw^- - F(x, sw^-) dx \\
& \leq (\tfrac{1}{2} - \tfrac{1}{2^*}) \textstyle\int_\Omega |w^+|^{2^*} dx + (\tfrac{1}{2} - \tfrac{1}{2^*}) \int_\Omega |w^-|^{2^*} dx \\
& \quad + \lambda \textstyle\int_\Omega \tfrac{1}{2} f(x, w^+)w^+ - F(x, w^+) dx + \lambda \int_\Omega \tfrac{1}{2} f(x, w^-)w^- - F(x, w^-) dx \\
& = \lim_{n\to\infty} \{ (\tfrac{1}{2} - \tfrac{1}{2^*}) \textstyle\int_\Omega |w_n^+|^{2^*} dx + (\tfrac{1}{2} - \tfrac{1}{2^*}) \int_\Omega |w_n^-|^{2^*} dx \\
& \quad + \lambda \textstyle\int_\Omega \tfrac{1}{2} f(x, w_n^+)w_n^+ - F(x, w_n^+) dx + \lambda \int_\Omega \tfrac{1}{2} f(x, w_n^-)w_n^- - F(x, w_n^-) dx \} \\
& = \lim_{n\to\infty} \{ \mathcal{J}_\lambda(w_n) - \tfrac{1}{2}\langle \mathcal{J}_\lambda'(w_n), w_n \rangle \} = \lim_{n\to\infty} \mathcal{J}_\lambda(w_n) = C_\lambda.
\end{aligned}$$

Therefore, we have proved that $\mathcal{J}_\lambda(tw^+ + sw^-) = C_\lambda$ and $t = s = 1$, that is, $w = w^+ + w^- \in \mathcal{M}$ and $\mathcal{J}_\lambda(w^+ + w^-) = C_\lambda$.

Finally we prove that w is a critical point of $\mathcal{J}_\lambda$ for $\lambda > \lambda_*$. If w is not a critical point of $\mathcal{J}_\lambda$ for $\lambda > \lambda_*$, then there are $\alpha_0 < 0$ and $v_0 \in E$ such that $\langle \mathcal{J}_\lambda'(w), v_0 \rangle = 2\alpha_0$. So there is $\delta \in (0, \frac{1}{2})$ such that

$$\langle \mathcal{J}_\lambda'(tw^+ + sw^- + \epsilon v_0), v_0 \rangle \leq \alpha_0, \qquad \forall\, (t, s, \epsilon) \in \{(t, s, \epsilon) : \ |t-1| + |s-1| \leq \delta, \ 0 \leq \epsilon \leq \delta\}. \tag{3.6}$$

Let $D := \{(t, s) \in \mathbb{R}^2 : \ |t-1| \leq \delta, \ |s-1| \leq \delta\}$. Choosing a continuous function $\eta : \ D \to [0, 1]$ such that

$$\eta(t, s) = \begin{cases} 1, & \text{if } |t-1| \leq \dfrac{\delta}{4} \text{ and } |s-1| \leq \dfrac{\delta}{4}, \\ 0, & \text{if } |t-1| \geq \dfrac{\delta}{2} \text{ or } |s-1| \geq \dfrac{\delta}{2}. \end{cases} \tag{3.7}$$

Let $Q(t,s) := tw^+ + sw^- + \delta\eta(t,s)v_0$ and

$$H(t,s) := (\langle \mathcal{J}'_\lambda(Q(t,s)), [Q(t,s)]^+\rangle,\ \langle \mathcal{J}'_\lambda(Q(t,s)), [Q(t,s)]^-\rangle),\ \forall\ (t,s) \in D.$$

Then $Q \in C(D, E)$ and $H \in C(D, \mathbb{R}^2)$. If $|t-1| = \delta$ or $|s-1| = \delta$, $\eta(t,s) = 0$, then $H(t,s) = (\langle \mathcal{J}'_\lambda(tw^+ + sw^-), tw^+\rangle,\ \langle \mathcal{J}'_\lambda(tw^+ + sw^-), sw^-\rangle) \neq (0,0)$ in view of $(t,s) \neq (1,1)$. As a consequence, the Brouwer's degree $deg(H, int(D), (0,0))$ is well defined. By using the homotopy invariance and the normalization, we have $deg(H, int(D), (0,0)) = 1$. Thus there exists a pair $(\bar{t}, \bar{s}) \in int(D)$ such that $H(\bar{t}, \bar{s}) = (0,0)$, so $Q(\bar{t}, \bar{s}) \in \mathcal{M}$ and $\mathcal{J}_\lambda(Q(\bar{t}, \bar{s})) \geq C_\lambda$.

On the other hand, from (3.6) and (3.7) we have

$$\begin{aligned} \mathcal{J}_\lambda(Q(\bar{t},\bar{s})) &= \mathcal{J}_\lambda(\bar{t}w^+ + \bar{s}w^-) + \int_0^1 \langle \mathcal{J}'_\lambda(\bar{t}w^+ + \bar{s}w^- + \theta\delta\eta(\bar{t},\bar{s})v_0), \delta\eta(\bar{t},\bar{s})v_0 d\theta\rangle \\ &\leq \mathcal{J}_\lambda(\bar{t}w^+ + \bar{s}w^-) + \delta\eta(\bar{t},\bar{s})\alpha_0. \end{aligned} \tag{3.8}$$

If $\bar{t}$ or $\bar{s}$ is not equal to 1, By using Lemma 2.2, the term on the right-hand is strictly less than C_λ. If $\bar{t} = \bar{s} = 1$, by $\eta(\bar{t}, \bar{s}) = 1$ and $\alpha_0 < 0$, we also have a contradiction. Thus, w is a sign-changing of (1.1) for $\lambda > \lambda_*$.

Proof of Theorem 1.2 We take $K(x) = |x|^{-(N+2s)}$, then it is obvious that $K(x)$ satisfies the conditions $(K_1), (K_2)$ and problem (1.1) turns into problem (1.2). By using Lemma 5 in [2], we can obtain that $E \subseteq H^s(\mathbb{R}^N)$. Thus, the assertion of Theorem 1.2 follows from Theorem 1.1.

4. Conclusions

We have established the existence theorems of sign-changing solution for problem (1.1) and problem (1.2) under much weaker conditions (Theorem 1.1 and Theorem 1.2). In comparison with previous works, this paper has several new features. Firstly, we consider the more general nonlinear term without (AR) condition. Secondly, the nonlinear term involves critical growth. Thirdly, we do not require the continuous differentiability of the nonlinear term with respect to the second argument. Finally, the existence of a least energy sign-changing solution is obtained by using constrained minimization method and topological degree theory. Therefore the previous related results in [19, 20] are improved and generalized. There have been no previous studies considering the existence of sign-changing solutions for problem (1.1) and problem (1.2) involving critical growth to the best of our knowledge.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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