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Research article

Ω-result for the index of composition of an integral ideal

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Abstract: Every nonzero integral ideal can be expressed as the product of finite prime ideals in Dedekind domain. For each integral ideal $\mathfrak{A}$, it is essential to measure the multiplicity of its prime ideal factors. We define $\lambda(\mathfrak{A}) := \frac{\log N(\mathfrak{A})}{\log \gamma(\mathfrak{A})}$ to be the index of composition of $\mathfrak{A}$, where $\gamma(\mathfrak{A}) = \prod_{\mathfrak{P}|\mathfrak{A}} N(\mathfrak{P})$ and $N(\mathfrak{A})$ is the norm of ideal $\mathfrak{A}$. In this paper, we obtain an Ω-result for the mean value of the index of composition of integral ideal.

Keywords: index of composition; asymptotic formula; prime ideal
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1. Introduction

Let K be an algebraic number filed of degree d and O_K be the ring of integers of K. For each integral ideal $\mathfrak{A} \in O_K$, $\mathfrak{A} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_g^{e_g}$, where the $\mathfrak{P}_i (i = 1, ..., g)$ are prime ideals of O_K, this expression is unique up to the order of the factors. Motivated by [17], we define $\lambda(\mathfrak{A}) := \frac{\log N(\mathfrak{A})}{\log \gamma(\mathfrak{A})}$ be the index of composition of $\mathfrak{A}$, where $N(\mathfrak{A})$ is the norm of ideal $\mathfrak{A}$ and $\gamma(\mathfrak{A}) = \prod_{\mathfrak{P}|\mathfrak{A}} N(\mathfrak{P})$. We write $\lambda(\mathfrak{A}) = \gamma(\mathfrak{A}) = 1$ if $\mathfrak{A} = O_K$. The index of composition of an integral ideal measures the multiplicity of its prime factors.

Before stating our main results, we introduce some notations. The Dedekind zeta-function for the field K is defined by

$$\zeta_K(s) = \sum_{\mathfrak{A} \neq 0} \frac{1}{N^s(\mathfrak{A})} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \ \Re s > 1,$$

where a_n is the number of integral ideals of K with norm n. $\zeta_K(s)$ can be analytically continued to the whole complex plane, $s = 1$ is a simple pole with residue

$$\rho_K = \frac{2^{r_1}(2\pi)^{r_2}R_K}{\omega_K\sqrt{|d(K)|}},$$

where r_1, r_2 denote the number of real and complex places respectively, R_K is the regular of K, $d(K)$ is the discriminant of K and ω_K is the order of the group of units. We know that $a_n \le (\tau(n))^d$, where $\tau(n)$ is the number of divisors of n. Let

$$C(x) := \sum_{n\le x} a_n = \rho_K x + \Delta(x). \tag{1.1}$$

Then (see [5, 8, 9, 11])

$$\Delta(x) = O(x^{\theta_d+\epsilon}),$$

where

$$\theta_d := \begin{cases} \frac{131}{416}, & \text{if } d = 2,\\ \frac{43}{96}, & \text{if } d = 3,\\ 1 - \frac{2}{d} + \frac{8}{d(5d+2)}, & \text{if } d = 4, 5, 6,\\ 1 - \frac{2}{d} + \frac{3}{2d^2}, & \text{if } d \ge 7.\\ 1 - \frac{3}{d+6}, & \text{if } d \ge 10. \end{cases}$$

In [15], Zhang and Zhai obtained a series of results about the mean value of $\lambda^{\pm k}(\mathfrak{A})$. The results imply that the average order of $\lambda(\mathfrak{A})$ is ρ_K. Also they found that the mean value of $\lambda^{-1}(\mathfrak{A})$ has a close connection with the zero free region of $\zeta_K(s)$ and got that

$$\sum_{N(\mathfrak{A})\le x} \lambda^{-1}(\mathfrak{A}) = \rho_K x + C_1 \int_2^x \frac{1}{\log z}dz + O_K\left(x^{\vartheta_d+\varepsilon}\right),$$

where

$$\vartheta_d := \begin{cases} \frac{1}{2}, & \text{if } d = 2, 3\\ 1 - \frac{2}{d} + \frac{8}{d(5d+2)}, & \text{if } d = 4, 5, 6,\\ 1 - \frac{2}{d} + \frac{3}{2d^2}, & \text{if } d \ge 7. \end{cases}$$

If K is a quadratic or cubic number field, Zhang and Zhai [16] proved the asymptotic formula

$$\sum_{N(\mathfrak{A})\le x} \lambda^{-1}(\mathfrak{A}) = \rho_K x + C_1 \int_2^x \frac{1}{\log z}dz + C_2 \int_2^x \frac{z^{-\frac{1}{2}}}{\log z}dz + O_K(R(x)), \tag{1.2}$$

where

$$R(x) = x^{\frac{1}{2}} \exp(-c \log^{\frac{1}{3}} x (\log\log x)^{-\frac{1}{3}}),$$

and C_1, C_2 are computable constants, $c > 0$ is a positive constant. Assuming the Riemann Hypothesis for $\zeta_K(s)$ is true, Zhang and Zhai [16] used the estimation of exponential sum and convolution method to get

$$R(x) = \begin{cases} x^{5/12+\varepsilon}, & \text{if } d = 2,\\ x^{73/156+\varepsilon}, & \text{if } d = 3. \end{cases}$$

It is natural to consider how well the main term of $\sum_{N(\mathfrak{A})\le x}\lambda^{-1}(\mathfrak{A})$ approximates it, that is, what can be said about Ω- result (for example, see [6, 7, 18]). In this paper, we shall get the following result.

Theorem 1.1. Let K be a quadratic or cubic number field. Then we have

$$\sum_{N(\mathfrak{A})\le x}\lambda^{-1}(\mathfrak{A}) =\rho_K x + C_1\int_2^x \frac{1}{\log z}dz + C_2\int_2^x \frac{z^{-\frac{1}{2}}}{\log z}dz + C_3\int_2^x \frac{z^{-\frac{2}{3}}}{\log z}dz + R_0(x),$$

where $C_i (i = 1, 2, 3)$ are computable constants and the error term $R_0(x)$ satisfies

$$\int_1^Y R_0(x)dx = \Omega\left(\frac{Y^{\frac{5}{4}}}{\log Y}\right). \tag{1.3}$$

Remark. In order to prove Theorem 1.1, we will follow the line in Pintz [12] and Nowak [10], using Mellin transform and constructing an auxiliary function $g(s)$ with some properties. Because $\lambda(\mathfrak{A})$ is not multiplicative, it is not easy to get the generating series of $\lambda^{-1}(\mathfrak{A})$. Instead we shall study the mean value of $\sum_{N(\mathfrak{A})\le x}\log\gamma(\mathfrak{A})$. In fact, as a function of z, $\gamma^z(\mathfrak{A})$ is regular for $|z| \le \varepsilon$, then we can differentiate it and set $z = 0$ to get the mean value and generating series for $\log\gamma(\mathfrak{A})$. Theorem 1.1 can follows from the lower bound for the error term of $\sum_{N(\mathfrak{A})\le x}\log\gamma(\mathfrak{A})$.

It is easy to see that (1.3) implies the following Ω-result.

Theorem 1.2. Let K be a quadratic or cubic number field. Then we have

$$R_0(x) = \Omega\left(\frac{x^{1/4}}{\log x}\right).$$

Notation. Throughout the paper ε always denotes a fixed but sufficiently small positive constant. We write $f(x) \ll g(x)$, or $f(x) = O(g(x))$, to mean that $|f(x)| \le Cg(x)$. $\sum_{n\sim N}$ denote that the sum over $N < n \le 2N$. $f(x) = \Omega(g(x))$ means that there exists a suitable constant $C > 0$ such that $|f(x)| > Cg(x)$ holds for a sequence $x = x_n$ such that $\lim_{n\to\infty} x_n = \infty$.

2. The mean value of $\log\gamma(\mathfrak{A})$

In this section, suppose $\varepsilon > 0$ is a small positive constant, z is a complex number such that $|z| \le \varepsilon$. Let $s = \sigma + it$ be a complex number with $\Re(s - z) > 1$. Define

$$G(s, z) := \sum_{\mathfrak{A}}\gamma^z(\mathfrak{A})N^{-s}(\mathfrak{A}).$$

Lemma 2.1. For $|z \le \varepsilon$, we have

$$G(s, z) = \frac{\zeta_K(s - z)\zeta_K(2s - z)\zeta_K(3s - z)\zeta_K(4s - z)\zeta_K(4s - 3z)}{\zeta_K(2s - 2z)\zeta_K(3s - 2z)\zeta_K(4s - 2z)}G_1(s, z),$$

where $G_1(s, z)$ can be expanded into a Dirichlet series of s, which is absolutely convergent for $\sigma > \frac{1}{5}+\varepsilon$.

Proof. By Euler product representation, we have

$$G(s,z) = \prod_{\mathfrak{P}} \left(1 + \frac{N^z(\mathfrak{P})}{N^s(\mathfrak{P})} + \frac{N^z(\mathfrak{P})}{N^{2s}(\mathfrak{P})} + \frac{N^z(\mathfrak{P})}{N^{3s}(\mathfrak{P})} + \cdots \right) = \zeta_K(s-z)G^*(s,z),$$

where $\zeta_K(s)$ is the Dedekind ζ-function, and

$$\begin{aligned} G^*(s,z) : &= \prod_{\mathfrak{P}} \left(1 - \frac{N^z(\mathfrak{P})}{N^s(\mathfrak{P})}\right) \prod_{\mathfrak{P}} \left(1 + \frac{N^z(\mathfrak{P})}{N^s(\mathfrak{P})} + \frac{N^z(\mathfrak{P})}{N^{2s}(\mathfrak{P})} + \frac{N^z(\mathfrak{P})}{N^{3s}(\mathfrak{P})} + \cdots \right) \\ &= \zeta_K(2s-z)G_1^*(s,z), \end{aligned}$$

where

$$G_1^*(s,z) = \prod_{\mathfrak{P}} \left(1 - \frac{N^{2z}(\mathfrak{P})}{N^{2s}(\mathfrak{P})} - \frac{N^{2z}(\mathfrak{P})}{N^{4s}(\mathfrak{P})} + \frac{N^{3z}(\mathfrak{P})}{N^{4s}(\mathfrak{P})} - \cdots \right).$$

Arguing similarly, we can get

$$G(s,z) = \frac{\zeta_K(s-z)\zeta_K(2s-z)\zeta_K(3s-z)\zeta_K(4s-z)\zeta_K(4s-3z)}{\zeta_K(2s-2z)\zeta_K(3s-2z)\zeta_K(4s-2z)} G_1(s,z),$$

where

$$G_1(s,z) = \prod_{\mathfrak{P}} \left(1 + \frac{N^{z}(\mathfrak{P})}{N^{5s}(\mathfrak{P})} - \frac{N^{2z}(\mathfrak{P})}{N^{5s}(\mathfrak{P})} + \frac{N^{3z}(\mathfrak{P})}{N^{5s}(\mathfrak{P})} - \cdots \right).$$

By the similar method as before, we know that $G_1(s,z)$ can be written as the product of Dedekind ζ-functions. If we note that $|z| \le \varepsilon$, then $G_1(s,z)$can be expanded to a Dirichlet series, which is absolutely convergent for $\Re s > 1/5 + \varepsilon$. □

To obtain the mean value of $\sum_{N(\mathfrak{A})\le x} \log \gamma(\mathfrak{A})$, we need the following Lemma.

Lemma 2.2. If $\sigma > 1$, then

$$\zeta_K(s) \ll \log(|t|+2). \tag{2.1}$$

Let $k \ge 0$ be an integer. Uniformly for $\frac{1}{2} \le \sigma \le 1$, we have

$$\zeta_K^{(k)}(s) \ll (|t|+2)^{\frac{d}{3}(1-\sigma)} \log^{k+1}(|t|+2), \tag{2.2}$$

where $\zeta_K^{(k)}(s)$ is the k-th derivative of $\zeta_K(s)$.

Proof. The order of $\zeta_K(s)$ can be found in [16, Lemma 2.3]. For the cases $k \ge 1$ of (2.2), we use the Cauchy derivative formula to get

$$\zeta_K^{(k)}(s) = \frac{k!}{2\pi i} \int_{|z-s|=R} \frac{\zeta_K(z)}{(z-s)^{k+1}} dz.$$

Let $z = s + Re^{i\theta}$ and $R = 1/\log(|t|+2)$. Then the above formula can be written as

$$\zeta_K^{(k)}(s) = \frac{k!}{2\pi} \int_0^{2\pi} \frac{\zeta_K(s+Re^{i\theta})}{(Re^{i\theta})^k} d\theta \ll \log^k(|t|+2)|\zeta_K(s+Re^{i\theta})|.$$

Therefore Lemma 2.2 follows from the order of $\zeta_K(s)$ ($k = 0$ in (2.2)). □

Proposition 2.3. We have

$$\sum_{N(\mathfrak{A})\le x}\log\gamma(\mathfrak{A})=\rho_K x\log x+c_1x+c_2x^{\frac{1}{2}}+c_3x^{\frac{1}{3}}+E_0(x),$$

where c_i $(1\le i\le 3)$ are computable constants, and for $Y\to\infty$, the error term $E_0(x)$ satisfies

$$\int_1^Y |E_0(x)|dx \gg Y^{\frac{5}{4}}.$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}G(s,z)&=\sum_{\mathfrak{A}}\gamma^z(\mathfrak{A})N^{-s}(\mathfrak{A})\\&=\frac{\zeta_K(s-z)\zeta_K(2s-z)\zeta_K(3s-z)\zeta_K(4s-z)\zeta_K(4s-3z)}{\zeta_K(2s-2z)\zeta_K(3s-2z)\zeta_K(4s-2z)}G_1(s,z).\end{aligned}\tag{2.3}$$

Firstly, we take the first partial derivative with respect to z form both sides of (2.3), and put $z=0$ to get

$$\begin{aligned}\frac{\partial G(s,z)}{\partial z}\bigg|_{z=0}&=\sum_{\mathfrak{A}}\frac{\gamma^z(\mathfrak{A})\log\gamma(\mathfrak{A})}{N^s(\mathfrak{A})}\bigg|_{z=0}=\sum_{\mathfrak{A}}\frac{\log\gamma(\mathfrak{A})}{N^s(\mathfrak{A})}\\&=\frac{\zeta_K(s)\zeta_K(4s)G_1(s,0)\{\zeta_K^{'}(2s)\zeta_K(3s)+\zeta_K(2s)\zeta_K^{'}(3s)\}}{\zeta_K(2s)\zeta_K(3s)}+G_2(s),\end{aligned}$$

where

$$G_2(s)=\zeta_K(s)\zeta_K(4s)G_1^{'}(s,0)-\zeta_K^{'}(s)\zeta_K(4s)G_1(s,0)-2\zeta_K(s)\zeta_K^{'}(4s)G_1(s,0),$$

and $G_1(s,0)$ is absolutely convergent for $\Re s>\frac{1}{5}+\varepsilon$. We can easily see that $\sum_{\mathfrak{A}}\frac{\log\gamma(\mathfrak{A})}{N^s(\mathfrak{A})}$ has a pole of order 2 at $s=1$ and poles of order 1 at $s=\frac{1}{2},\frac{1}{3}$, which prompts us to consider that $\sum_{N(\mathfrak{A})\le x}\log\gamma(\mathfrak{A})$ should have the following asymptotic formula

$$\sum_{N(\mathfrak{A})\le x}\log\gamma(\mathfrak{A})=\rho_K x\log x+c_1x+c_2x^{\frac{1}{2}}+c_3x^{\frac{1}{3}}+E_0(x),$$

where $E_0(x)=O(x^{\frac{1}{4}+\varepsilon})$.

Following the idea of Pintz [12] and Nowak [10], we use the Mellin transform to get

$$\begin{aligned}H(s)&:=\int_1^\infty E_0(x)x^{-s-1}dx\\&=\int_1^\infty\left(\sum_{N(\mathfrak{A})\le x}\log\gamma(\mathfrak{A})-\rho_K x\log x+c_1x+c_2x^{\frac{1}{2}}+c_3x^{\frac{1}{3}}\right)\\&\quad\times x^{-s-1}dx\end{aligned}\tag{2.4}$$

for $\Re s > 1$. Now we deal with the first term of (2.4). By the partial integration and (2.4), we have

$$\begin{aligned}
&\int_1^\infty \sum_{N(\mathfrak{A})\le x} \frac{\log\gamma(\mathfrak{A})}{x^{s+1}}dx \\
&= -\frac{1}{s}\left(\sum_{N(\mathfrak{A})\le x} \log\gamma(\mathfrak{A})x^{-s}\Big|_1^\infty - \int_1^\infty x^{-s} d\left(\sum_{N(\mathfrak{A})\le x}\log\gamma(\mathfrak{A})\right)\right) \\
&= \frac{1}{s}\sum_{\mathfrak{A}} \frac{\log\gamma(\mathfrak{A})}{N^{s+1}(\mathfrak{A})} \\
&= \frac{\zeta_K(s)\zeta_K(4s)G_1(s,0)(\zeta_K'(2s)\zeta_K(3s)+\zeta(2s)\zeta_K'(3s))}{s\zeta_K(2s)\zeta_K(3s)} + \frac{G_2(s)}{s}.
\end{aligned}$$

Let $K(s) = \zeta_K(s)\zeta_K(4s)G_1(s,0)(\zeta_K'(2s)\zeta_K(3s) + \zeta(2s)\zeta_K'(3s))$. After computing another four terms of (2.4), we can get

$$\begin{aligned}
H(s) &= \frac{K(s)}{s\zeta_K(2s)\zeta_K(3s)} + \frac{G_2(s)}{s} - \frac{\rho_K}{(s-1)^2} - \frac{c_1}{s-1} - \frac{c_2}{s-\frac{1}{2}} - \frac{c_3}{s-\frac{1}{3}} \\
&= \frac{F(s)}{s(s-1)^2\zeta_K(2s)\zeta_K(3s)(2s-1)^2(3s-1)^2(4s-1)^2},
\end{aligned}$$

where

$$\begin{aligned}
F(s) =&\{K(s) + G_2(s)\zeta_K(2s)\zeta_K(3s)\}(s-1)^2(2s-1)^2(3s-1)^2(4s-1)^2 \\
&- s\zeta_K(2s)\zeta_K(3s)(2s-1)(3s-1)(4s-1)^2M(s),
\end{aligned}$$

here $M(s) = (2s-1)(3s-1)(\rho_K + c_1(s-1)) + (s-1)^2(2c_2(3s-1) + 3c_3(2s-1))$. It is easy to see that $F(s)$ is an entire function for $\Re s > \frac{1}{5} + \epsilon$. We choose $z_0 = \frac{1}{4} + i\beta_0$, according to the results of [1] (or [2–4,13,14]), we get $2z_0$ is a single zero of the Dedekind ζ-function and $\zeta_K(z_0)\zeta_K(3z_0)G_1(z_0,0) \neq 0$. In addition,

$$\zeta_K'(2z_0) \neq 0,\ \zeta_K(4z_0) = \zeta_K(1 + i4\beta_0) \neq 0.$$

We write

$$g(s) := \frac{s(s-1)^2\zeta_K(2s)(2s-1)^2\zeta_K(3s)(3s-1)^2(4s-1)^2}{(s-z_0)(s+2)^{13}},$$

which is regular in $\Re s > -2$, and

$$g(s)H(s) = \frac{F(s)}{(s-z_0)(s+2)^{13}}$$

is regular in $\Re s > \frac{1}{5} + \epsilon$ apart from a simple pole at $s = z_0$, since

$$\begin{aligned}
F(z_0) =&\zeta_K(z_0)\zeta_K'(2z_0)\zeta_K(3z_0)\zeta_K(4z_0)G_1(z_0,0) \\
&\times (z_0-1)^2(2z_0-1)^2(3z_0-1)^2(4z_0-1)^2 \neq 0.
\end{aligned}$$

Using the order of $\zeta_K(s)$ and $\zeta_K'(s)$ (Lemma 2.2), we know that the integrals

$$\int_{\beta-i\infty}^{\beta+i\infty} |g(s)|ds,\ \int_{\beta-i\infty}^{\beta+i\infty} |g(s)H(s)|ds$$

converge for $\beta \in \{\frac{1}{5}, 2\}$ as $|t| \to \infty$. Now, for $\eta > 0$, we define a weight function

$$\omega(\eta) := \int_{2-i\infty}^{2+i\infty} g(s)\eta^{s+1} ds$$

which satisfies

$$\omega(\eta) = \begin{cases} O(1), & \eta \geq 1, \\ 0, & \text{if } 0 < \eta < 1. \end{cases} \tag{2.5}$$

Therefore,

$$\begin{aligned} V(Y) :&= \frac{1}{Y}\int_{1}^{\infty} E_0(x)\omega\left(\frac{Y}{x}dx\right) \\ &= \frac{1}{Y}\int_{1}^{\infty} E_0(x)\left(\int_{2-i\infty}^{2+i\infty} g(s)\left(\frac{Y}{x}\right)^{s+1} ds\right)dx \\ &= \int_{2-i\infty}^{2+i\infty} g(s)Y^s\left(\int_{1}^{\infty} E_0(x)x^{-s-1}dx\right)ds \\ &= \int_{2-i\infty}^{2+i\infty} g(s)H(s)Y^s ds. \end{aligned}$$

For Y large, we shift the line of integration to $\Re s = \frac{1}{5}$, then we have

$$\begin{aligned} V(Y) &= 2\pi i \mathrm{Res}_{s=z_0}(g(s)H(s)Y^s) + \int_{\frac{1}{5}-i\infty}^{\frac{1}{5}+i\infty} g(s)H(s)Y^s ds \\ &= 2\pi i\alpha_0 Y^{z_0} + O(Y^{\frac{1}{5}}), \end{aligned}$$

where

$$\begin{aligned} \alpha_0 &= \frac{F(z_0)}{(z_0+2)^{13}} \\ &= \frac{\zeta_K(z_0)\zeta_K'(2z_0)\zeta_K(3z_0)\zeta_K(4z_0)G_1(z_0,0)(z_0-1)^2(2z_0-1)^2(3z_0-1)^2(4z_0-1)^2}{(z_0+2)^{13}}. \end{aligned} \tag{2.6}$$

By (2.6), we can evident that

$$|V(Y)| \gg |Y^{z_0}| = Y^{\frac{1}{4}}$$

as $Y \to \infty$. On the other hand, by (2.5), we can obtain

$$|V(Y)| = \left|\frac{1}{Y}\int_{1}^{Y} E_0(x)\omega\left(\frac{Y}{x}\right)dx\right| \ll \frac{1}{Y}\int_{1}^{Y} E_0(x)dx.$$

Consequently, for $Y \to \infty$, we have

$$\frac{1}{Y}\int_{1}^{Y} |E_0(x)|dx \gg Y^{\frac{1}{4}}.$$

□

3. Proof of Theorem 1.1

Now we prove Theorem 1.1. From Proposition 2.3, we get

$$\sum_{N(\mathfrak{A})\le x}\log(\gamma(\mathfrak{A}))=\rho_K x\log x+c_1x+c_2x^{\frac{1}{2}}+c_3x^{\frac{1}{3}}+E_0(x), \tag{3.1}$$

with

$$\int_1^Y |E_0(x)|dx \gg Y^{\frac{5}{4}},$$

where $Y\to\infty$. Using partial integration, we get

$$\begin{aligned}\sum_{N(\mathfrak{A})\le x}\lambda^{-1}(\mathfrak{A})=\sum_{2\le N(\mathfrak{A})\le x}\frac{\log\gamma(\mathfrak{A})}{\log N(\mathfrak{A})}&=\int_{2^-}^{x}\frac{1}{\log t}d\left(\sum_{2\le N(\mathfrak{A})\le t}\log\gamma(\mathfrak{A})\right)\\ &=\rho_K x+C_1\int_2^x\frac{1}{\log t}dt+C_2\int_2^x\frac{t^{-\frac{1}{2}}}{\log t}dt\\ &\quad+C_3\int_2^x\frac{t^{-\frac{2}{3}}}{\log t}dt+R_0(x),\end{aligned}$$

where

$$R_0(x)=\int_2^x\frac{dE_0(t)}{\log t}=\int_2^x\frac{E_0'(t)}{\log t}dt. \tag{3.2}$$

Taking the derivative of the above formula we get $E_0'(x)=R_0'(x)\log x$. Integrating both sides with respect to x, we have

$$E_0(x)=\int_1^x R_0'(t)\log t dt=R_0(x)\log x-\int_1^x\frac{R_0(t)}{t}dt. \tag{3.3}$$

And also

$$\begin{aligned}\int_1^Y|E_0(x)|dx&=\int_1^Y\left|R_0(x)\log x-\int_1^x\frac{R_0(t)}{t}dt\right|dx\\ &\le\int_1^Y|R_0(x)\log x|dx+\int_1^Y\left|\int_1^x\frac{R_0(t)}{t}dt\right|dx.\end{aligned} \tag{3.4}$$

We prove Theorem 1.1 by contradiction. Suppose that

$$\int_1^Y|R_0(x)|dx\le\epsilon\frac{Y^{\frac{5}{4}}}{\log Y}, \tag{3.5}$$

where ϵ is a small constant. Thus we have

$$\begin{aligned}\int_1^Y|E_0(x)|dx&\le\log Y\int_1^Y|R_0(x)|dx+\int_1^Y\left(\int_1^x\frac{R_0(t)}{t}dt\right)dx\\ &\le\epsilon Y^{\frac{5}{4}}+\int_1^Y\left(\int_1^x\frac{|R_0(t)|}{t}dt\right)dx.\end{aligned} \tag{3.6}$$

We use dyadic arguments to the inner integral to get

$$\begin{aligned}\int_1^x \frac{|R_0(t)|}{t}dt &= \sum_{j=0}^{\log\frac{x}{2}-1} \int_{2^{-j-1}x}^{2^{-j}x} \frac{|R_0(t)|}{t}dt \\ &\leq \sum_{j=0}^{\log x} \frac{1}{2^{-j-1}x} \int_{2^{-j-1}x}^{2^{-j}x} |R_0(t)|dt.\end{aligned} \tag{3.7}$$

By (3.5), we can obtain

$$\int_1^x \frac{|R_0(t)|}{t}dt \leq \sum_{j=0}^{\log x} \frac{\epsilon}{2^{-j-1}x} \cdot \frac{(2^{-j}x)^{\frac{5}{4}}}{\log(2^{-j}x)} \leq \epsilon x^{\frac{1}{4}}. \tag{3.8}$$

Inserting (3.8) into (3.6), we have

$$\int_1^Y |E_0(x)|dx \leq \epsilon Y^{\frac{5}{4}},$$

which contradict with Proposition 2.3. Then we have

$$\int_1^Y |R_0(x)|dx = \Omega\left(\frac{Y^{\frac{5}{4}}}{\log Y}\right).$$

4. Conclusions

For each integral ideal $\mathfrak{A}$, it is essential to measure the multiplicity of its prime ideal factors. In this paper, we define $\lambda(\mathfrak{A}) := \frac{\log N(\mathfrak{A})}{\log \gamma(\mathfrak{A})}$ to be the index of composition of $\mathfrak{A}$ and consider how well the main term of $\sum_{N(\mathfrak{A})\leq x} \lambda^{-1}(\mathfrak{A})$ approximates it, that is, what can be said about Ω-results for the index of composition of integral ideal. The results imply that the average order of $\lambda(\mathfrak{A})$ is ρ_K.

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Conflict of interest

The authors declare there is no conflict of interest.

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