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Research article

Sharp refined quadratic estimations of Shafer's inequalities

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Abstract: In this paper, using the power series expansions of $(\tan x)^k (k = 1, 2, 3)$ and the monotonicity of a function involving the Riemann's zeta function, we sharpen the quadratic estimations of Shafer's inequalities which is refined by Nishizawa [5].

Keywords: Shafer's inequalities; upper bound for arctangent; lower bound for arctangent; best constants
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1. Introduction

Shafer [1–3] established the following result for arctangent function, which is known as Shafer's inequality:
Theorem 1. Let $x > 0$. Then

$$\arctan x > \frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}} \tag{1.1}$$

holds.

The author of this paper [4] derived an upper bound for $\arctan x$, and obtained the following result.
Theorem 2. (The double Shafer inequality, [4, Theorem 3]) Let $x > 0$. Then

$$\frac{8x}{3 + \sqrt{25 + \frac{80}{3}x^2}} < \arctan x < \frac{8x}{3 + \sqrt{25 + \frac{256}{\pi^2}x^2}}. \tag{1.2}$$

Furthermore, $80/3$ and $256/\pi^2$ are the best constants in (1.2).

During the past few years, the inequalities on inverse trigonometric functions have been the subject of intensive research. In particular, some interesting generalizations and improvements relating to (1.1) and (1.2) can be found in the literature [5–13]. Using the main result of [14], Nishizawa [5] gave the following conclusion, which does not contain (1.2).

Theorem 3. ([5, Theorem 2.1]) For $x > 0$, we have

$$\frac{\pi^2 x}{4 + \sqrt{(\pi^2 - 4)^2 + (2\pi x)^2}} < \arctan x < \frac{\pi^2 x}{4 + \sqrt{32 + (2\pi x)^2}}, \tag{1.3}$$

where the constants $(\pi^2 - 4)^2$ and 32 are the best possible.

In this paper, using the power series expansions of $(\tan x)^k (k = 1, 2, 3)$ and the monotonicity of a function involving the Riemann's zeta function, we sharpen the quadratic estimations of Shafer's inequality (1.3) as follows.

Theorem 4. Let $p > 0$. Then we have

(i) the inequality

$$\frac{\pi^2 x}{4 + \sqrt{p + (2\pi x)^2}} < \arctan x \tag{1.4}$$

holds for all $x \in (0, \infty)$ if and only if $p \in [(\pi^2 - 4)^2, \infty)$;

(ii) the inequality

$$\arctan x < \frac{\pi^2 x}{4 + \sqrt{p + (2\pi x)^2}} \tag{1.5}$$

holds for all $x \in (0, \infty)$ if $p \in (0, 32]$.

Theorem 5. The inequality

$$\arctan x < \frac{8\pi^2 x - \pi\sqrt{(4\pi^2 x^2 + p - 16)^2 - 16(p - 16)}}{2(16 - p - 4\pi^2 x^2)} \tag{1.6}$$

holds for all $x \in (0, \infty)$ if and only if $p \in (32, \infty)$.

Theorem 6. The inequality

$$(\pi^2 x - 4\arctan x)^2 - (\arctan x)^2 (p + (2\pi x)^2) < \frac{\pi^2}{4}(32 - p) \tag{1.7}$$

holds for all $x \in (0, \infty)$ if $p \in (0, 16 + 4\pi^2/3]$.

Obviously, Theorem 3 is a straightforward consequence of Theorem 4.

2. Lemmas

Lemma 1. ([14, Lemma 2.1; 15, Lemma 2.1]) The function

$$(1 - \frac{1}{2^n})\zeta(n), \ n = 1, 2, \cdots \tag{2.1}$$

is decreasing, where $\zeta(n)$ is Riemann's zeta function.

Lemma 2. ([16, Theorem 3.4]) Let $\zeta(n)$ be Riemann's zeta function and B_{2n} the even-indexed Bernoulli numbers. Then

$$\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|, \ n = 1, 2, \cdots . \tag{2.2}$$

Lemma 3. Let B_{2n} be the even-indexed Bernoulli numbers. Then for $n = 1, 2, \cdots$,

$$\frac{|B_{2n}|}{|B_{2n+2}|} > \frac{\pi^2(2^{2n+2}-1)}{(2n+2)(2n+1)(2^{2n}-1)} \tag{2.3}$$

holds.

Proof. Since $(1 - 1/2^{2n})\zeta(2n)$ is decreasing by Lemma 1, it follows that

$$\frac{2^{2n+2}-1}{4}\zeta(2n+2) < (2^{2n}-1)\zeta(2n),\ n = 1, 2, \cdots. \tag{2.4}$$

Using the representation (2.2), the inequality (2.4) becomes

$$\frac{\pi^2(2^{2n+2}-1)}{(2n+2)!}\,|B_{2n+2}| < \frac{(2^{2n}-1)}{(2n)!}\,|B_{2n}|\,,\ n = 1, 2, \cdots, \tag{2.5}$$

that is, the inequality (2.3) holds. □

Lemma 4. Let $|t| < \pi/2$. Then

$$\tan t = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|t^{2n-1}, \tag{2.6}$$

$$(\tan t)^2 = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|t^{2n-2}, \tag{2.7}$$

$$(\tan t)^3 = \sum_{n=2}^{\infty} \frac{(2n+1)\,2^{2n}\left[2(2^{2n+2}-1)(2n)|B_{2n+2}| - (2^{2n}-1)\,(2n+2)\,|B_{2n}|\right]}{(2n+2)!}t^{2n-1} \tag{2.8}$$

hold.

Proof. From [17, p.133] we have

$$\tan t = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|t^{2n-1},\ \ |t| < \frac{\pi}{2}.$$

Then we obtan

$$\begin{aligned}(\tan t)^2 &= (\sec t)^2 - 1 = (\tan t)' - 1\\ &= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|t^{2n-2} - 1\\ &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|t^{2n-2},\ \ |t| < \frac{\pi}{2},\end{aligned}$$

and

$$
\begin{aligned}
(\tan t)^3 &= \frac{1}{2}((\tan t)^2)' - \tan t \\
&= \frac{1}{2}\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{(2n)!}|B_{2n}|t^{2n-3} \\
&\quad -\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|t^{2n-1} \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n+2}(2^{2n+2}-1)(2n+1)(2n)}{(2n+2)!}|B_{2n+2}|t^{2n-1} \\
&\quad -\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|t^{2n-1} \\
&:= \sum_{n=1}^{\infty}\frac{u(n)}{(2n+2)!}t^{2n-1} = \sum_{n=2}^{\infty}\frac{u(n)}{(2n+2)!}t^{2n-1},
\end{aligned}
$$

where

$$
u(n) = (2n+1)\,2^{2n}\left[2(2^{2n+2}-1)(2n)|B_{2n+2}| - (2^{2n}-1)\,(2n+2)\,|B_{2n}|\right]
$$

with $u(1) = 0$. This completes the proof of Lemma 4. □

Lemma 5. The function

$$
g(t) = 16 - 4\pi^2 + (\pi^4 - 4\pi^2)\frac{\tan t}{t} + \pi^4\frac{\tan^3 t}{t} - 4\pi^2 t\tan^3 t - 8\pi^2\tan^2 t - 4\pi^2 t\tan t \tag{2.9}
$$

is decreasing on $(0, \pi/2)$.

Proof. From (2.6)–(2.8), we have

$$
\begin{aligned}
g(t) &= 16 - 4\pi^2 + (\pi^4 - 4\pi^2)\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|t^{2n-2} \\
&\quad +\pi^4\sum_{n=2}^{\infty}\frac{u(n)}{(2n+2)!}t^{2n-2} - 4\pi^2\sum_{n=2}^{\infty}\frac{u(n)}{(2n+2)!}t^{2n} \\
&\quad -8\pi^2\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)\,(2n-1)}{(2n)!}|B_{2n}|t^{2n-2} - 4\pi^2\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|t^{2n} \\
&= (\pi^2-4)^2 - \sum_{n=2}^{\infty}\frac{\pi^2(2n+1)(2n)q(n)}{(2n+2)!}t^{2n-2},
\end{aligned}
$$

where

$$
q(n) = 2^{2n+1}\left[\left(2^{2n}-1\right)(2n+2)(2n+1)|B_{2n}| - \pi^2\left(2^{2n+2}-1\right)|B_{2n+2}|\right]. \tag{2.10}
$$

By Lemma 3 and (2.10) we get that $q(n) > 0$ for $n = 2, 3, \cdots$. This leads to

$$
g'(t) = -\sum_{n=2}^{\infty}\frac{\pi^2(2n+1)(2n)q(n)}{(2n+2)!}(2n-2)t^{2n-3} < 0, \tag{2.11}
$$

so $g(t)$ is decreasing on $(0,\pi/2)$. □

Lemma 6. Let $t\in(0,\pi/2)$, $p>32$. Then

$$-\frac{1}{4}\pi^2(p-32)< (\pi^2\tan t-4t)^2-t^2(p+(2\pi\tan t)^2 \tag{2.12}$$

holds if and only if

$$t<\frac{8\pi^2\tan t-\pi\sqrt{(4\pi^2\tan^2 t+p-16)^2-16(p-16)}}{2(16-p-4\pi^2\tan^2 t)}. \tag{2.13}$$

Proof. The inequality (2.12) is equivalent to

$$(16-p-4\pi^2\tan^2 t)t^2-(8\pi^2\tan t)t+\left[\pi^4\tan^2 t+\frac{1}{4}\pi^2(p-32)\right]>0. \tag{2.14}$$

Let

$$a(t)=16-p-4\pi^2\tan^2 t,\ \ b(t)=-(8\pi^2\tan t),$$

and

$$c(t)=\pi^4\tan^2 t+\frac{1}{4}\pi^2(p-32).$$

Then

$$\begin{aligned} & b^2(t)-4a(t)c(t)\\ = & (-8\pi^2\tan t)^2-4(16-p-4\pi^2\tan^2 t)(\pi^4\tan^2 t+\frac{1}{4}\pi^2(p-32))\\ = & \pi^2\left[(4\pi^2\tan^2 t+p-16)^2-16(p-16)\right]>0, \end{aligned}$$

and the inequality (2.14) is equivalent to

$$(16-p-4\pi^2\tan^2 t)(t-T_1(t))(t-T_2(t))>0, \tag{2.15}$$

where

$$T_1(t)=\frac{8\pi^2\tan t+\pi\sqrt{(4\pi^2\tan^2 t+p-16)^2-16(p-16)}}{2(16-p-4\pi^2\tan^2 t)},$$

$$T_2(t)=\frac{8\pi^2\tan t-\pi\sqrt{(4\pi^2\tan^2 t+p-16)^2-16(p-16)}}{2(16-p-4\pi^2\tan^2 t)}.$$

Since for $0<t<\pi/2$,

$$16-p-4\pi^2\tan^2 t<0,\ \ t-T_1(t)>0,$$

from (2.15) we have

$$t<T_2(t),$$

that is, the inequality (2.13) is true. □

3. Proofs of Theorems 4–6

Let $t = \arctan x$. Then $t \in (0, \pi/2)$. We investigate the maximum and minimum values of the function

$$G(t) = (\pi^2 \tan t - 4t)^2 - t^2 \left[p + (2\pi \tan t)^2 \right]$$

on the interval $(0, \pi/2)$.

We can compute

$$G'(t) = 2t \left[g(t) - p \right],$$

where

$$g(t) = 16 - 4\pi^2 + \left(\pi^4 - 4\pi^2\right) \frac{\tan t}{t} + \frac{\pi^4}{t} \tan^3 t - 4\pi^2 t \tan^3 t - 8\pi^2 \tan^2 t - 4\pi^2 t \tan t.$$

By Lemma 5 we obtain that

$$\max_{t\in(0,\pi/2)} g(t) = g(0^+) = \left(\pi^2 - 4\right)^2, \quad \min_{t\in(0,\pi/2)} g(t) = g\left(\left(\frac{\pi}{2}\right)^-\right) = \frac{4}{3}\pi^2 + 16.$$

We consider the following three cases.

Case 1: When $p \geq \max_{t\in(0,\pi/2)} g(t) = \left(\pi^2 - 4\right)^2 \simeq 34.452$, we have $G'(t) \leq 0$, and the function $G(t)$ is decreasing on $(0, \pi/2)$. In view of

$$G(0^+) = 0, \; G\left(\left(\frac{\pi}{2}\right)^-\right) = -\frac{\pi^2}{4}(p - 32),$$

we obtain

$$-\frac{\pi^2}{4}(p - 32) = G\left(\frac{\pi}{2}\right) < G(t) < G(0^+) = 0. \tag{3.1}$$

Since the three functions $\pi^2 \tan t - 4t$, t, and $p + (2\pi \tan t)^2$ are positive on $(0, \pi/2)$, the right-hand side inequality of (3.1) leads to (1.4) while the left-hand side one leads to (1.6) by Lemma 6.

Case 2: When $p \leq \min_{t\in(0,\pi/2)} g(t) = 4\pi^2/3 + 16 \simeq 29.159$, we have $G'(t) \geq 0$, so the function $G(t)$ is increasing on $(0, \pi/2)$. Then

$$0 = G(0^+) < G(t) < G\left(\left(\frac{\pi}{2}\right)^-\right) = -\frac{\pi^2}{4}(p - 32). \tag{3.2}$$

The left-hand side inequality of (3.2) leads to (1.5) while the right-hand side one is the inequality (1.7).

Case 3: When $4\pi^2/3 + 16 < p < \left(\pi^2 - 4\right)^2$, we set $g(t) - p := q(t)$. Since $q(0^+) = g(0^+) - p = \left(\pi^2 - 4\right)^2 - p > 0$ and $q((\pi/2)^-) = g((\pi/2)^-) - p = 4\pi^2/3 + 16 - p < 0$, there is a unique point $\xi \in (0, \pi/2)$ such that $G'(t) > 0$ for all $t \in (0, \xi)$ and $G'(t) < 0$ for all $t \in (\xi, \pi/2)$. So we have

$$\min\left(G(0^+), G\left(\left(\frac{\pi}{2}\right)^-\right)\right) < G(t) < G(\xi). \tag{3.3}$$

Subcase 3.1: If $p \leq 32$, we have $G(0^+) \leq G((\pi/2)^-)$ and $\min\left(G(0^+), G((\pi/2)^-)\right) = G(0^+)$. The left-hand side inequality of (3.3) leads to (1.5).

Subcase 3.2: If $p > 32$, we have $G((\pi/2)^-) < G(0^+)$ and $\min\left(G(0^+), G((\pi/2)^-)\right) = G((\pi/2)^-)$. The left-hand side inequality of (3.3) leads to (1.6) by Lemma 6.

So the proofs of Theorems 4–6 are complete.

Remark 1. Let us notice that proofs of all inequalities in new Theorems 4–6 also can be obtained forming appropriated mixed trigonometric polynomial function based on the function $g(t)$ by methods and algorithms developed in [18, 19].

4. Conclusions

We have established some sharp inequalities of Shafer-type for all $x \in (0, \infty)$:

$$\frac{\pi^2 x}{4 + \sqrt{p + (2\pi x)^2}} < \arctan x, \ (\pi^2 - 4)^2 \le p < \infty,$$

$$\arctan x < \frac{\pi^2 x}{4 + \sqrt{p + (2\pi x)^2}}, \ 0 < p \le 32,$$

$$\arctan x < \frac{8\pi^2 x + \pi\sqrt{(4\pi^2x^2 + p - 16)^2 - 16(p - 16)}}{2(16 - p - 4\pi^2x^2)}, \ 32 < p < \infty,$$

$$(\pi^2 x - 4\arctan x)^2 - (\arctan x)^2(p + (2\pi x)^2) < \frac{\pi^2}{4}(32 - p), \ 0 < p \le 16 + \frac{4}{3}\pi^2.$$

The above inequalities improve and develop the known famous results.

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Conflict of interest

The author declares no conflict of interest in this paper.

References

1. R. E. Shafer, On quadratic approximation, *SIAM J. Numer. Anal.*, **11** (1974), 447–460.
2. R. E. Shafer, Analytic inequalities obtained by quadratic approximation, *Publ. Elektroteh. Fak. Ser. Mat. Fiz.*, (1977), 96–97.
3. R. E. Shafer, On quadratic approximation, II, *Publ. Elektroteh. Fak. Ser. Mat. Fiz.*, (1978), 163–170.
4. L. Zhu, On a quadratic estimate of Shafer, *J. Math. Inequal.*, **2** (2008), 571–574.

5. Y. Nishizawa, Refined quadratic estimations of Shafer's inequality, *J. Inequal. Appl.*, **2017** (2017), 1–11.

6. B. N. Guo, Q. M. Luo, F. Qi, Sharpening and generalizations of Shafer-Fink's double inequality for the arc sine function, *Filomat*, **27** (2013), 261–265.

7. B. J. Maleševic, Application of λ-method on Shafer-Fink's inequality, *Publ. Elektroteh. Fak. Ser. Mat.*, (1997), 103–105.

8. B. J. Maleševic, An application of λ-method on inequalities of Shafer-Fink's type, *Math. Inequal. Appl.*, **10** (2007), 529–534.

9. Y. Nishizawa, Sharpening of Jordan's type and Shafer-Fink's type inequalities with exponential approximations, *Appl. Math. Comput.*, **269** (2015), 146–154.

10. J. L. Sun, C. P. Chen, Shafer-type inequalities for inverse trigonometric functions and Gauss lemniscate functions, *J. Inequal. Appl.*, **2016** (2016), 1–9.

11. L. Zhu, On Shafer-Fink inequalities, *Math. Inequal. Appl.*, **8** (2005), 571–574.

12. L. Zhu, On Shafer-Fink-type inequality, *J. Inequal. Appl.*, **2007** (2007), 1–4.

13. L. Zhu, New inequalities of Shafer-Fink type for arc hyperbolic sine, *J. Inequal. Appl.*, **2008** (2008), 1–5.

14. L. Zhu, A refinement of the Becker-Stark inequalities, *Math. Notes*, **93** (2013), 421–425.

15. L. Zhu, J. Hua, Sharpening the Becker-Stark inequalities, *J. Inequal. Appl.*, **2010** (2010), 931275.

16. W. Scharlau, H. Opolka, *From fermat to Minkowski*, Springer-Verlag New York Inc., 1985

17. A. Jeffrey, *Handbook of mathematical formulas and integrals*, 3Eds., Elsevier Academic Press, 2004

18. B. J. Maleševic, M. Makragic, A method for proving some inequalities on mixed trigonometric polynomial functions, *J. Math. Inequal.*, **10** (2016), 849–876.

19. B. J. Maleševic, T. Lutovac, B. Banjac, A proof of an open problem of Yusuke Nishizawa for a power-exponential function, *J. Math. Inequal.*, **12** (2018), 473–485.

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