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Research article

Endpoint estimates for multilinear fractional singular integral operators on Herz and Herz type Hardy spaces

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Abstract: The boundedness of singular and fractional integral operator on Lebesgue and Hardy spaces have been well studied. The theory of Herz space and Herz type Hardy space, as a local version of Lebesgue and Hardy space, have been developed. The main purpose of this paper is to establish the endpoint continuity properties of some multilinear operators related to certain non-convolution type fractional singular integral operators on Herz and Herz type Hardy spaces and the endpoint estimates for the multilinear operators on Herz and Herz type Hardy spaces are obtained.

Keywords: multilinear operator; fractional singular integral operators; BMO space; Herz space; Herz type Hardy space
Mathematics Subject Classification: 42B20, 42B25

1. Introduction and notations

Let T be the Calderón-Zygmund singular integral operator and b be a locally integrable function on R^n. The commutator generated by b and T is defined by $[b, T]f = bT(f) - T(bf)$. The investigation of the commutator begins with Coifman-Rochberg-Weiss pioneering study and classical result (see [6]). The classical result of Coifman, Rochberg and Weiss (see [6]) states that the commutator $[b, T]f = T(bf) - bTf$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ if and only if $b \in BMO(R^n)$. The major reason for considering the problem of commutators is that the boundedness of commutator can produces some characterizations of function spaces (see [1,6]). Chanillo (see [1]) proves a similar result when T is replaced by the fractional integral operator. In [11], the boundedness properties of the commutators for the extreme values of p are obtained. In recent years, the theory of Herz space and Herz type Hardy space, as a local version of Lebesgue space and Hardy space, have been developed (see [8,9,12,13]). The main purpose of this paper is to establish the endpoint continuity properties of some multilinear operators related to certain non-convolution type fractional singular integral operators on Herz and

Herz type Hardy spaces.

First, let us introduce some notations (see [8–10,12,13,15]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{Q\ni x}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, f is said to belong to $BMO(R^n)$ if $f^{\#} \in L^\infty$ and define $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$; We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$\|f\|_{CMO} = \sup_{r>1}|Q(0,r)|^{-1}\int_Q |f(y) - f_Q|dy < \infty.$$

It is well-known that (see [9,10])

$$\|f\|_{CMO} \approx \sup_{r>1}\inf_{c\in C}|Q(0,r)|^{-1}\int_Q |f(x) - c|dx.$$

For $k \in Z$, define $B_k = \{x \in R^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and $\tilde{\chi}_k$ the characteristic function of C_k for $k \ge 1$ and $\tilde{\chi}_0$ the characteristic function of B_0.

Definition 1. Let $0 < p < \infty$ and $\alpha \in R$.

(1) The homogeneous Herz space $\dot{K}_p^\alpha(R^n)$ is defined by

$$\dot{K}_p^\alpha(R^n) = \{f \in L_{loc}^p(R^n \setminus \{0\}) : \|f\|_{\dot{K}_p^\alpha} < \infty\},$$

where

$$\|f\|_{\dot{K}_p^\alpha} = \sum_{k=-\infty}^{\infty} 2^{k\alpha}\|f\chi_k\|_{L^p};$$

(2) The nonhomogeneous Herz space $K_p^\alpha(R^n)$ is defined by

$$K_p^\alpha(R^n) = \{f \in L_{loc}^p(R^n) : \|f\|_{K_p^\alpha} < \infty\},$$

where

$$\|f\|_{K_p^\alpha} = \sum_{k=0}^{\infty} 2^{k\alpha}\|f\tilde{\chi}_k\|_{L^p}.$$

If $\alpha = n(1 - 1/p)$, we denote that $\dot{K}_p^\alpha(R^n) = \dot{K}_p(R^n)$, $K_p^\alpha(R^n) = K_p(R^n)$.

Definition 2. Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that

$$\|f\|_{B_p^\delta} = \sup_{d>1} d^{-n(1/p-\delta/n)}\|f\chi_{Q(0,d)}\|_{L^p} < \infty.$$

Definition 3. Let $1 < p < \infty$.

(1) The homogeneous Herz type Hardy space $H\dot{K}_p(R^n)$ is defined by

$$H\dot{K}_p(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_p(R^n)\},$$

where

$$\|f\|_{H\dot{K}_p} = \|G(f)\|_{\dot{K}_p}.$$

(2) The nonhomogeneous Herz type Hardy space $HK_p(R^n)$ is defined by

$$HK_p(R^n) = \{f \in S'(R^n) : G(f) \in K_p(R^n)\},$$

where

$$\|f\|_{HK_p} = \|G(f)\|_{K_p}.$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 4. Let $1 < p < \infty$. A function $a(x)$ on R^n is called a central $(n(1-1/p), p)$-atom (or a central $(n(1-1/p), p)$-atom of restrict type), if

1) $\mathrm{Supp} a \subset B(0,d)$ for some $d > 0$ (or for some $d \geq 1$),
2) $\|a\|_{L^p} \leq |B(0,d)|^{1/p-1}$,
3) $\int a(x)dx = 0$.

Lemma 1. (see [9,13]) Let $1 < p < \infty$. A temperate distribution f belongs to $H\dot{K}_p(R^n)$(or $HK_p(R^n)$) if and only if there exist central $(n(1-1/p), p)$-atoms(or central $(n(1-1/p), p)$-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$)in the $S'(R^n)$ sense, and

$$\|f\|_{H\dot{K}_p}(\ or\ \|f\|_{HK_p}) \approx \sum_j |\lambda_j|.$$

2. Theorems

In this paper, we will consider a class of multilinear operators related to some non-convolution type singular integral operators, whose definition are following.

Let m be a positive integer and A be a function on R^n. We denote that

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\beta| \leq m} \frac{1}{\beta!} D^\beta A(y)(x-y)^\beta$$

and

$$Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\beta| = m} \frac{1}{\beta!} D^\beta A(x)(x-y)^\beta.$$

Definition 5. Fixed $\varepsilon > 0$ and $0 < \delta < n$. Let $T_\delta : S \to S'$ be a linear operator. T_δ is called a fractional singular integral operator if there exists a locally integrable function $K(x, y)$ on $R^n \times R^n$ such that

$$T_\delta(f)(x) = \int_{R^n} K(x, y) f(y) dy$$

for every bounded and compactly supported function f, where K satisfies:

$$|K(x, y)| \leq C|x-y|^{-n+\delta}$$

and

$$|K(y, x) - K(z, x)| + |K(x, y) - K(x, z)| \leq C|y-z|^\varepsilon |x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \le |x-z|$. The multilinear operator related to the fractional singular integral operator T_δ is defined by

$$T_\delta^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f(y)dy;$$

We also consider the variant of T_δ^A, which is defined by

$$\tilde{T}_\delta^A(f)(x) = \int_{R^n} \frac{Q_{m+1}(A;x,y)}{|x-y|^m} K(x,y) f(y)dy.$$

Note that when $m=0$, T_δ^A is just the commutators of T_δ and A (see [1,6,11,14]). It is well known that multilinear operator, as a non-trivial extension of commutator, is of great interest in harmonic analysis and has been widely studied by many authors (see [3–5]). In [7], the weighted $L^p(p>1)$-boundedness of the multilinear operator related to some singular integral operator are obtained. In [2], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operator are obtained. In this paper, we will study the endpoint continuity properties of the multilinear operators T_δ^A and $\tilde{T}_\delta^A$ on Herz and Herz type Hardy spaces.

Now we state our results as following.

Theorem 1. Let $0<\delta<n$, $1<p<n/\delta$ and $D^\beta A \in BMO(R^n)$ for all β with $|\beta|=m$. Suppose that T_δ^A is the same as in Definition 5 such that T_δ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p,q \in (1,+\infty]$ with $1/q = 1/p - \delta/n$. Then T_δ^A is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$.

Theorem 2. Let $0<\delta<n$, $1<p<n/\delta$, $1/q = 1/p-\delta/n$ and $D^\beta A \in BMO(R^n)$ for all β with $|\beta|=m$. Suppose that $\tilde{T}_\delta^A$ is the same as in Definition 5 such that $\tilde{T}_\delta^A$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p,q \in (1,+\infty)$ with $1/q = 1/p-\delta/n$. Then $\tilde{T}_\delta^A$ is bounded from $H\dot{K}_p(R^n)$ to $\dot{K}_q^\alpha(R^n)$ with $\alpha = n(1-1/p)$.

Theorem 3. Let $0<\delta<n$, $1<p<n/\delta$ and $D^\beta A \in BMO(R^n)$ for all β with $|\beta|=m$. Suppose that $\tilde{T}_\delta^A$ is the same as in Definition 5 such that $\tilde{T}_\delta^A$ is bounded from $L^p(R^n)$ to $L^q(R^n)$ for any $p,q \in (1,+\infty)$ with $1/q = 1/p-\delta/n$. Then the following two statements are equivalent:

(i) $\tilde{T}_\delta^A$ is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$;

(ii) for any cube Q and $z \in 3Q \setminus 2Q$, there is

$$\frac{1}{|Q|}\int_Q \left| \sum_{|\beta|=m} \frac{1}{\beta!} |D^\beta A(x) - (D^\beta A)_Q| \int_{(4Q)^c} K_\beta(z,y) f(y)dy \right| dx \le C\|f\|_{B_p^\delta},$$

where $K_\beta(z,y) = \frac{(z-y)^\beta}{|z-y|^m} K(z,y)$ for $|\beta|=m$.

Remark. Theorem 2 is also hold for nonhomogeneous Herz and Herz type Hardy space.

3. Proofs of theorems

To prove the theorem, we need the following lemma.

Lemma 2. (see [5]) Let A be a function on R^n and $D^\beta A \in L^q(R^n)$ for $|\beta|=m$ and some $q>n$. Then

$$|R_m(A;x,y)| \le C|x-y|^m \sum_{|\beta|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\beta A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Proof of Theorem 1. It suffices to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|}\int_Q |T_\delta^A(f)(x) - C_Q|dx \le C\|f\|_{B_p^\delta}$$

holds for any cube $Q = Q(0, d)$ with $d > 1$. Fix a cube $Q = Q(0, d)$ with $d > 1$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A)_{\tilde{Q}} x^\beta$, then $R_{m+1}(A; x, y) = R_{m+1}(\tilde{A}; x, y)$ and $D^\beta \tilde{A} = D^\beta A - (D^\beta A)_{\tilde{Q}}$ for all β with $|\beta| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} T_\delta^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x-y|^m} K(x, y) f(y) dy = \int_{R^n} \frac{R_m(\tilde{A}; x, y)}{|x-y|^m} K(x, y) f_1(y) dy \\ &\quad - \sum_{|\beta|=m} \frac{1}{\beta!} \int_{R^n} \frac{K(x, y)(x-y)^\beta}{|x-y|^m} D^\beta \tilde{A}(y) f_1(y) dy + \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x-y|^m} K(x, y) f_2(y) dy, \end{aligned}$$

then

$$\begin{aligned} &\frac{1}{|Q|}\int_Q \left|T_\delta^A(f)(x) - T_\delta^{\tilde{A}}(f_2)(0)\right| dx \le \frac{1}{|Q|}\int_Q \left|T_\delta\left(\frac{R_m(\tilde{A}; x, \cdot)}{|x-\cdot|^m} f_1\right)(x)\right| dx \\ &+ \sum_{|\beta|=m} \frac{1}{\beta!} \frac{1}{|Q|} \int_Q \left|T_\delta\left(\frac{(x-\cdot)^\beta}{|x-\cdot|^m} D^\beta \tilde{A} f_1\right)(x)\right| dx + \left|T_\delta^{\tilde{A}}(f_2)(x) - T_\delta^{\tilde{A}}(f_2)(0)\right| dx \\ := \;& I + II + III. \end{aligned}$$

For I, note that for $x \in Q$ and $y \in \tilde{Q}$, using Lemma 2, we get

$$R_m(\tilde{A}; x, y) \le C|x-y|^m \sum_{|\beta|=m} \|D^\beta A\|_{BMO},$$

thus, by the $L^p(R^n)$ to $L^q(R^n)$-boundedness of T_δ^A for $1 < p, q < \infty$ with $1/q = 1/p - \delta/n$, we get

$$\begin{aligned} I &\le \frac{C}{|Q|}\int_Q \left|T_\delta\left(\sum_{|\beta|=m} \|D^\beta A\|_{BMO} f_1\right)(x)\right| dx \\ &\le C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left(\frac{1}{|Q|}\int_Q |T_\delta(f_1)(x)|^q dx\right)^{1/q} \\ &\le C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} |Q|^{-1/q} \|f_1\|_{L^p} \\ &\le C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} r^{-n(1/p-\delta/n)} \|f\chi_{\tilde{Q}}\|_{L^p} \\ &\le C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \|f\|_{B_p^\delta}. \end{aligned}$$

For II, taking $1 < s < p$ such that $1/r = 1/s - \delta/n$, by the (L^s, L^r)-boundedness of T_δ and Holder's inequality, we gain

$$II \le \frac{C}{|Q|}\int_Q \left|T_\delta\left(\sum_{|\beta|=m} (D^\beta A - (D^\beta A)_{\tilde{Q}}) f_1\right)(x)\right| dx$$

$$
\begin{aligned}
&\leq C \sum_{|\beta|=m} \left(\frac{1}{|Q|} \int_Q \left| T_\delta \left((D^\beta A - (D^\beta A)_{\tilde{Q}} \right) f_1)(x) \right|^r dx \right)^{1/r} \\
&\leq C|Q|^{-1/r} \sum_{|\beta|=m} \|(D^\beta A - (D^\beta A)_{\tilde{Q}}) f_1\|_{L^s} \\
&\leq C|Q|^{-1/r} \|f_1\|_{L^p} \sum_{|\beta|=m} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^\beta A(y) - (D^\beta A)_{\tilde{Q}}|^{ps/(p-s)} dy \right)^{(p-s)/(ps)} |Q|^{(p-s)/(ps)} \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} r^{-n/q} \|f \chi_{\tilde{Q}}\|_{L^p} \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \|f\|_{B_p^\delta}.
\end{aligned}
$$

To estimate III, we write

$$
\begin{aligned}
T_\delta^{\tilde{A}}(f_2)(x) - T_\delta^{\tilde{A}}(f_2)(0) &= \int_{R^n} \left[\frac{K(x,y)}{|x-y|^m} - \frac{K(0,y)}{|y|^m} \right] R_m(\tilde{A}; x, y) f_2(y) dy \\
&\quad + \int_{R^n} \frac{K(0,y) f_2(y)}{|y|^m} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; 0, y)] dy \\
&\quad - \sum_{|\beta|=m} \frac{1}{\beta!} \int_{R^n} \left(\frac{K(x,y)(x-y)^\beta}{|x-y|^m} - \frac{K(0,y)(-y)^\beta}{|y|^m} \right) D^\beta \tilde{A}(y) f_2(y) dy \\
&:= III_1 + III_2 + III_3.
\end{aligned}
$$

By Lemma 2 and the following inequality (see [15])

$$
|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{BMO} \; for \; Q_1 \subset Q_2,
$$

we know that, for $x \in Q$ and $y \in 2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}$,

$$
\begin{aligned}
|R_m(\tilde{A}; x, y)| &\leq C|x-y|^m \sum_{|\beta|=m} (\|D^\beta A\|_{BMO} + |(D^\beta A)_{\tilde{Q}(x,y)} - (D^\beta A)_{\tilde{Q}}|) \\
&\leq Ck|x-y|^m \sum_{|\beta|=m} \|D^\beta A\|_{BMO}.
\end{aligned}
$$

Note that $|x-y| \sim |y|$ for $x \in Q$ and $y \in R^n \setminus \tilde{Q}$, we obtain, by the condition of K,

$$
\begin{aligned}
|III_1| &\leq C \int_{R^n} \left(\frac{|x|}{|y|^{m+n+1-\delta}} + \frac{|x|^\varepsilon}{|y|^{m+n+\varepsilon-\delta}} \right) |R_m(\tilde{A}; x, y)| |f_2(y)| dy \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} k \left(\frac{|x|}{|y|^{n+1-\delta}} + \frac{|x|^\varepsilon}{|y|^{n+\varepsilon-\delta}} \right) |f(y)| dy \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \sum_{k=1}^\infty k(2^{-k} + 2^{-\varepsilon k})(2^k r)^{-n(1/p-\delta/n)} \|f \chi_{2^k \tilde{Q}}\|_{L^p} \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \sum_{k=1}^\infty k(2^{-k} + 2^{-\varepsilon k}) \|f\|_{B_p^\delta}
\end{aligned}
$$

$$\leq \quad C\sum_{|\beta|=m}\|D^{\beta}A\|_{BMO}\|f\|_{B_p^\delta}.$$

For III_2, by the formula (see [5]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\gamma|<m}\frac{1}{\gamma!}R_{m-|\gamma|}(D^{\gamma}\tilde{A};x,x_0)(x-y)^{\gamma}$$

and Lemma 2, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\leq C\sum_{|\gamma|<m}\sum_{|\beta|=m}|x-x_0|^{m-|\gamma|}|x-y|^{|\gamma|}\|D^{\beta}A\|_{BMO},$$

thus, similar to the estimates of III_1, we get

$$|III_2|\leq C\sum_{|\beta|=m}\|D^{\beta}A\|_{BMO}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{|x|}{|y|^{n+1-\delta}}|f(y)|dy\leq C\sum_{|\beta|=m}\|D^{\beta}A\|_{BMO}\|f\|_{B_p^\delta}.$$

For III_3, by Holder's inequality, similar to the estimates of III_1, we get

$$\begin{aligned}
|III_3| &\leq C\sum_{|\beta|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left(\frac{|x|}{|y|^{n+1-\delta}}+\frac{|x|^{\varepsilon}}{|y|^{n+\varepsilon-\delta}}\right)|D^{\beta}\tilde{A}(y)||f(y)|dy\\
&\leq C\sum_{|\beta|=m}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})(2^k r)^{-n(1/p-\delta/n)}\left(|2^k\tilde{Q}|^{-1}\int_{2^k\tilde{Q}}|D^{\beta}A(y)-(D^{\beta}A)_{\tilde{Q}}|^{p'}dy\right)^{1/p'}\|f\chi_{2^k\tilde{Q}}\|_{L^p}\\
&\leq C\sum_{|\beta|=m}\|D^{\beta}A\|_{BMO}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})(2^k r)^{-n(1/p-\delta/n)}\|f\chi_{2^k\tilde{Q}}\|_{L^p}\\
&\leq C\sum_{|\beta|=m}\|D^{\beta}A\|_{BMO}\|f\|_{B_p^\delta}.
\end{aligned}$$

Thus

$$III\leq C\sum_{|\beta|=m}\|D^{\beta}A\|_{BMO}\|f\|_{B_p^\delta},$$

which together with the estimates for I and II yields the desired result. This finishes the proof of Theorem 1.

Proof of Theorem 2. Let $f\in H\dot{K}_p(R^n)$, by Lemma 1, $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, where $a_j's$ are the central $(n(1-1/p),p)$-atom with supp$a_j\subset B_j=B(0,2^j)$ and $\|f\|_{H\dot{K}_p}\approx\sum_j|\lambda_j|$. We write

$$\begin{aligned}
\|\tilde{T}^A_{\delta}(f)\|_{\dot{K}^{\alpha}_q} &= \sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\|\chi_k\tilde{T}^A_{\delta}(f)\|_{L^q}\\
&\leq \sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=-\infty}^{k-1}|\lambda_j|\|\chi_k\tilde{T}^A_{\delta}(a_j)\|_{L^q}+\sum_{k=-\infty}^{\infty}2^{kn(1-1/p)}\sum_{j=k}^{\infty}|\lambda_j|\|\chi_k\tilde{T}^A_{\delta}(a_j)\|_{L^q}\\
&= J+JJ.
\end{aligned}$$

For JJ, by the (L^p, L^q)-boundedness of $\tilde{T}_\delta^A$ for $1/q = 1/p - \delta/n$, we get

$$\begin{aligned} JJ &\leq C\sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} |\lambda_j| \|a_j\|_{L^p} \leq C\sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=k}^{\infty} |\lambda_j| 2^{jn(1/p-1)} \\ &\leq C\sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=-\infty}^{j} 2^{(k-j)n(1-1/p)} \leq C\sum_{j=-\infty}^{\infty} |\lambda_j| \leq C\|f\|_{H\dot{K}_p}. \end{aligned}$$

To obtain the estimate of J, we denote that $\tilde{A}(x) = A(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A)_{2B_j} x^\beta$. Then $Q_m(A;x,y) = Q_m(\tilde{A};x,y)$ and $Q_{m+1}(A;x,y) = R_m(A;x,y) - \sum_{|\beta|=m} \frac{1}{\beta!}(x-y)^\beta D^\beta A(x)$. We write, by the vanishing moment of a and for $x \in C_k$ with $k \geq j+1$,

$$\begin{aligned} \tilde{T}_\delta^A(a_j)(x) &= \int_{R^n} \frac{K(x,y)R_m(A;x,y)}{|x-y|^m} a_j(y)dy - \sum_{|\beta|=m} \frac{1}{\beta!} \int_{R^n} \frac{K(x,y)D^\beta \tilde{A}(x)(x-y)^\beta}{|x-y|^m} a_j(y)dy \\ &= \int_{R^n} \left[\frac{K(x,y)}{|x-y|^m} - \frac{K(x,0)}{|x|^m}\right] R_m(\tilde{A};x,y)a_j(y)dy \\ &\quad + \int_{R^n} \frac{K(x,0)}{|x|^m} [R_m(\tilde{A};x,y) - R_m(\tilde{A};x,0)] a_j(y)dy \\ &\quad - \sum_{|\beta|=m} \frac{1}{\beta!} \int_{R^n} \left[\frac{K(x,y)(x-y)^\beta}{|x-y|^m} - \frac{K(x,0)x^\beta}{|x|^m}\right] D^\beta \tilde{A}(x) a_j(y)dy. \end{aligned}$$

Similar to the proof of Theorem 1, we obtain

$$\begin{aligned} |\tilde{T}_\delta^A(a_j)(x)| &\leq C\int_{R^n} \left[\frac{|y|}{|x|^{m+n+1-\delta}} + \frac{|y|^\varepsilon}{|x|^{m+n+\varepsilon-\delta}}\right] |R_m(\tilde{A};x,y)||a_j(y)|dy \\ &\quad + C\sum_{|\beta|=m} \int_{R^n} \left[\frac{|y|}{|x|^{n+1-\delta}} + \frac{|y|^\varepsilon}{|x|^{n+\varepsilon-\delta}}\right] |D^\beta \tilde{A}(x)||a_j(y)|dy \\ &\leq C\sum_{|\beta|=m} \|D^\beta A\|_{BMO} \left[\frac{2^j}{2^{k(n+1-\delta)}} + \frac{2^{j\varepsilon}}{2^{k(n+\varepsilon-\delta)}}\right] + C\sum_{|\beta|=m} \left[\frac{2^j}{2^{k(n+1-\delta)}} + \frac{2^{j\varepsilon}}{2^{k(n+\varepsilon-\delta)}}\right] |D^\beta \tilde{A}(x)|, \end{aligned}$$

thus

$$\begin{aligned} J &\leq C\sum_{|\beta|=m} \|D^\beta A\|_{BMO} \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=-\infty}^{k-1} |\lambda_j| \left[\frac{2^j}{2^{k(n+1-\delta)}} + \frac{2^{j\varepsilon}}{2^{k(n+\varepsilon-\delta)}}\right] 2^{kn/q} \\ &\quad + C\sum_{|\beta|=m} \sum_{k=-\infty}^{\infty} 2^{kn(1-1/p)} \sum_{j=-\infty}^{k-1} |\lambda_j| \left[\frac{2^j}{2^{k(n+1-\delta)}} + \frac{2^{j\varepsilon}}{2^{k(n+\varepsilon-\delta)}}\right] \left(\int_{B_k} |D^\beta \tilde{A}(x)|^q dx\right)^{1/q} \\ &\leq C\sum_{|\beta|=m} \|D^\beta A\|_{BMO} \sum_{k=-\infty}^{\infty} 2^{kn(1-\delta/n)} \sum_{j=-\infty}^{k-1} |\lambda_j| \left[\frac{2^j}{2^{k(n+1-\delta)}} + \frac{2^{j\varepsilon}}{2^{k(n+\varepsilon-\delta)}}\right] \\ &\leq C\sum_{|\beta|=m} \|D^\beta A\|_{BMO} \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=j+1}^{\infty} [2^{j-k} + 2^{(j-k)\varepsilon}] \end{aligned}$$

$$
\begin{aligned}
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \sum_{j=-\infty}^{\infty} |\lambda_j| \\
&\leq C \sum_{|\beta|=m} \|D^\beta A\|_{BMO} \|f\|_{H\dot{K}_p}.
\end{aligned}
$$

This completes the proof of Theorem 2.

Proof of Theorem 3. For any cube $Q = Q(0, r)$ with $r > 1$, let $f \in B_p^\delta$ and $\tilde{A}(x) = A(x) - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A)_{\tilde{Q}} x^\beta$. We write, for $f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2$ and $z \in 3Q \setminus 2Q$,

$$
\begin{aligned}
\tilde{T}_\delta^A(f)(x) &= \tilde{T}_\delta^A(f_1)(x) + \int_{R^n} \frac{R_m(\tilde{A}; x, y)}{|x-y|^m} K(x, y) f_2(y) dy \\
&\quad - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A(x) - (D^\beta A)_Q)(T_{\delta,\beta}(f_2)(x) - T_{\delta,\beta}(f_2)(z)) \\
&\quad - \sum_{|\beta|=m} \frac{1}{\beta!}(D^\beta A(x) - (D^\beta A)_Q) T_{\delta,\beta}(f_2)(z) \\
&= I_1(x) + I_2(x) + I_3(x, z) + I_4(x, z),
\end{aligned}
$$

where $T_{\delta,\beta}$ is the singular integral operator with the kernel $\frac{(x-y)^\beta}{|x-y|^m} K(x, y)$ for $|\beta| = m$. Note that $(I_4(\cdot, z))_Q = 0$, we have

$$
\tilde{T}_\delta^A(f)(x) - (\tilde{T}_\delta^A(f))_Q = I_1(x) - (I_1(\cdot))_Q + I_2(x) - I_2(z) - [I_2(\cdot) - I_2(z)]_Q - I_3(x, z) + (I_3(x, z))_Q - I_4(x, z).
$$

By the (L^p, L^q)-bounded of $\tilde{T}_\delta^A$, we get

$$
\frac{1}{|Q|} \int_Q |I_1(x)| dx \leq \left(\frac{1}{|Q|} \int_Q |\tilde{T}_\delta^A(f_1)(x)|^q dx \right)^{1/q} \leq C|Q|^{-1/q} \|f_1\|_{L^p} \leq C\|f\|_{B_p^\delta}.
$$

Similar to the proof of Theorem 1, we obtain

$$
|I_2(x) - I_2(z)| \leq C\|f\|_{B_p^\delta}
$$

and

$$
\frac{1}{|Q|} \int_Q |I_3(x, z)| dx \leq C\|f\|_{B_p^\delta}.
$$

Then integrating in x on Q and using the above estimates, we obtain the equivalence of the estimate

$$
\frac{1}{|Q|} \int_Q |\tilde{T}_\delta^A(f)(x) - (\tilde{T}_\delta^A(f))_Q| dx \leq C\|f\|_{B_p^\delta}
$$

and the estimate

$$
\frac{1}{|Q|} \int_Q |I_4(x, z)| dx \leq C\|f\|_{B_p^\delta}.
$$

This completes the proof of Theorem 3.

4. Applications

In this section we shall apply the theorems of the paper to some particular operators such as the Calderón-Zygmund singular integral operator and fractional integral operator.

Application 1. Calderón-Zygmund singular integral operator.

Let T be the Calderón-Zygmund operator defined by (see [10,11,15])

$$T(f)(x) = \int_{R^n} K(x,y)f(y)dy,$$

the multilinear operator related to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y)f(y)dy.$$

Then it is easily to see that T satisfies the conditions in Theorems 1–3, thus the conclusions of Theorems 1–3 hold for T^A.

Application 2. Fractional integral operator with rough kernel.

For $0 < \delta < n$, let T_δ be the fractional integral operator with rough kernel defined by (see [2,7])

$$T_\delta f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\delta}} f(y)dy,$$

the multilinear operator related to T_δ is defined by

$$T_\delta^A f(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^{m+n-\delta}} \Omega(x-y)f(y)dy,$$

where Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$ and $\Omega \in Lip_\varepsilon(S^{n-1})$ for some $0 < \varepsilon \le 1$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\varepsilon$. Then T_δ satisfies the conditions in Theorem 1. In fact, for $suppf \subset (2Q)^c$ and $x \in Q = Q(x_0, d)$, by the condition of Ω, we have (see [16])

$$\left| \frac{\Omega(x-y)}{|x-y|^{n-\delta}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-\delta}} \right| \le C \left(\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}} + \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} \right),$$

thus, the conclusions of Theorems 1–3 hold for T_δ^A.

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Conflict of interest

The authors declare that they have no competing interests.

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