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Research article

Double series expansions for π

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Abstract: We use some properties of gamma functions and a summation formula for Kampé de Fériet function $F^{0:3;3}_{1:1;1}$ to give many double series expansions for $1/\pi$ and π.

Keywords: Gamma function; double hypergeometric function; double series expansions for π
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1. Introduction

In [18] Ramanujan showed a total of 17 series for $1/\pi$ but he did not indicate how he arrived at these series. The Borwein brothers [5] gave rigorous proofs of Ramanujan's series for the first time and also obtained many new series for $1/\pi$. Till now, many new Ramanujan's-type series for $1/\pi$ have been published, (see, for example, [4, 6, 8]). Chu [7], Liu [15, 16] and Wei et al. [21, 22] gave many π-formula with free parameters by means of gamma functions and hypergeometric series. Guillera [10] proved a kind of bilateral semi-terminating series related to Ramanujan-like series for negative powers of π. Moreover, Guillera and Zudilin [11] outlined an elementary method for proving numerical hypergeometric identities, in particular, Ramanujan-type identities for $1/\pi$. Recently, q-analogues of Ramanujan-type series for $1/\pi$ have caught the interests of many authors (see, for example, [9, 12–14, 20, 21]).

Although various definitions for gamma functions are used in the literature, we adopt the following definition [23, p.76]

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

where γ is the Euler constant defined as

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right).$$

It is easy to verify that $\Gamma(1) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(z+1) = z\Gamma(z)$. It follows that for every positive integer n, $\Gamma(n) = (n-1)!$.

For any complex α, we define the general rising shifted factorial by

$$(z)_\alpha = \Gamma(z+\alpha)/\Gamma(z). \tag{1.1}$$

Obviously, $(z)_0 = 1$. For every positive integer n, we have

$$(z)_n = \Gamma(z+n)/\Gamma(z) = z(z+1)\cdots(z+n-1)$$

and

$$(z)_{-n} = \Gamma(z-n)/\Gamma(z) = \frac{1}{(z-1)(z-2)\ldots(z-n)}.$$

For convenience, we use the following compact notations

$$(a_1, a_2, \ldots, a_m)_n = (a_1)_n(a_2)_n \ldots (a_m)_n$$

and

$$(a)_{(n_1,n_2,\ldots,n_m)} = (a)_{n_1}(a)_{n_2} \ldots (a)_{n_m}.$$

Following [1, 3], the hypergeometric series is defined by

$${}_{r+1}F_s\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix}; z\right] = \sum_{k=0}^{\infty} \frac{(a_0, a_1, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} \frac{z^k}{k!},$$

where $a_i, b_j (i = 0, 1, \ldots, r, j = 1, 2, \ldots, s)$ are complex numbers such that no zero factors appear in the denominators of the summand on the right hand side.

We let $F^{p:r;u}_{q:s;v}$ $(p, q, r, s, u, v \in \mathbb{N}_0 = \{0, 1, 2, \ldots\})$ denote a general (Kampé de Fériet's) double hypergeometric function defined by (see [2, 19])

$$F^{p:r;u}_{q:s;v}\left[\begin{matrix} \alpha_1, \ldots, \alpha_p : & a_1, \ldots, a_r; & c_1, \ldots, c_u; \\ \beta_1, \ldots, \beta_q : & b_1, \ldots, b_s; & d_1, \ldots, d_v; \end{matrix} x, y\right]$$
$$= \sum_{m,n=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_p)_{m+n}(a_1, \ldots, a_r)_m(c_1, \ldots, c_u)_n}{(\beta_1, \ldots, \beta_q)_{m+n}(b_1, \ldots, b_s)_m(d_1, \ldots, d_v)_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

where, for convergence of the double hypergeometric series,

$$p + r \le q + s + 1 \quad \text{and} \quad p + u \le q + v + 1,$$

with equality only when $|x|$ and $|y|$ are appropriately constrained (see, for details, [19, Eq 1.3(29), p.27]).

There exist numerous identities for such series. For example, we have

Theorem 1.1 (See [17, (30)]) If $Re(e-d) > 0$ and $Re(d+e-a-b-c) > 0$, then

$$F^{0:3;3}_{1:1;1}\left[\begin{matrix} - : & a, b, c; & d-a, d-b, d-c; \\ d : & e; & d+e-a-b-c; \end{matrix} 1, 1\right] = \frac{\Gamma(e)\Gamma(e+d-a-b-c)\Gamma(e-d)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}.$$

In [15], Liu used the general rising shifted factorial and the Gauss summation formula to prove the following four-parameter series expansions formula, which implies infinitely many Ramanujan type series for $1/\pi$ and π.

Theorem 1.2 For any complex α and $Re(c-a-b)>0$, we have

$$\sum_{n=0}^{\infty}\frac{(\alpha)_{a+n}(1-\alpha)_{b+n}}{n!\Gamma(c+n+1)}=\frac{(\alpha)_a(1-\alpha)_b\Gamma(c-a-b)}{(\alpha)_{c-b}(1-\alpha)_{c-a}}\cdot\frac{\sin\pi\alpha}{\pi}.$$

Motivated by Liu's work, in this paper we derive the following result from Theorem 1.1 which enables us to give many double series expansions for $1/\pi$ and π. To the best of our knowledge, most of the results in this paper have not previously appeared.

Theorem 1.3 If $d\in\mathbb{N}_0, Re(e-d+\sigma-\delta)>0$ and $Re(d+e-a-b-c+\delta+\sigma-\alpha-\beta-\gamma)>0$, then

$$\sum_{m,n=0}^{\infty}\frac{(\alpha)_{a+m}(\beta)_{b+m}(\gamma)_{c+m}(\delta-\alpha)_{d-a+n}(\delta-\beta)_{d-b+n}(\delta-\gamma)_{d-c+n}}{m!n!(\delta+d)_{m+n}(\sigma)_{e+m}(\delta+\sigma-\alpha-\beta-\gamma)_{d+e-a-b-c+n}}$$
$$=\frac{(\alpha)_a(\beta)_b(\gamma)_c(\delta-\alpha)_{d-a}(\delta-\beta)_{d-b}(\delta-\gamma)_{d-c}(\sigma-\delta)_{e-d}}{(\sigma-\alpha)_{e-a}(\sigma-\beta)_{e-b}(\sigma-\gamma)_{e-c}}\cdot\frac{\Gamma(\sigma)\Gamma(\sigma-\delta)\Gamma(\delta+\sigma-\alpha-\beta-\gamma)}{\Gamma(\sigma-\alpha)\Gamma(\sigma-\beta)\Gamma(\sigma-\gamma)}.$$

Several examples of such formulae are

$$\sum_{m,n=0}^{\infty}\frac{(\frac{1}{2})_m^3(\frac{1}{2})_n^2}{m!n!(m+n)!(m+1)!(2n+1)}=\frac{4}{\pi},$$

$$\sum_{m,n=0}^{\infty}\frac{(-\frac{1}{2})_m^3(\frac{3}{2})_n^3}{m!n!(m+n)!(n+3)!(\frac{1}{2})_{m+1}}=\pi,$$

and

$$\sum_{m,n=0}^{\infty}\frac{(-\frac{2}{3})_m^2(\frac{1}{3})_n^3}{m!n!(n+1)!(2-3m)(-\frac{1}{3})_{m+n}}=\frac{\sqrt{3}\pi}{3}.$$

The remainder of the paper is organized as follows. In section 2 we give the proof of Theorem 1.3. Sections 3 and 4 are devoted to the double series expansions for $1/\pi$ and π, respectively.

2. Proof of Theorem 1.3

First of all, by making use of (1.1), Theorem 1.3 can be restated as follows:

$$\sum_{m,n=0}^{\infty}\frac{\Gamma(a+m)\Gamma(b+m)\Gamma(c+m)\Gamma(d-a+n)\Gamma(d-b+n)\Gamma(d-c+n)}{m!n!\Gamma(d+m+n)\Gamma(e+m)\Gamma(d+e-a-b-c+n)}$$
$$=\frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d-a)\Gamma(d-b)\Gamma(d-c)\Gamma(e-d)}{\Gamma(d)\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}. \tag{2.1}$$

From (1.1) it is easy to see that

$$\begin{aligned}
&\Gamma(a+\alpha+m)=(\alpha)_{a+m}\Gamma(\alpha),\ \Gamma(b+\beta+m)=(\beta)_{b+m}\Gamma(\beta),\ \Gamma(c+\gamma+m)=(\gamma)_{c+m}\Gamma(\gamma),\\
&\Gamma(d-a+\delta-\alpha+n)=(\delta-\alpha)_{d-a+n}\Gamma(\delta-\alpha),\ \Gamma(d-b+\delta-\beta+n)=(\delta-\beta)_{d-b+n}\Gamma(\delta-\beta),\\
&\Gamma(d-c+\delta-\gamma+n)=(\delta-\gamma)_{d-c+n}\Gamma(\delta-\gamma),\ \Gamma(d+\delta+m+n)=(\delta)_{d+m+n}\Gamma(\delta)\\
&\Gamma(e+m+\sigma)=(\sigma)_{e+m}\Gamma(\sigma),\ \Gamma(a+\alpha)=(\alpha)_a\Gamma(\alpha),\ \Gamma(b+\beta)=(\beta)_b\Gamma(\beta),\ \Gamma(c+\gamma)=(\gamma)_c\Gamma(\gamma),\\
&\Gamma(d-a+\delta-\alpha)=(\delta-\alpha)_{d-a}\Gamma(\delta-\alpha),\ \Gamma(d-b+\delta-\beta)=(\delta-\beta)_{d-b}\Gamma(\delta-\beta),\\
&\Gamma(d-c+\delta-\gamma)=(\delta-\gamma)_{d-c}\Gamma(\delta-\gamma),\ \Gamma(e-d+\sigma-\delta)=(\sigma-\delta)_{e-d}\Gamma(\sigma-\delta),\\
&\Gamma(d+\delta)=(\delta)_d\Gamma(\delta),\quad \Gamma(e-a+\sigma-\alpha)=(\sigma-\alpha)_{e-a}\Gamma(\sigma-\alpha),\\
&\Gamma(e-b+\sigma-\beta)=(\sigma-\beta)_{e-b}\Gamma(\sigma-\beta),\ \Gamma(e-c+\sigma-\gamma)=(\sigma-\gamma)_{e-c}\Gamma(\sigma-\gamma),\\
&\Gamma(d+e-a-b-c+\delta+\sigma-\alpha-\beta-\gamma)\\
&\ =(\delta+\sigma-\alpha-\beta-\gamma)_{d+e-a-b-c}\Gamma(\delta+\sigma-\alpha-\beta-\gamma).
\end{aligned}$$

and we realize that $(\delta)_{d+m+n}=(\delta)_d(\delta+d)_{m+n}$ when $d\in\mathbb{N}_0$. Replacing(a,b,c,d,e) by $(a+\alpha,b+\beta,c+\gamma,d+\delta,e+\sigma)$ in (2.1) and substituting above identities into the resulting equation, we get the desired result.

3. Double series expansions for $1/\pi$

In this section we will use Theorem 1.3 to prove the following double series expansion formula for $1/\pi$.

Theorem 3.1 If $d\in\mathbb{N}_0, Re(e-d+1)>0$ and $Re(d+e-a-b-c+\frac{3}{2})>0$, then

$$\sum_{m,n=0}^{\infty}\frac{(\frac{1}{2})_{(a+m,b+m,c+m,d-a+n,d-b+n,d-c+n)}}{m!n!(d+1)_{m+n}(2)_{e+m}(\frac{3}{2})_{d+e-a-b-c+n}}=\frac{(\frac{1}{2})_{(a,b,c,d-a,d-b,d-c)}(1)_{e-d}}{(\frac{3}{2})_{(e-a,e-b,e-c)}}\cdot\frac{4}{\pi}.$$

Proof. Let $(\alpha,\beta,\gamma,\delta,\sigma)=(\frac{1}{2},\frac{1}{2},\frac{1}{2},1,2)$ in Theorem 1.3. We find that

$$\sum_{m,n=0}^{\infty}\frac{(\frac{1}{2})_{(a+m,b+m,c+m,d-a+n,d-b+n,d-c+n)}}{m!n!(d+1)_{m+n}(2)_{e+m}(\frac{3}{2})_{d+e-a-b-c+n}}=\frac{(\frac{1}{2})_{(a,b,c,d-a,d-b,d-c)}(1)_{e-d}}{(\frac{3}{2})_{(e-a,e-b,e-c)}}\cdot\frac{\Gamma(2)\Gamma(1)\Gamma(\frac{3}{2})}{\Gamma^3(\frac{3}{2})}. \tag{3.1}$$

Substituting $\Gamma(\frac{3}{2})=\frac{\sqrt{\pi}}{2}$ into (3.1) we obtain the result immediately. □

Putting $(a,b,c)=(0,0,0)$ in Theorem 3.1 we get the following general double summation formula for $1/\pi$ with two free parameters.

Corollary 3.2 If $d\in\mathbb{N}_0, Re(e-d+1)>0$ and $Re(d+e+\frac{3}{2})>0$, then

$$\sum_{m,n=0}^{\infty}\frac{(\frac{1}{2})^3_{(m,d+n)}}{m!n!(d+1)_{m+n}(2)_{e+m}(\frac{3}{2})_{d+e+n}}=\frac{4(\frac{1}{2})^3_d(1)_{e-d}}{\pi(\frac{3}{2})^3_e}.$$

Setting $d=0$ and $e=k\in\mathbb{N}_0$ in Corallary 3.2 we have the following result.

Proposition 3.3 Let k be a nonnegative integer. Then

$$\sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})^3_{(m,n)}}{m!n!(m+n)!(m+k+1)!(\frac{3}{2}+k)_n} = \frac{4k!}{\pi(\frac{3}{2})^2_k}.$$

Example 3.1 ($k = 0$ in Proposition 3.3).

$$\sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})^3_m(\frac{1}{2})^2_n}{m!n!(m+n)!(m+1)!(2n+1)} = \frac{4}{\pi}.$$

If $d = e = k \in \mathbb{N}_0$ in Corollary 3.2 we achieve

Proposition 3.4 Let k be a nonnegative integer. Then

$$\sum_{m,n=0}^{\infty} \frac{(\frac{1}{2})^3_{(m,n+k)}}{m!n!(k+1)_{m+n}(m+k+1)!(\frac{3}{2})_{n+2k}} = \frac{4}{\pi(2k+1)^3}.$$

If we put $k = 0$ into Proposition 3.4, then we can also get Example 3.1.

4. Double series expansions for π

In this section we will prove the following theorem, which allows us to derive infinitely double series expansions for π.

Theorem 4.1 If $d \in \mathbb{N}_0, Re(e-d-\sigma+1) > 0$ and $Re(d+e-a-b-c+2) > 0$, then

$$\sum_{m,n=0}^{\infty} \frac{(\sigma-1)_{(a+m,b+m,c+m)}(\sigma)_{(d-a+n,d-b+n,d-c+n)}}{m!n!(2\sigma+d-1)_{m+n}(\sigma)_{e+m}(2)_{d+e-a-b-c+n}}$$
$$= \frac{(\sigma-1)_{(a,b,c)}(\sigma)_{(d-a,d-b,d-c)}(1-\sigma)_{e-d}}{(1)_{(e-a,e-b,e-c)}} \cdot \frac{\pi}{\sin\sigma\pi}.$$

Proof. Let $(\alpha,\beta,\gamma,\delta) = (\sigma-1,\sigma-1,\sigma-1,2\sigma-1)$ in Theorem 1.3. We obtain that

$$\sum_{m,n=0}^{\infty} \frac{(\sigma-1)_{(a+m,b+m,c+m)}(\sigma)_{(d-a+n,d-b+n,d-c+n)}}{m!n!(2\sigma+d-1)_{m+n}(\sigma)_{e+m}(2)_{d+e-a-b-c+n}}$$
$$= \frac{(\sigma-1)_{(a,b,c)}(\sigma)_{(d-a,d-b,d-c)}(1-\sigma)_{e-d}}{(1)_{(e-a,e-b,e-c)}} \cdot \frac{\Gamma(\sigma)\Gamma(1-\sigma)\Gamma(2)}{\Gamma^3(1)}. \tag{4.1}$$

Combining $\Gamma(\sigma)\Gamma(1-\sigma) = \frac{\pi}{\sin\sigma\pi}$ with (4.1) we get the desired result immediately. □

Putting $a = b = c = 0$ in Theorem 4.1 we obtain the following equation.

Corollary 4.2 If $d \in \mathbb{N}_0, Re(e-d-\sigma+1) > 0$ and $Re(d+e+2) > 0$, then

$$\sum_{m,n=0}^{\infty} \frac{(\sigma-1)^3_m(\sigma)^3_{d+n}}{m!n!(2\sigma+d-1)_{m+n}(\sigma)_{e+m}(2)_{d+e+n}} = \frac{(\sigma)^3_d(1-\sigma)_{e-d}}{(1)^3_e} \cdot \frac{\pi}{\sin\sigma\pi}.$$

4.1. $\sigma = \frac{1}{2}$ in Corollary 4.2

Letting $\sigma = \frac{1}{2}$ in Corollary 4.2, we get the following proposition.
Proposition 4.3 If $d \in \mathbb{N}_0, Re(e - d + \frac{1}{2}) > 0$ and $Re(d + e + 2) > 0$, then

$$\sum_{m,n=0}^{\infty} \frac{(-\frac{1}{2})_m^3 (\frac{1}{2})_{d+n}^3}{m!n!(d)_{m+n}(\frac{1}{2})_{e+m}(2)_{d+e+n}} = \frac{(\frac{1}{2})_d^3 (\frac{1}{2})_{e-d}}{(1)_e^3}\pi.$$

When we set $d = 1$ and $e = k \in \mathbb{N} = \{1, 2, 3 \ldots\}$ in Proposition 4.3 we obtain
Proposition 4.4 If k is a positive integer, then

$$\sum_{m,n=0}^{\infty} \frac{(-\frac{1}{2})_m^3 (\frac{3}{2})_n^3}{m!n!(m+n)!(n+k+2)!(\frac{1}{2})_{m+k}} = \frac{\pi(\frac{1}{2})_{k-1}}{(k!)^3}.$$

Example 4.1 ($k = 1$ in Proposition 4.4).

$$\sum_{m,n=0}^{\infty} \frac{(-\frac{1}{2})_m^3 (\frac{3}{2})_n^3}{m!n!(m+n)!(n+3)!(\frac{1}{2})_{m+1}} = \pi.$$

4.2. $\sigma = \frac{1}{3}$ in Corollary 4.2

Putting $\sigma = \frac{1}{3}$ in Corollary 4.2, we get the following proposition.
Proposition 4.5 If $d \in \mathbb{N}_0, Re(e - d + \frac{2}{3}) > 0$ and $Re(d + e + 2) > 0$, then

$$\sum_{m,n=0}^{\infty} \frac{(-\frac{2}{3})_m^3 (\frac{1}{3})_{d+n}^3}{m!n!(d-\frac{1}{3})_{m+n}(\frac{1}{3})_{e+m}(2)_{d+e+n}} = \frac{2\sqrt{3}\pi(\frac{1}{3})_d^3 (\frac{2}{3})_{e-d}}{3(1)_e^3}.$$

When we set $d = 0$ and $e = k \in \mathbb{N}_0$ in Proposition 4.5 we obtain
Proposition 4.6 If k is a nonnegative integer, then

$$\sum_{m,n=0}^{\infty} \frac{(-\frac{2}{3})_m^3 (\frac{1}{3})_n^3}{m!n!(-\frac{1}{3})_{m+n}(\frac{1}{3})_{m+k}(n+k+1)!} = \frac{2\sqrt{3}\pi(\frac{2}{3})_k}{3k!^3}.$$

Example 4.2 ($k = 0$ in Proposition 4.6).

$$\sum_{m,n=0}^{\infty} \frac{(-\frac{2}{3})_m^2 (\frac{1}{3})_n^3}{m!n!(n+1)!(2-3m)(-\frac{1}{3})_{m+n}} = \frac{\sqrt{3}\pi}{3}.$$

Setting $d = e = k \in \mathbb{N}_0$ in Proposition 4.5, we get
Proposition 4.7 If k is a nonnegative integer, then

$$\sum_{m,n=0}^{\infty} \frac{(-\frac{2}{3})_m^3 (\frac{1}{3}+k)_n^3}{m!n!(n+2k+1)!(k-\frac{1}{3})_{m+n}(\frac{1}{3})_{m+k}} = \frac{2\sqrt{3}\pi}{3k!^3}.$$

Therefore, Example 4.2 can also be deduced by fixing $k = 0$ in the above equation.
Example 4.3 ($k = 1$ in Proposition 4.7).

$$\sum_{m,n=0}^{\infty} \frac{(-\frac{2}{3})_m^3 (\frac{4}{3})_n^3}{m!n!(n+3)!(\frac{2}{3})_{m+n}(\frac{4}{3})_m} = \frac{2\sqrt{3}\pi}{9}.$$

5. Conclusions

Double series expansions for $1/\pi$ and π with several free parameters are established and many interesting formulas are obtained. A point that should be stressed is that there is an important connection between the summation formulas for double hypergeometric functions and double series expansions for the powers of π.

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Conflicts of interest

The author declares that there is no conflict of interest in this paper.

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