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Research article

Strong and weak measurable optimal controls

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Abstract: Sufficient conditions for weak and strong minima in an optimal control problem of Lagrange are provided. This sufficiency theory is applicable for problems containing fixed end-points, nonlinear dynamics, nonlinear isoperimetric inequality and equality constraints together with nonlinear mixed time-state-control pointwise inequality and equality restrictions. The presence of purely measurable optimal controls is a fundamental component of this theory.

Keywords: calculus of variations; optimal control; inequality and equality constraints; sufficiency; strong minima; weak minima; measurable optimal controls
Mathematics Subject Classification: 49K15

1. Introduction

In this paper we establish and prove two new sufficiency theorems for weak and strong minima for an optimal control problem of Lagrange with fixed end-points, nonlinear dynamics, nonlinear isoperimetric inequality and equality constraints and pointwise mixed nonlinear time-state-control inequality and equality restrictions. The proof of the sufficiency theorems is independent of classical methods used to obtain sufficiency in optimal control problems of this type, see for example [32], where the insertion of the original optimal control problem in a Banach space is a fundamental component in order to obtain the corresponding sufficiency theory; [16], where the construction of a bounded solution of a matrix Riccati equation is crucial in this sufficiency approach; or [8, 19], where a verification function and a quadratic function satisfying a Hamilton-Jacobi inequality is an indispensable tool in the sufficiency treatments of these theories. Concretely, the sufficiency theorems of this article state that if an admissible process satisfies a first order sufficient condition related with Pontryagin maximum principle, a similar hypothesis of the necessary Legendre-Clebsch condition, the positivity of a quadratic integral on the cone of critical directions, and several conditions of Weierstrass of some functions, where one of them plays a similar role to the Hamiltonian of the problem, then, the previously mentioned admissible process is a strict local minimum. The set of active indices of the corresponding mixed time-state-control inequality constraints must be piecewise constant on the underlying time interval of consideration, the Lagrange multipliers of the inequality constraints must be nonnegative and in fact they have to be zero whenever the associated index of the Lagrange multiplier is inactive. Additionally, the proposed optimal controls need not be continuous on the underlying interval of time but *only measurable*, see for example [7–9, 13–19, 21, 22, 25–27, 29, 32], where the continuity of the proposed optimal control is a crucial assumption in some sufficiency optimal control theories having the same degree of generality as the problems studied in this article. In contrast, in Examples 2.3 and 2.4, we show how two purely measurable optimal controls comprised with the proposed optimal processes satisfy all the hypotheses of Theorems 2.1 and 2.2 becoming in this way strict local minima.

Additionally, it is worth mentioning that in these new sufficiency theorems for local minima presented in this paper, all the premises that must be satisfied by an admissible process to become an optimal process, are imposed in the hypotheses established in the theorems, in contrast, with other second order necessary and sufficiency theories which depend upon the verifiability of some preliminary assumptions, see for example [2, 3, 5, 6, 11, 24], where the necessary second order conditions for optimality depend on some previous hypotheses involving the full rankness of a matrix whose nature arises from the linear independence of vectors whose role are the gradients of the active inequality and equality constraints and where further assumptions involving some notions of regularity or normality of a solution are fundamental hypotheses; or [28], where the corresponding sufficiency theory for optimality depends upon the existence of a continuous function dominating the norm of one of the partial derivatives of the dynamic of the problem. Another remarkable feature presented in this theory concerns the fact that our sufficiency treatment not only provides sufficient conditions for strict local minima but they allow measuring the deviation between admissible costs and optimal costs. This deviation involves a functional playing the role of the square of a norm of the Banach space L^1, see for example [1, 23], where similar estimations of the growth of the objective functional around the optimal control are established.

On the other hand, it is worth pointing out the existence of some recent optimal control theories which also study optimal control problems with functional inequality or equality restrictions such as the isoperimetric constraints of this paper. Concretely, in [20], necessary optimality conditions for a Mayer optimal control problem involving semilinear unbounded evolution inclusions and inequality and equality Lipschitzian restrictions are obtained by constructing a sequence of discrete approximations and proving that the optimal solutions of discrete approximations converge uniformly to a given optimal process for the primary continuous-time problem. In [12], necessary and sufficient optimality conditions of Mayer optimal control problems involving differential inclusions and functional inequality constraints are presented and the authors study Mayer optimal control problems with higher order differential inclusions and inequality functional constraints. The necessary conditions for optimality obtained in [12], are important generalizations of associated problems for a first order differential inclusions of optimality settings established in [4, 10, 20]. The sufficiency conditions obtained in [12] include second order discrete inclusions with inequality end-point constraints. The use of convex and nonsmooth analysis plays a crucial role in this related sufficiency treatment. Moreover, one of the fundamental novelties of the work provided in [12] concerns the derivation of sufficient optimality conditions for Mayer optimal control problems having m-th order ordinary differential inclusions with $m \geq 3$.

The paper is organized as follows. In Section 2, we pose the problem we shall deal with together with some basic definitions, the statement of the main results and two examples illustrating the sufficiency theorems of the article. Section 3 is devoted to state one auxiliary lemma on which the proof of Theorem 2.1, given in the same section, is strongly based. Section 4 is dedicated to state another auxiliary result on which the proof of Theorem 2.2, once again given in the same section, is based.

2. The problem and the main results for local minima

Suppose we are given an interval $T := [t_0, t_1]$ in $\mathbf{R}$, two fixed points ξ_0, ξ_1 in $\mathbf{R}^n$, functions L, L_γ ($\gamma = 1, \ldots, K$) mapping $T \times \mathbf{R}^n \times \mathbf{R}^m$ to $\mathbf{R}$, two functions f and $\varphi = (\varphi_1, \ldots, \varphi_s)$ mapping $T \times \mathbf{R}^n \times \mathbf{R}^m$ to $\mathbf{R}^n$ and $\mathbf{R}^s$ respectively. Let

$$\mathcal{A} := \{(t, x, u) \in T \times \mathbf{R}^n \times \mathbf{R}^m \mid \varphi_\alpha(t, x, u) \leq 0 \, (\alpha \in R), \ \varphi_\beta(t, x, u) = 0 \, (\beta \in S)\}$$

where $R := \{1, \ldots, r\}$ and $S := \{r + 1, \ldots, s\}$ ($r = 0, 1, \ldots, s$). If $r = 0$ then $R = \emptyset$ and we disregard statements involving φ_α. Similarly, if $r = s$ then $S = \emptyset$ and we disregard statements involving φ_β.

Let $\{\Lambda_n\}$ be a sequence of measurable functions and let Λ be a measurable function. We shall say that the sequence of measurable functions $\{\Lambda_n\}$ converges almost uniformly to a function Λ on T, if given $\epsilon > 0$, there exists a measurable set $\Upsilon_\epsilon \subset T$ with $m(\Upsilon_\epsilon) < \epsilon$ such that $\{\Lambda_n\}$ converges uniformly to Λ on $T \setminus \Upsilon_\epsilon$. We will also denote uniform convergence by $\Lambda_n \stackrel{\text{u}}{\longrightarrow} \Lambda$, almost uniform convergence by $\Lambda_n \stackrel{\text{au}}{\longrightarrow} \Lambda$, strong convergence in L^p by $\Lambda_n \stackrel{L^p}{\longrightarrow} \Lambda$ and weak convergence in L^p by $\Lambda_n \stackrel{L^p}{\rightharpoonup} \Lambda$. From now on we shall not relabel the subsequences of a given sequence since this fact will not alter our results.

It will be assumed throughout the paper that L, L_γ ($\gamma = 1, \ldots, K$), f and φ have first and second derivatives with respect to x and u. Also, if we denote by $b(t, x, u)$ either $L(t, x, u)$, $L_\gamma(t, x, u)$ ($\gamma = 1, \ldots, K$), $f(t, x, u)$, $\varphi(t, x, u)$ or any of their partial derivatives of order less or equal than two with respect to x and u, we shall assume that if $\mathcal{B}$ is any bounded subset of $T \times \mathbf{R}^n \times \mathbf{R}^m$, then $|b(\mathcal{B})|$ is a bounded subset of $\mathbf{R}$. Additionally, we shall assume that if $\{(\Phi_q, \Psi_q)\}$ is any sequence in $AC(T; \mathbf{R}^n) \times L^\infty(T; \mathbf{R}^m)$ such that for some $\Upsilon \subset T$ measurable and some $(\Phi_0, \Psi_0) \in AC(T; \mathbf{R}^n) \times L^\infty(T; \mathbf{R}^m)$, $(\Phi_q, \Psi_q) \stackrel{L^\infty}{\longrightarrow} (\Phi_0, \Psi_0)$ on Υ, then for all $q \in \mathbf{N}$, $b(\cdot, \Phi_q(\cdot), \Psi_q(\cdot))$ is measurable on Υ and

$$b(\cdot, \Phi_q(\cdot), \Psi_q(\cdot)) \stackrel{L^\infty}{\longrightarrow} b(\cdot, \Phi_0(\cdot), \Psi_0(\cdot)) \text{ on } \Upsilon.$$

Note that all conditions given above are satisfied if the functions L, L_γ ($\gamma = 1, \ldots, K$), f and φ and their first and second derivatives with respect to x and u are continuous on $T \times \mathbf{R}^n \times \mathbf{R}^m$.

The fixed end-point optimal control problem we shall deal with, denoted by (P), is that of minimizing the functional

$$I(x, u) := \int_{t_0}^{t_1} L(t, x(t), u(t)) dt$$

over all couples (x, u) with $x \colon T \to \mathbf{R}^n$ absolutely continuous and $u \colon T \to \mathbf{R}^m$ essentially bounded, satisfying the constraints

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \text{ (a.e. in } T). \\ x(t_0) = \xi_0, \ \ x(t_1) = \xi_1. \\ I_i(x, u) := \int_{t_0}^{t_1} L_i(t, x(t), u(t))dt \leq 0 \ (i = 1, \ldots, k). \\ I_j(x, u) := \int_{t_0}^{t_1} L_j(t, x(t), u(t))dt = 0 \ (j = k + 1, \ldots, K). \\ (t, x(t), u(t)) \in \mathcal{A} \ (t \in T). \end{cases}$$

Denote by $\mathcal{X}$ the space of all absolutely continuous functions mapping T to $\mathbf{R}^n$ and by $\mathcal{U}_c := L^\infty(T; \mathbf{R}^c)$ ($c \in \mathbf{N}$). Elements of $\mathcal{X} \times \mathcal{U}_m$ will be called *processes* and a process (x, u) is *admissible* if it satisfies the constraints. A process (x, u) *solves* (P) if it is admissible and $I(x, u) \leq I(y, v)$ for all admissible processes (y, v). An admissible process (x, u) is called a *strong minimum* of (P) if it is a minimum of I with respect to the norm

$$\|x\| := \sup_{t \in T} |x(t)|,$$

that is, if for any $\epsilon > 0$, $I(x, u) \leq I(y, v)$ for all admissible processes (y, v) satisfying $\|y - x\| < \epsilon$. An admissible process (x, u) is called a *weak minimum* of (P) if it is a minimum of I with respect to the norm

$$\|(x, u)\| := \|x\| + \|u\|_\infty,$$

that is, if for any $\epsilon > 0$, $I(x, u) \leq I(y, v)$ for all admissible processes (y, v) satisfying $\|(y, v) - (x, u)\| < \epsilon$. It is a *strict minimum* if $I(x, u) = I(y, v)$ only in case $(x, u) = (y, v)$. Note that the crucial difference between strong and weak minima is that in the former, if I affords a strong minimum at (x_0, u_0), then, if (x, u) is admissible and it is sufficiently close to (x_0, u_0), in the sense that the quantity $\|x - x_0\|_\infty$ is sufficiently small, then $I(x, u) \geq I(x_0, u_0)$, meanwhile for the latter, if (x, u) is admissible and it is sufficiently close to (x_0, u_0), in the sense that the quantities $\|x - x_0\|_\infty$, $\|u - u_0\|_\infty$ are sufficiently small, then $I(x, u) \geq I(x_0, u_0)$.

The following definitions will be useful in order to continue with the development of this theory.

• For any $(x, u) \in \mathcal{X} \times \mathcal{U}_m$ we shall use the notation $(\tilde{x}(t))$ to represent $(t, x(t), u(t))$. Similarly $(\tilde{x}_0(t))$ represents $(t, x_0(t), u_0(t))$. Throughout the paper the notation "$*$" will denote transpose.

• Given K real numbers $\lambda_1, \ldots, \lambda_K$, consider the functional $I_0 \colon \mathcal{X} \times \mathcal{U}_m \to \mathbf{R}$ defined by

$$I_0(x, u) := I(x, u) + \sum_{\gamma=1}^{K} \lambda_\gamma I_\gamma(x, u) = \int_{t_0}^{t_1} L_0(\tilde{x}(t))dt,$$

where $L_0 \colon T \times \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is given by

$$L_0(t, x, u) := L(t, x, u) + \sum_{\gamma=1}^{K} \lambda_\gamma L_\gamma(t, x, u).$$

• For all $(t, x, u, \rho, \mu) \in T \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^s$, set

$$H(t, x, u, \rho, \mu) := \rho^* f(t, x, u) - L_0(t, x, u) - \mu^* \varphi(t, x, u).$$

Given $\rho \in \mathcal{X}$ and $\mu \in \mathcal{U}_s$ define, for all $(t, x, u) \in T \times \mathbf{R}^n \times \mathbf{R}^m$,

$$F_0(t, x, u) := -H(t, x, u, \rho(t), \mu(t)) - \dot{\rho}^*(t)x$$

and let

$$J_0(x,u) := \rho^*(t_1)\xi_1 - \rho^*(t_0)\xi_0 + \int_{t_0}^{t_1} F_0(\tilde{x}(t))dt.$$

• Consider the *first variations* of J_0 and I_γ ($\gamma = 1,\dots,K$) with respect to $(x,u) \in \mathcal{X} \times \mathcal{U}_m$ over $(y,v) \in \mathcal{X} \times L^2(T;\mathbf{R}^m)$ which are given, respectively, by

$$J_0'((x,u);(y,v)) := \int_{t_0}^{t_1} \{F_{0x}(\tilde{x}(t))y(t) + F_{0u}(\tilde{x}(t))v(t)\}dt,$$

$$I_\gamma'((x,u);(y,v)) := \int_{t_0}^{t_1} \{L_{\gamma x}(\tilde{x}(t))y(t) + L_{\gamma u}(\tilde{x}(t))v(t)\}dt.$$

The *second variation of* J_0 with respect to $(x,u) \in \mathcal{X} \times \mathcal{U}_m$ over $(y,v) \in \mathcal{X} \times L^2(T;\mathbf{R}^m)$ is given by

$$J_0''((x,u);(y,v)) := \int_{t_0}^{t_1} 2\Omega_0(\tilde{x}(t);t,y(t),v(t))dt$$

where, for all $(t,y,v) \in T \times \mathbf{R}^n \times \mathbf{R}^m$,

$$2\Omega_0(\tilde{x}(t);t,y,v) := y^*F_{0xx}(\tilde{x}(t))y + 2y^*F_{0xu}(\tilde{x}(t))v + v^*F_{0uu}(\tilde{x}(t))v.$$

• Denote by E_0 the *Weierstrass excess function* of F_0, given by

$$E_0(t,x,u,v) := F_0(t,x,v) - F_0(t,x,u) - F_{0u}(t,x,u)(v-u).$$

Similarly, the *Weierstrass excess function* of L_γ ($\gamma = 1,\dots,K$) corresponds to

$$E_\gamma(t,x,u,v) := L_\gamma(t,x,v) - L_\gamma(t,x,u) - L_{\gamma u}(t,x,u)(v-u).$$

• For all $(x,u) \in \mathcal{X} \times L^1(T;\mathbf{R}^m)$ let

$$D(x,u) := \max\{D_1(x), D_2(u)\}$$

where

$$D_1(x) := V(x(t_0)) + \int_{t_0}^{t_1} V(\dot{x}(t))dt \quad \text{and} \quad D_2(u) := \int_{t_0}^{t_1} V(u(t))dt,$$

where $V(\pi) := (1+|\pi|^2)^{1/2} - 1$ with $\pi := (\pi_1,\dots,\pi_n)^* \in \mathbf{R}^n$ or $\pi := (\pi_1,\dots,\pi_m)^* \in \mathbf{R}^m$.

Finally, for all $(t,x,u) \in T \times \mathbf{R}^n \times \mathbf{R}^m$, denote by

$$\mathcal{I}_a(t,x,u) := \{\alpha \in R \mid \varphi_\alpha(t,x,u) = 0\},$$

the set of active indices of (t,x,u) with respect to the mixed inequality constraints. For all $(x,u) \in \mathcal{X} \times \mathcal{U}_m$, denote by

$$i_a(x,u) := \{i = 1,\dots,k \mid I_i(x,u) = 0\},$$

the set of active indices of (x,u) with respect to the isoperimetric inequality constraints. Given $(x,u) \in \mathcal{X} \times \mathcal{U}_m$, let $\mathcal{Y}(x,u)$ be the set of all $(y,v) \in \mathcal{X} \times L^2(T;\mathbf{R}^m)$ satisfying

$$\begin{cases} \dot{y}(t) = f_x(\tilde{x}(t))y(t) + f_u(\tilde{x}(t))v(t) \text{ (a.e. in } T), \ y(t_i) = 0 \ (i = 0,1). \\ I_i'((x,u);(y,v)) \le 0 \ (i \in i_a(x,u)), \ I_j'((x,u);(y,v)) = 0 \ (j = k+1,\ldots,K). \\ \varphi_{\alpha x}(\tilde{x}(t))y(t) + \varphi_{\alpha u}(\tilde{x}(t))v(t) \le 0 \text{ (a.e. in } T, \alpha \in \mathcal{I}_a(\tilde{x}(t))). \\ \varphi_{\beta x}(\tilde{x}(t))y(t) + \varphi_{\beta u}(\tilde{x}(t))v(t) = 0 \text{ (a.e. in } T, \beta \in S). \end{cases}$$

The set $\mathcal{Y}(x,u)$ is called the cone of *critical directions* along (x,u).

Now we are in a position to state the main results of the article, two sufficiency results for strict local minima of problem (P). Given an admissible process (x_0,u_0) where the proposed optimal controls u_0 need not be continuous but only measurable, the hypotheses include, two conditions related with Pontryagin maximum principle, a similar assumption of the necessary Legendre-Clebsch condition, the positivity of the second variation on the cone of critical directions and some conditions involving the Weierstrass functions delimiting problem (P). It is worth observing that the sufficiency theorems not only give sufficient conditions for strict local minima but also provides some information concerning the deviation between optimal and feasible costs. In the measure of this deviation are involved the functionals D_i $(i = 1,2)$ which play the role of the square of the norm of the Banach space L^1.

The following theorem provides sufficient conditions for a strict strong minimum of problem (P).

Theorem 2.1 *Let (x_0,u_0) be an admissible process. Assume that $\mathcal{I}_a(\tilde{x}_0(\cdot))$ is piecewise constant on T, suppose that there exist $\rho \in \mathcal{X}$, $\mu \in \mathcal{U}_s$ with $\mu_\alpha(t) \ge 0$ and $\mu_\alpha(t)\varphi_\alpha(\tilde{x}_0(t)) = 0$ $(\alpha \in R, t \in T)$, two positive numbers δ, ϵ, and multipliers $\lambda_1,\ldots,\lambda_K$ with $\lambda_i \ge 0$ and $\lambda_i I_i(x_0,u_0) = 0$ $(i = 1,\ldots,k)$ such that*

$$\dot{\rho}(t) = -H_x^*(\tilde{x}_0(t),\rho(t),\mu(t)) \ (a.e. \ in \ T),$$

$$H_u^*(\tilde{x}_0(t),\rho(t),\mu(t)) = 0 \ (t \in T),$$

and the following holds:

(i) $H_{uu}(\tilde{x}_0(t),\rho(t),\mu(t)) \le 0$ *(a.e. in T).*

(ii) $J_0''((x_0,u_0);(y,v)) > 0$ *for all* $(y,v) \ne (0,0)$, $(y,v) \in \mathcal{Y}(x_0,u_0)$.

(iii) *If (x,u) is admissible with $\|x - x_0\| < \epsilon$, then*

a. $E_0(t,x(t),u_0(t),u(t)) \ge 0$ *(a.e. in T).*

b. $\int_{t_0}^{t_1} E_0(t,x(t),u_0(t),u(t))dt \ge \delta \max\{\int_{t_0}^{t_1} V(\dot{x}(t) - \dot{x}_0(t))dt, \int_{t_0}^{t_1} V(u(t) - u_0(t))dt\}$.

c. $\int_{t_0}^{t_1} E_0(t,x(t),u_0(t),u(t))dt \ge \delta|\int_{t_0}^{t_1} E_\gamma(t,x(t),u_0(t),u(t))dt|$ $(\gamma = 1,\ldots,K)$.

In this case, there exist $\theta_1,\theta_2 > 0$ such that if (x,u) is admissible with $\|x - x_0\| < \theta_1$,

$$I(x,u) \ge I(x_0,u_0) + \theta_2 D(x - x_0, u - u_0).$$

In particular, (x_0,u_0) is a strict strong minimum of (P).

The theorem below gives sufficient conditions for weak minima of problem (P).

Theorem 2.2 *Let (x_0,u_0) be an admissible process. Assume that $\mathcal{I}_a(\tilde{x}_0(\cdot))$ is piecewise constant on T,* *suppose that there exist* $\rho \in \mathcal{X}$, $\mu \in \mathcal{U}_s$ *with* $\mu_\alpha(t) \geq 0$ *and* $\mu_\alpha(t)\varphi_\alpha(\tilde{x}_0(t)) = 0$ $(\alpha \in R, t \in T)$, *two positive numbers* δ, ϵ, *and multipliers* $\lambda_1, \ldots, \lambda_K$ *with* $\lambda_i \geq 0$ *and* $\lambda_i I_i(x_0, u_0) = 0$ $(i = 1, \ldots, k)$ *such that*

$$\dot{\rho}(t) = -H_x^*(\tilde{x}_0(t), \rho(t), \mu(t)) \text{ (a.e. in } T),$$

$$H_u^*(\tilde{x}_0(t), \rho(t), \mu(t)) = 0 \ (t \in T),$$

and the following holds:

(i) $H_{uu}(\tilde{x}_0(t), \rho(t), \mu(t)) \leq 0$ (*a.e. in* T).

(ii) $J_0''((x_0, u_0); (y, v)) > 0$ *for all* $(y, v) \neq (0, 0)$, $(y, v) \in \mathcal{Y}(x_0, u_0)$.

(iii) *If* (x, u) *is admissible with* $\|(x, u) - (x_0, u_0)\| < \epsilon$, *then*

a′. $\int_{t_0}^{t_1} E_0(t, x(t), u_0(t), u(t))dt \geq \delta \int_{t_0}^{t_1} V(u(t) - u_0(t))dt$.

b′. $\int_{t_0}^{t_1} E_0(t, x(t), u_0(t), u(t))dt \geq \delta |\int_{t_0}^{t_1} E_\gamma(t, x(t), u_0(t), u(t))dt|$ $(\gamma = 1, \ldots, K)$.

In this case, there exist $\theta_1, \theta_2 > 0$ *such that if* (x, u) *is admissible with* $\|(x, u) - (x_0, u_0)\| < \theta_1$,

$$I(x, u) \geq I(x_0, u_0) + \theta_2 D_2(u - u_0).$$

In particular, (x_0, u_0) *is a strict weak minimum of* (P).

Examples 2.3 and 2.4 illustrate Theorems 2.1 and 2.2 respectively. It is worth mentioning that the sufficiency theory of [28] cannot be applied in both examples. Indeed, if f denotes the dynamic of the problems, as one readily verifies, in both examples, we have that

$$f_u(t, x, u) = (u_2, u_1) \text{ for all } (t, x, u) \in [0, 1] \times \mathbf{R} \times \mathbf{R}^2,$$

and hence, it does not exist a continuous function $\psi\colon [0, 1] \times \mathbf{R} \to \mathbf{R}$ such that

$$|f_u(t, x, u)| \leq \psi(t, x) \text{ for all } (t, x, u) \in [0, 1] \times \mathbf{R} \times \mathbf{R}^2.$$

Example 2.3 *Let* $u_{02}\colon [0, 1] \to \mathbf{R}$ *be any measurable function whose codomain belongs to the set* $\{-1, 1\}$.

Consider problem (P) of minimizing

$$I(x, u) := \int_0^1 \{\sinh(u_1(t)) + u_1^2(t)\cos(2\pi u_2(t)) - x^2(t)\}dt$$

over all couples (x, u) *with* $x\colon [0, 1] \to \mathbf{R}$ *absolutely continuous and* $u\colon [0, 1] \to \mathbf{R}^2$ *essentially bounded satisfying the constraints*

$$\begin{cases} x(0) = x(1) = 0. \\ \dot{x}(t) = u_1(t)u_2(t) + \frac{1}{2}x(t) \text{ (a.e. in } [0, 1]). \\ I_1(x, u) := \int_0^1 \{\frac{1}{4}x^2(t) + x(t)u_1(t)u_2(t)\}dt \leq 0. \\ (t, x(t), u(t)) \in \mathcal{A} \ (t \in [0, 1]) \end{cases}$$

where

$$\mathcal{A} := \{(t, x, u) \in [0, 1] \times \mathbf{R} \times \mathbf{R}^2 \mid u_1 \geq 0,\ (u_2 - u_{02}(t))^2 \leq 1,\ u_2^2 = 1\}.$$

For this case, $T = [0,1]$, $n = 1$, $m = 2$, $r = 2$, $s = 3$, $k = K = 1$, $\xi_0 = \xi_1 = 0$,

$$L(t,x,u) = \sinh(u_1) + u_1^2\cos(2\pi u_2) - x^2, \quad f(t,x,u) = u_1u_2 + \tfrac{1}{2}x,$$

$$L_1(t,x,u) = \tfrac{1}{4}x^2 + xu_1u_2, \quad L_0(t,x,u) = \sinh(u_1) + u_1^2\cos(2\pi u_2) - x^2 + \lambda_1[\tfrac{1}{4}x^2 + xu_1u_2],$$

$$\varphi_1(t,x,u) = -u_1, \quad \varphi_2(t,x,u) = (u_2 - u_{02}(t))^2 - 1, \quad \varphi_3(t,x,u) = u_2^2 - 1.$$

Clearly, L, L_1, f and $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ satisfy the hypotheses imposed in the statement of the problem. Also, as one readily verifies, the process $(x_0, u_0) = (x_0, u_{01}, u_{02}) \equiv (0, 0, u_{02})$ is admissible. Moreover,

$$\begin{aligned} H(t,x,u,\rho,\mu) &= \rho u_1u_2 + \tfrac{1}{2}\rho x - \sinh(u_1) - u_1^2\cos(2\pi u_2) + x^2 - \lambda_1[\tfrac{1}{4}x^2 + xu_1u_2] \\ &\quad + \mu_1u_1 - \mu_2[(u_2 - u_{02}(t))^2 - 1] - \mu_3[u_2^2 - 1], \end{aligned}$$

$$H_x(t,x,u,\rho,\mu) = \tfrac{1}{2}\rho + 2x - \lambda_1[\tfrac{1}{2}x + u_1u_2],$$

$$H_u(t,x,u,\rho,\mu) = \begin{pmatrix} \rho u_2 - \cosh(u_1) - 2u_1\cos(2\pi u_2) - \lambda_1xu_2 + \mu_1 \\ \rho u_1 + 2\pi u_1^2\sin(2\pi u_2) - \lambda_1xu_1 - 2\mu_2(u_2 - u_{02}(t)) - 2\mu_3u_2 \end{pmatrix}^*.$$

Therefore, if we set $\rho \equiv 0$, $\mu_1 \equiv 1$, $\mu_2 = \mu_3 \equiv 0$ and $\lambda_1 = 0$, we have

$$\dot\rho(t) = -H_x(\tilde x_0(t), \rho(t), \mu(t)) \text{ (a.e. in } T), \quad H_u(\tilde x_0(t), \rho(t), \mu(t)) = (0,0) \ (t \in T),$$

and hence the first order sufficient conditions involving the Hamiltonian of problem (P) are verified. Moreover, if we set $R := \{1, 2\}$, observe that

$$\lambda_1 \geq 0, \quad \lambda_1 I_1(x_0, u_0) = 0,$$

$$\mu_\alpha(t) \geq 0, \quad \mu_\alpha(t)\varphi_\alpha(\tilde x_0(t)) = 0 \quad (\alpha \in R,\ t \in T).$$

Additionally, $\mathcal{I}_a(\tilde x_0(\cdot)) \equiv \{1\}$ is constant on T. Also, it is readily seen that for all $t \in T$,

$$H_{uu}(\tilde x_0(t), \rho(t), \mu(t)) = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix},$$

and so condition (i) of Theorem 2.1 is satisfied. Observe that, for all $t \in T$,

$$f_x(\tilde x_0(t)) = \tfrac{1}{2}, \quad f_u(\tilde x_0(t)) = (u_{02}(t), 0), \quad L_{1x}(\tilde x_0(t)) = 0, \quad L_{1u}(\tilde x_0(t)) = (0,0),$$

$$\varphi_{1x}(\tilde x_0(t)) = 0, \quad \varphi_{1u}(\tilde x_0(t)) = (-1, 0), \quad \varphi_{3x}(\tilde x_0(t)) = 0, \quad \varphi_{3u}(\tilde x_0(t)) = (0, 2u_{02}(t)).$$

Thus, $\mathcal{Y}(x_0, u_0)$ is given by all $(y, v) \in \mathcal{X} \times L^2(T; \mathbf{R}^2)$ satisfying

$$\begin{cases} y(0) = y(1) = 0. \\ \dot y(t) = \tfrac{1}{2}y(t) + u_{02}(t)v_1(t) \text{ (a.e. in } T). \\ -v_1(t) \leq 0 \text{ (a.e. in } T). \\ 2u_{02}(t)v_2(t) = 0 \text{ (a.e. in } T). \end{cases}$$

Moreover, note that, for all $(t, x, u) \in T \times \mathbf{R} \times \mathbf{R}^2$,

$$F_0(t,x,u) = -H(t,x,u,\rho(t),\mu(t)) - \dot\rho(t)x = \sinh(u_1) + u_1^2\cos(2\pi u_2) - x^2 - u_1,$$

and so, for all $t \in T$,

$$F_{0xx}(\tilde{x}_0(t)) = -2, \quad F_{0xu}(\tilde{x}_0(t)) = (0,0), \quad F_{0uu}(\tilde{x}_0(t)) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Consequently, we have

$$\begin{aligned} \tfrac{1}{2}J_0''((x_0,u_0);(y,v)) &= \int_0^1 \{v_1^2(t) - y^2(t)\}dt = \int_0^1 \{(\dot{y}(t) - \tfrac{1}{2}y(t))^2 - y^2(t)\}dt \\ &= \int_0^1 \{\dot{y}^2(t) - y(t)\dot{y}(t) - \tfrac{3}{4}y^2(t)\}dt = \int_0^1 \{\dot{y}^2(t) - \tfrac{3}{4}y^2(t)\}dt > 0 \end{aligned}$$

for all $(y,v) \neq (0,0)$, $(y,v) \in \mathcal{Y}(x_0,u_0)$. Hence, condition (ii) of Theorem 2.1 is verified.

Additionally, observe that for all (x,u) admissible and all $t \in T$,

$$E_0(t,x(t),u_0(t),u(t)) = \sinh(u_1(t)) + u_1^2(t)\cos(2\pi u_2(t)) - u_1(t) \geq u_1^2(t)\cos(2\pi u_{02}(t)) = u_1^2(t) \geq 0,$$

and then, condition (iii)(a) of Theorem 2.1 is satisfied for any $\epsilon > 0$. Now, if (x,u) is admissible, then

$$u - u_0 = (u_1 - u_{01}, u_2 - u_{02}) = (u_1, u_{02} - u_{02}) = (u_1, 0)$$

and so, if (x,u) is admissible,

$$\int_0^1 E_0(t,x(t),u_0(t),u(t))dt \geq \int_0^1 u_1^2(t)dt \geq \int_0^1 V(u_1(t))dt = \int_0^1 V(u(t)-u_0(t))dt.$$

Also, if (x,u) is admissible, then

$$\begin{aligned} \int_0^1 E_0(t,x(t),u_0(t),u(t))dt &\geq \int_0^1 u_1^2(t)dt = \int_0^1 \{(u_1(t)u_2(t) + \tfrac{1}{2}x(t))^2 - x(t)u_1(t)u_2(t) - \tfrac{1}{4}x^2(t)\}dt \\ &= \int_0^1 \{\dot{x}^2(t) - x(t)u_1(t)u_2(t) - \tfrac{1}{4}x^2(t)\}dt \geq \int_0^1 \dot{x}^2(t)dt \geq \int_0^1 V(\dot{x}(t) - \dot{x}_0(t))dt. \end{aligned}$$

Therefore, if (x,u) is admissible, then

$$\int_0^1 E_0(t,x(t),u_0(t),u(t))dt \geq \max\Big\{\int_0^1 V(\dot{x}(t)-\dot{x}_0(t))dt, \int_0^1 V(u(t)-u_0(t))dt\Big\},$$

and hence, condition (iii)(b) of Theorem 2.1 is verified for any $\epsilon > 0$ and $\delta = 1$. Finally, if (x,u) is admissible, note that

$$\begin{aligned} &\Big|\int_0^1 E_1(t,x(t),u_0(t),u(t))dt\Big| = \Big|\int_0^1 x(t)u_1(t)u_2(t)dt\Big| = \Big|\int_0^1 \{x(t)\dot{x}(t) - \tfrac{1}{2}x^2(t)\}dt\Big| \\ = &\Big|\int_0^1 -\tfrac{1}{2}x^2(t)dt\Big| = \frac{1}{2}\int_0^1 x^2(t)dt \leq \int_0^1 \dot{x}^2(t)dt = \int_0^1 (u_1(t)u_2(t) + \tfrac{1}{2}x(t))^2 dt \\ = &\int_0^1 u_1^2(t)dt + \int_0^1 \{x(t)u_1(t)u_2(t) + \tfrac{1}{4}x^2(t)\}dt \leq \int_0^1 u_1^2(t)dt + \int_0^1 x(t)\dot{x}(t)dt \end{aligned}$$

$$= \int_0^1 u_1^2(t)dt \leq \int_0^1 E_0(t, x(t), u_0(t), u(t))dt,$$

implying that condition (iii)(c) of Theorem 2.1 holds for any $\epsilon > 0$ and $\delta = 1$. By Theorem 2.1, (x_0, u_0) is a strict strong minimum of (P).

Example 2.4 *Consider problem (P) of minimizing*

$$I(x, u) := \int_0^1 \{\sinh(u_1(t) + u_1(t)x^3(t)) + \tfrac{1}{2}u_1^2(t)\cos(2\pi u_2(t)) - \cosh(x(t)) + 1\}dt$$

over all couples (x, u) *with* $x\colon [0, 1] \to \mathbf{R}$ *absolutely continuous and* $u\colon [0, 1] \to \mathbf{R}^2$ *essentially bounded satisfying the constraints*

$$\begin{cases} x(0) = x(1) = 0. \\ \dot{x}(t) = u_1(t)u_2(t) + x(t) \ (a.e. \ in \ [0, 1]). \\ I_1(x, u) := \int_0^1 \{\sin(u_1(t)) - \sinh(u_1(t) + u_1(t)x^3(t))\}dt \leq 0. \\ (t, x(t), u(t)) \in \mathcal{A} \ (t \in [0, 1]) \end{cases}$$

where

$$\mathcal{A} := \{(t, x, u) \in [0, 1] \times \mathbf{R} \times \mathbf{R}^2 \mid \sin(u_1) \geq 0, \ u_2^2 = 1\}.$$

For this case, $T = [0, 1]$, $n = 1$, $m = 2$, $r = 1$, $s = 2$, $k = K = 1$, $\xi_0 = \xi_1 = 0$,

$$L(t, x, u) = \sinh(u_1 + u_1x^3) + \tfrac{1}{2}u_1^2\cos(2\pi u_2) - \cosh(x) + 1, \quad f(t, x, u) = u_1u_2 + x,$$

$$L_1(t, x, u) = \sin(u_1) - \sinh(u_1 + u_1x^3),$$

$$L_0(t, x, u) = \sinh(u_1 + u_1x^3) + \tfrac{1}{2}u_1^2\cos(2\pi u_2) - \cosh(x) + 1 + \lambda_1[\sin(u_1) - \sinh(u_1 + u_1x^3)],$$

$$\varphi_1(t, x, u) = -\sin(u_1), \quad \varphi_2(t, x, u) = u_2^2 - 1.$$

Clearly, L, L_1, f and $\varphi = (\varphi_1, \varphi_2)$ satisfy the hypotheses imposed in the statement of the problem.
Let $u_{02}\colon T \to \mathbf{R}$ be any measurable function whose codomain belongs to the set $\{-1, 1\}$.
Clearly, the process $(x_0, u_0) = (x_0, u_{01}, u_{02}) \equiv (0, 0, u_{02})$ is admissible.
Moreover,

$$\begin{aligned} H(t, x, u, \rho, \mu) &= \rho u_1u_2 + \rho x - \sinh(u_1 + u_1x^3) - \tfrac{1}{2}u_1^2\cos(2\pi u_2) + \cosh(x) - 1 \\ &\quad -\lambda_1[\sin(u_1) - \sinh(u_1 + u_1x^3)] + \mu_1\sin(u_1) - \mu_2[u_2^2 - 1], \end{aligned}$$

$$H_x(t, x, u, \rho, \mu) = \rho - 3x^2u_1\cosh(u_1 + u_1x^3) + \sinh(x) + 3\lambda_1x^2u_1\cosh(u_1 + u_1x^3),$$

$$\begin{aligned} H_{u_1}(t, x, u, \rho, \mu) &= \rho u_2 - [1 + x^3]\cosh(u_1 + u_1x^3) - u_1\cos(2\pi u_2) \\ &\quad -\lambda_1[\cos(u_1) - \{1 + x^3\}\cosh(u_1 + u_1x^3)] + \mu_1\cos(u_1), \end{aligned}$$

$$H_{u_2}(t, x, u, \rho, \mu) = \rho u_1 + \pi u_1^2\sin(2\pi u_2) - 2\mu_2u_2.$$

Therefore, if we set $\rho \equiv 0$, $\mu_1 \equiv 1$, $\mu_2 \equiv 0$ and $\lambda_1 = 0$, we have

$$\dot{\rho}(t) = -H_x(\tilde{x}_0(t), \rho(t), \mu(t)) \text{ (a.e. in } T), \quad H_u(\tilde{x}_0(t), \rho(t), \mu(t)) = (0, 0) \ (t \in T),$$

and hence the first order sufficient conditions involving the Hamiltonian of problem (P) are verified. Additionally, observe that

$$\lambda_1 \geq 0, \quad \lambda_1 I_1(x_0, u_0) = 0,$$

$$\mu_1(t) \geq 0, \quad \mu_1(t)\varphi_1(\tilde{x}_0(t)) = 0 \quad (t \in T).$$

Also, $\mathcal{I}_a(\tilde{x}_0(\cdot)) \equiv \{1\}$ is constant on T. Moreover, it is readily seen that for all $t \in T$,

$$H_{uu}(\tilde{x}_0(t), \rho(t), \mu(t)) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and so condition (i) of Theorem 2.2 is satisfied. Observe that, for all $t \in T$,

$$f_x(\tilde{x}_0(t)) = 1, \quad f_u(\tilde{x}_0(t)) = (u_{02}(t), 0), \quad L_{1x}(\tilde{x}_0(t)) = 0, \quad L_{1u}(\tilde{x}_0(t)) = (0, 0),$$

$$\varphi_{1x}(\tilde{x}_0(t)) = 0, \quad \varphi_{1u}(\tilde{x}_0(t)) = (-1, 0), \quad \varphi_{2x}(\tilde{x}_0(t)) = 0, \quad \varphi_{2u}(\tilde{x}_0(t)) = (0, 2u_{02}(t)).$$

Thus, $\mathcal{Y}(x_0, u_0)$ is given by all $(y, v) \in \mathcal{X} \times L^2(T; \mathbf{R}^2)$ satisfying

$$\begin{cases} y(0) = y(1) = 0. \\ \dot{y}(t) = y(t) + u_{02}(t)v_1(t) \text{ (a.e. in } T). \\ -v_1(t) \leq 0 \text{ (a.e. in } T). \\ 2u_{02}(t)v_2(t) = 0 \text{ (a.e. in } T). \end{cases}$$

Also, note that, for all $(t, x, u) \in T \times \mathbf{R} \times \mathbf{R}^2$,

$$F_0(t, x, u) = -H(t, x, u, \rho(t), \mu(t)) - \dot{\rho}(t)x = \sinh(u_1 + u_1 x^3) + \tfrac{1}{2}u_1^2 \cos(2\pi u_2) - \cosh(x) + 1 - \sin(u_1),$$

and so, for all $t \in T$,

$$F_{0xx}(\tilde{x}_0(t)) = -1, \quad F_{0xu}(\tilde{x}_0(t)) = (0, 0), \quad F_{0uu}(\tilde{x}_0(t)) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Consequently, we have

$$\begin{aligned} J_0''((x_0, u_0); (y, v)) &= \int_0^1 \{v_1^2(t) - y^2(t)\}dt = \int_0^1 \{(\dot{y}(t) - y(t))^2 - y^2(t)\}dt \\ &= \int_0^1 \{\dot{y}^2(t) - 2y(t)\dot{y}(t)\}dt = \int_0^1 \dot{y}^2(t)dt > 0 \end{aligned}$$

for all $(y, v) \neq (0, 0)$, $(y, v) \in \mathcal{Y}(x_0, u_0)$. Hence, condition (ii) of Theorem 2.2 is verified.

Additionally, observe that for any $\epsilon \in (0, 1)$, all (x, u) admissible satisfying $\|(x, u) - (x_0, u_0)\| < \epsilon$ and all $t \in T$,

$$\begin{aligned} E_0(t, x(t), u_0(t), u(t)) &= \sinh(u_1(t) + u_1(t)x^3(t)) + \tfrac{1}{2}u_1^2(t)\cos(2\pi u_2(t)) - \sin(u_1(t)) \\ &= \sinh(u_1(t) + u_1(t)x^3(t)) + \tfrac{1}{2}u_1^2(t)\cos(2\pi u_{02}(t)) - \sin(u_1(t)) \\ &= \sinh(u_1(t) + u_1(t)x^3(t)) + \tfrac{1}{2}u_1^2(t) - \sin(u_1(t)). \end{aligned}$$

Therefore, for any $\epsilon \in (0,1)$ and all (x,u) admissible satisfying $\|(x,u)-(x_0,u_0)\| < \epsilon$,

$$\int_0^1 E_0(t,x(t),u_0(t),u(t))dt = \int_0^1 \{\sinh(u_1(t)+u_1(t)x^3(t)) - \sin(u_1(t)) + \tfrac{1}{2}u_1^2(t)\}dt$$

$$\geq \int_0^1 \tfrac{1}{2}u_1^2(t)dt \geq \int_0^1 V(u_1(t))dt = \int_0^1 V(u(t)-u_0(t))dt,$$

and hence condition (iii)(a′) of Theorem 2.2 is satisfied for any $\epsilon \in (0,1)$ and $\delta = 1$.

Finally, if (x,u) is admissible, note that

$$\int_0^1 E_0(t,x(t),u_0(t),u(t))dt \geq \left|\int_0^1 \{\sinh(u_1(t)+u_1(t)x^3(t)) - \sin(u_1(t))\}dt\right| = \left|\int_0^1 E_1(t,x(t),u_0(t),u(t))dt\right|,$$

implying that condition (iii)(b′) of Theorem 2.2 holds for any $\epsilon > 0$ and $\delta = 1$. By Theorem 2.2, (x_0,u_0) is a strict weak minimum of (P).

3. Proof of Theorem 2.1

In this section we shall prove Theorem 2.1. We first state an auxiliary result whose proof is given in Lemmas 2–4 of [31].

In the following lemma we shall assume that we are given $z_0 := (x_0,u_0) \in \mathcal{X} \times L^1(T;\mathbf{R}^m)$ and a subsequence $\{z_q := (x_q,u_q)\}$ in $\mathcal{X} \times L^1(T;\mathbf{R}^m)$ such that

$$\lim_{q\to\infty} D(z_q - z_0) = 0 \quad \text{and} \quad d_q := [2D(z_q - z_0)]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

For all $q \in \mathbf{N}$, set

$$y_q := \frac{x_q - x_0}{d_q} \quad \text{and} \quad v_q := \frac{u_q - u_0}{d_q}.$$

For all $q \in \mathbf{N}$, define

$$W_q := \max\{W_{1q}, W_{2q}\}$$

where

$$W_{1q} := [1 + \tfrac{1}{2}V(\dot{x}_q - \dot{x}_0)]^{1/2} \quad \text{and} \quad W_{2q} := [1 + \tfrac{1}{2}V(u_q - u_0)]^{1/2}.$$

As we mentioned in the introduction, we do not relabel the subsequences of a given sequence since as one readily verifies this fact will not alter our results.

Lemma 3.1

a. *For some $v_0 \in L^2(T;\mathbf{R}^m)$ and some subsequence of $\{z_q\}$, $v_q \xrightarrow{L^1} v_0$ on T. Even more, $u_q \xrightarrow{au} u_0$ on T.*

b. *There exist $\zeta_0 \in L^2(T;\mathbf{R}^n)$, $\bar{y}_0 \in \mathbf{R}^n$, and some subsequence of $\{z_q\}$, such that $\dot{y}_q \xrightarrow{L^1} \zeta_0$ on T. Moreover, if $y_0(t) := \bar{y}_0 + \int_{t_0}^t \zeta_0(\tau)d\tau$ $(t \in T)$, then $y_q \xrightarrow{u} y_0$ on T.*

c. *Let $\Upsilon \subset T$ be measurable and suppose that $W_q \xrightarrow{u} 1$ on Υ. Let $R_q, R_0 \in L^\infty(\Upsilon;\mathbf{R}^{m\times m})$, assume that $R_q \xrightarrow{u} R_0$ on Υ, $R_0(t) \geq 0$ $(t \in \Upsilon)$, and let v_0 be the function considered in condition (a) of Lemma 3.1. Then,*

$$\liminf_{q\to\infty} \int_\Upsilon v_q^*(t)R_q(t)v_q(t)dt \geq \int_\Upsilon v_0^*(t)R_0(t)v_0(t)dt.$$

Proof. The proof of Theorem 2.1 will be made by contraposition, that is, we shall assume that for all $\theta_1, \theta_2 > 0$, there exists an admissible process (x, u) such that

$$\|x - x_0\| < \theta_1 \quad \text{and} \quad I(x, u) < I(x_0, u_0) + \theta_2 D(x - x_0, u - u_0). \tag{1}$$

Also, we are going to assume that all the hypotheses of Theorem 2.1 are satisfied with the exception of hypothesis (ii) and we will obtain the negation of condition (ii) of Theorem 2.1. First of all, note that since

$$\mu_\alpha(t) \geq 0 \ (\alpha \in R, t \in T) \quad \text{and} \quad \lambda_i \geq 0 \ (i = 1, \ldots, k),$$

if (x, u) is admissible, then $I(x, u) \geq J_0(x, u)$. Also, since

$$\mu_\alpha(t)\varphi_\alpha(\tilde{x}_0(t)) = 0 \ (\alpha \in R, t \in T) \quad \text{and} \quad \lambda_i I_i(x_0, u_0) = 0 \ (i = 1, \ldots, k),$$

then $I(x_0, u_0) = J_0(x_0, u_0)$. Thus, (1) implies that for all $\theta_1, \theta_2 > 0$, there exists (x, u) admissible with $\|x - x_0\| < \theta_1$ and

$$J_0(x, u) < J_0(x_0, u_0) + \theta_2 D(x - x_0, u - u_0). \tag{2}$$

Let $z_0 := (x_0, u_0)$. Note that, for all admissible processes $z = (x, u)$,

$$J_0(z) = J_0(z_0) + J_0'(z_0; z - z_0) + \mathcal{K}_0(z) + \mathcal{E}_0(z) \tag{3}$$

where

$$\mathcal{E}_0(x, u) := \int_{t_0}^{t_1} E_0(t, x(t), u_0(t), u(t))dt,$$

$$\mathcal{K}_0(x, u) := \int_{t_0}^{t_1} \{M_0(t, x(t)) + [u^*(t) - u_0^*(t)]N_0(t, x(t))\}dt,$$

and the functions M_0 and N_0 are given by

$$M_0(t, y) := F_0(t, y, u_0(t)) - F_0(\tilde{x}_0(t)) - F_{0x}(\tilde{x}_0(t))(y - x_0(t)),$$

$$N_0(t, y) := F_{0u}^*(t, y, u_0(t)) - F_{0u}^*(\tilde{x}_0(t)).$$

We have,

$$M_0(t, y) = \tfrac{1}{2}[y^* - x_0^*(t)]P_0(t, y)(y - x_0(t)), \quad N_0(t, y) = Q_0(t, y)(y - x_0(t)),$$

where

$$P_0(t, y) := 2\int_0^1 (1 - \lambda)F_{0xx}(t, x_0(t) + \lambda[y - x_0(t)], u_0(t))d\lambda,$$

$$Q_0(t, y) := \int_0^1 F_{0ux}(t, x_0(t) + \lambda[y - x_0(t)], u_0(t))d\lambda.$$

Now, as in [28], choose $\nu > 0$ such that for all $z = (x, u)$ admissible with $\|x - x_0\| < 1$,

$$|\mathcal{K}_0(x, u)| \leq \nu\|x - x_0\|[1 + D(z - z_0)]. \tag{4}$$

Now, by (2), for all $q \in \mathbf{N}$ there exists $z_q := (x_q, u_q)$ admissible such that

$$\|x_q - x_0\| < \epsilon, \quad \|x_q - x_0\| < \frac{1}{q}, \quad J_0(z_q) - J_0(z_0) < \frac{1}{q}D(z_q - z_0). \tag{5}$$

The last inequality of (5) implies that $z_q \neq z_0$ and so for all $q \in \mathbf{N}$,

$$d_q := [2D(z_q - z_0)]^{1/2} > 0.$$

Since

$$\dot{\rho}(t) = -H_x^*(\tilde{x}_0(t), \rho(t), \mu(t)) \text{ (a.e. in } T), \quad H_u^*(\tilde{x}_0(t), \rho(t), \mu(t)) = 0 \ (t \in T),$$

it follows that $J_0'(z_0; (y, v)) = 0$ for all $(y, v) \in \mathcal{X} \times L^2(T; \mathbf{R}^m)$. With this in mind, by (3), condition (iii)(b) of Theorem 2.1, (4) and (5),

$$J_0(z_q) - J_0(z_0) = \mathcal{K}_0(z_q) + \mathcal{E}_0(z_q) \geq -\nu\|x_q - x_0\| + D(z_q - z_0)(\delta - \nu\|x_q - x_0\|).$$

By (5), for all $q \in \mathbf{N}$,

$$D(z_q - z_0)\left(\delta - \frac{1}{q} - \frac{\nu}{q}\right) < \frac{\nu}{q}$$

and hence

$$\lim_{q\to\infty} D(z_q - z_0) = 0.$$

For all $q \in \mathbf{N}$, define

$$y_q := \frac{x_q - x_0}{d_q} \quad \text{and} \quad v_q := \frac{u_q - u_0}{d_q}.$$

By condition (a) of Lemma 3.1, there exist $v_0 \in L^2(T; \mathbf{R}^m)$ and a subsequence of $\{z_q\}$ such that $v_q \xrightarrow{L^1} v_0$ on T. By condition (b) of Lemma 3.1, there exist $\zeta_0 \in L^2(T; \mathbf{R}^n)$, $\bar{y}_0 \in \mathbf{R}^n$ and a subsequence of $\{z_q\}$ such that, if for all $t \in T$, $y_0(t) := \bar{y}_0 + \int_{t_0}^t \zeta_0(\tau)d\tau$, then $y_q \xrightarrow{\text{u}} y_0$ on T.

We claim that

i. $J_0''(z_0; (y_0, v_0)) \leq 0$, $(y_0, v_0) \neq (0, 0)$.
ii. $\dot{y}_0(t) = f_x(\tilde{x}_0(t))y_0(t) + f_u(\tilde{x}_0(t))v_0(t)$ (a.e. in T), $y_0(t_i) = 0$ $(i = 0, 1)$.
iii. $I_i'(z_0; (y_0, v_0)) \leq 0$ $(i \in i_a(z_0))$, $I_j'(z_0; (y_0, v_0)) = 0$ $(j = k + 1, \ldots, K)$.
iv. $\varphi_{\alpha x}(\tilde{x}_0(t))y_0(t) + \varphi_{\alpha u}(\tilde{x}_0(t))v_0(t) \leq 0$ (a.e. in T, $\alpha \in \mathcal{I}_a(\tilde{x}_0(t))$).
v. $\varphi_{\beta x}(\tilde{x}_0(t))y_0(t) + \varphi_{\beta u}(\tilde{x}_0(t))v_0(t) = 0$ (a.e. in T, $\beta \in S$).

Indeed, the equalities $y_0(t_i) = 0$ $(i = 0, 1)$ follow from the definition of y_q, the admissibility of z_q and the fact that $y_q \xrightarrow{\text{u}} y_0$ on T.

For all $q \in \mathbf{N}$, we have

$$\frac{\mathcal{K}_0(z_q)}{d_q^2} = \int_{t_0}^{t_1} \left\{ \frac{M_0(t, x_q(t))}{d_q^2} + v_q^*(t)\frac{N_0(t, x_q(t))}{d_q} \right\} dt.$$

By condition (b) of Lemma 3.1,

$$\frac{M_0(\cdot, x_q(\cdot))}{d_q^2} \xrightarrow{L^\infty} \tfrac{1}{2}y_0^*(\cdot)F_{0xx}(\tilde{x}_0(\cdot))y_0(\cdot),$$

$$\frac{N_0(\cdot, x_q(\cdot))}{d_q} \xrightarrow{L^\infty} F_{0ux}(\tilde{x}_0(\cdot))y_0(\cdot),$$

both on T and, since $v_q \xrightarrow{L^1} v_0$ on T,

$$\tfrac{1}{2}J_0''(z_0;(y_0,v_0)) = \lim_{q\to\infty} \frac{\mathcal{K}_0(z_q)}{d_q^2} + \frac{1}{2}\int_{t_0}^{t_1} v_0^*(t)F_{0uu}(\tilde{x}_0(t))v_0(t)dt. \tag{6}$$

We have,

$$\liminf_{q\to\infty} \frac{\mathcal{E}_0(z_q)}{d_q^2} \geq \frac{1}{2}\int_{t_0}^{t_1} v_0^*(t)F_{0uu}(\tilde{x}_0(t))v_0(t)dt. \tag{7}$$

Indeed, by condition (a) of Lemma 3.1, we are able to choose $\Upsilon \subset T$ measurable such that $u_q \xrightarrow{\text{u}} u_0$ on Υ. Since z_q is admissible, then recalling the definition of W_q given in the beginning of this section, as one readily verifies, $W_q \xrightarrow{\text{u}} 1$ on Υ. Moreover, for all $t \in \Upsilon$ and $q \in \mathbf{N}$,

$$\frac{1}{d_q^2}E_0(t,x_q(t),u_0(t),u_q(t)) = \tfrac{1}{2}v_q^*(t)R_q(t)v_q(t)$$

where

$$R_q(t) := 2\int_0^1 (1-\lambda)F_{0uu}(t,x_q(t),u_0(t)+\lambda[u_q(t)-u_0(t)])d\lambda.$$

Clearly,

$$R_q(\cdot) \xrightarrow{\text{u}} R_0(\cdot) := F_{0uu}(\tilde{x}_0(\cdot)) \text{ on } \Upsilon.$$

By condition (i) of Theorem 2.1, $R_0(t) \geq 0$ ($t \in \Upsilon$). Additionally, by condition (iii)(a) of Theorem 2.1, for all $q \in \mathbf{N}$,

$$E_0(t,x_q(t),u_0(t),u_q(t)) \geq 0 \quad \text{(a.e. in } T),$$

and so, by condition (c) of Lemma 3.1,

$$\begin{aligned}\liminf_{q\to\infty} \frac{\mathcal{E}_0(z_q)}{d_q^2} &= \liminf_{q\to\infty} \frac{1}{d_q^2}\int_{t_0}^{t_1} E_0(t,x_q(t),u_0(t),u_q(t))dt \geq \liminf_{q\to\infty} \frac{1}{d_q^2}\int_{\Upsilon} E_0(t,x_q(t),u_0(t),u_q(t))dt \\ &= \frac{1}{2}\liminf_{q\to\infty}\int_{\Upsilon} v_q^*(t)R_q(t)v_q(t)dt \geq \frac{1}{2}\int_{\Upsilon} v_0^*(t)R_0(t)v_0(t)dt.\end{aligned}$$

As Υ can be chosen to differ from T by a set of an arbitrarily small measure and the function

$$t \mapsto v_0^*(t)R_0(t)v_0(t)$$

belongs to $L^1(T;\mathbf{R})$, this inequality holds when $\Upsilon = T$ and this establishes (7). By (3) and (5)–(7),

$$\tfrac{1}{2}J_0''(z_0;(y_0,v_0)) \leq \lim_{q\to\infty} \frac{\mathcal{K}_0(z_q)}{d_q^2} + \liminf_{q\to\infty} \frac{\mathcal{E}_0(z_q)}{d_q^2} = \liminf_{q\to\infty} \frac{J_0(z_q)-J_0(z_0)}{d_q^2} \leq 0.$$

If $(y_0,v_0) = (0,0)$, then

$$\lim_{q\to\infty} \frac{\mathcal{K}_0(z_q)}{d_q^2} = 0$$

and so, by condition (iii)(b) of Theorem 2.1,

$$\tfrac{1}{2}\delta \le \liminf_{q\to\infty} \frac{\mathcal{E}_0(z_q)}{d_q^2} \le 0,$$

which contradicts the positivity of δ.

For all $q \in \mathbf{N}$, we have

$$\dot{y}_q(t) = A_q(t)y_q(t) + B_q(t)v_q(t) \text{ (a.e. in } T), \quad y_q(t_0) = 0,$$

where

$$A_q(t) = \int_0^1 f_x(t, x_0(t) + \lambda[x_q(t) - x_0(t)], u_0(t))d\lambda,$$

$$B_q(t) = \int_0^1 f_u(t, x_q(t), u_0(t) + \lambda[u_q(t) - u_0(t)])d\lambda.$$

Since

$$A_q(\cdot) \xrightarrow{\text{u}} A_0(\cdot) := f_x(\tilde{x}_0(\cdot)), \quad B_q(\cdot) \xrightarrow{\text{u}} B_0(\cdot) := f_u(\tilde{x}_0(\cdot)),$$

$y_q \xrightarrow{\text{u}} y_0$ and $v_q \overset{L^1}{\rightharpoonup} v_0$ all on Υ, it follows that $\dot{y}_q \overset{L^1}{\rightharpoonup} A_0 y_0 + B_0 y_0$ on Υ. By condition (b) of Lemma 3.1, $\dot{y}_q \overset{L^1}{\rightharpoonup} \zeta_0 = \dot{y}_0$ on Υ. Therefore,

$$\dot{y}_0(t) = A_0(t)y_0(t) + B_0(t)v_0(t) \quad (t \in \Upsilon).$$

As Υ can be chosen to differ from T by a set of an arbitrarily small measure, then there cannot exist a subset of T of positive measure in which the functions y_0 and v_0 do not satisfy the differential equation $\dot{y}_0(t) = A_0(t)y_0(t) + B_0(t)v_0(t)$. Consequently,

$$\dot{y}_0(t) = A_0(t)y_0(t) + B_0(t)v_0(t) \quad \text{(a.e. in } T)$$

and (i) and (ii) of our claim are proved.

Finally, in order to obtain (iii)–(v) of our claim it is enough to copy the proofs of [28] from Eqs (8)–(15). □

4. Proof of Theorem 2.2

In this section we shall prove Theorem 2.2. We first state an auxiliary result which is an immediate consequence of Lemmas 3.1 and 3.2 of [30].

In the following lemma we shall assume that we are given $u_0 \in L^1(T; \mathbf{R}^m)$ and a sequence $\{u_q\}$ in $L^1(T; \mathbf{R}^m)$ such that

$$\lim_{q\to\infty} D_2(u_q - u_0) = 0 \quad \text{and} \quad d_{2q} := [2D_2(u_q - u_0)]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

For all $q \in \mathbf{N}$ define

$$v_{2q} := \frac{u_q - u_0}{d_{2q}}.$$

Lemma 4.1

a. *For some* $v_{02} \in L^2(T;\mathbf{R}^m)$ *and a subsequence of* $\{u_q\}$, $v_{2q} \overset{L^1}{\rightharpoonup} v_{02}$ *on* T.

b. *Let* $A_q \in L^\infty(T;\mathbf{R}^{n\times n})$ *and* $B_q \in L^\infty(T;\mathbf{R}^{n\times m})$ *be matrix functions for which there exist constants* $m_0, m_1 > 0$ *such that* $\|A_q\|_\infty \leq m_0$, $\|B_q\|_\infty \leq m_1$ $(q \in \mathbf{N})$, *and for all* $q \in \mathbf{N}$ *denote by* Y_q *the solution of the initial value problem*

$$\dot{y}(t) = A_q(t)y(t) + B_q(t)v_{2q}(t) \text{ (a.e. in } T), \quad y(t_0) = 0.$$

Then there exist $\sigma_0 \in L^2(T;\mathbf{R}^n)$ *and a subsequence of* $\{z_q\}$, *such that* $\dot{Y}_q \overset{L^1}{\rightharpoonup} \sigma_0$ *on* T, *and hence if* $Y_0(t) := \int_{t_0}^t \sigma_0(\tau)d\tau$ $(t \in T)$, *then* $Y_q \overset{u}{\longrightarrow} Y_0$ *on* T.

Proof. As we made with the proof of Theorem 2.1, the proof of Theorem 2.2 will be made by contraposition, that is, we shall assume that for all $\theta_1, \theta_2 > 0$, there exists an admissible process (x, u) such that

$$\|(x,u) - (x_0,u_0)\| < \theta_1 \quad \text{and} \quad I(x,u) < I(x_0,u_0) + \theta_2 D_2(u-u_0). \tag{8}$$

Once again, as we made with the proof of Theorem 2.1, (8) implies that for all $\theta_1, \theta_2 > 0$, there exists (x, u) admissible with

$$\|(x,u) - (x_0,u_0)\| < \theta_1 \quad \text{and} \quad J_0(x,u) < J_0(x_0,u_0) + \theta_2 D_2(u-u_0). \tag{9}$$

Let $z_0 := (x_0, u_0)$. As in the proof of Theorem 2.1, for all admissible processes $z = (x, u)$,

$$J_0(z) = J_0(z_0) + J_0'(z_0; z - z_0) + \mathcal{K}_0(z) + \mathcal{E}_0(z)$$

where $\mathcal{E}_0$ and K_0 are given as in the proof of Theorem 2.1.

Now, by (9), for all $q \in \mathbf{N}$ there exists $z_q := (x_q, u_q)$ admissible such that

$$\|z_q - z_0\| < \frac{1}{q}, \quad J_0(z_q) - J_0(z_0) < \frac{1}{q} D_2(u_q - u_0). \tag{10}$$

Since z_q is admissible, the last inequality of (10) implies that $u_q \neq u_0$ and so

$$d_{2q} := [2D_2(u_q - u_0)]^{1/2} > 0 \quad (q \in \mathbf{N}).$$

By the first relation of (10), we have

$$\lim_{q\to\infty} D_2(u_q - u_0) = 0.$$

For all $q \in \mathbf{N}$, define v_{2q} as in Lemma 4.1 and

$$Y_q := \frac{x_q - x_0}{d_{2q}} \quad \text{and} \quad W_{2q} := [1 + \tfrac{1}{2}V(u_q - u_0)]^{1/2}.$$

By condition (a) of Lemma 4.1, there exist $v_{02} \in L^2(T;\mathbf{R}^m)$ and a subsequence of $\{z_q\}$ such that $v_{2q} \overset{L^1}{\rightharpoonup} v_{02}$ on T. As in the proof of Theorem 2.1, for all $q \in \mathbf{N}$,

$$\dot{Y}_q(t) = A_q(t)Y_q(t) + B_q(t)v_{2q}(t), \quad Y_q(t_0) = 0 \quad \text{(a.e. in } T).$$

We have the existence of $m_0, m_1 > 0$ such that $\|A_q\|_\infty \leq m_0$ and $\|B_q\|_\infty \leq m_1$ ($q \in \mathbf{N}$). By condition (b) of Lemma 4.1, there exist $\sigma_0 \in L^2(T;\mathbf{R}^n)$ and a subsequence of $\{z_q\}$ such that, if $Y_0(t) := \int_{t_0}^t \sigma_0(\tau)d\tau$ ($t \in T$), then $Y_q \xrightarrow{\text{u}} Y_0$ on T. We claim that

i. $J_0''(z_0;(Y_0,v_{02})) \leq 0$, $(Y_0,v_{02}) \neq (0,0)$.
ii. $\dot{Y}_0(t) = f_x(\tilde{x}_0(t))Y_0(t) + f_u(\tilde{x}_0(t))v_{02}(t)$ (a.e. in T), $Y_0(t_i) = 0$ ($i = 0,1$).
iii. $I_i'(z_0;(Y_0,v_{02})) \leq 0$ ($i \in i_a(z_0)$), $I_j'(z_0;(Y_0,v_{02})) = 0$ ($j = k+1,\ldots,K$).
iv. $\varphi_{\alpha x}(\tilde{x}_0(t))Y_0(t) + \varphi_{\alpha u}(\tilde{x}_0(t))v_{02}(t) \leq 0$ (a.e. in T, $\alpha \in \mathcal{I}_a(\tilde{x}_0(t))$).
v. $\varphi_{\beta x}(\tilde{x}_0(t))Y_0(t) + \varphi_{\beta u}(\tilde{x}_0(t))v_{02}(t) = 0$ (a.e. in T, $\beta \in S$).

Indeed, for all $q \in \mathbf{N}$, we have

$$\frac{\mathcal{K}_0(z_q)}{d_{2q}^2} = \int_{t_0}^{t_1}\left\{\frac{M_0(t,x_q(t))}{d_{2q}^2} + v_{2q}^*(t)\frac{N_0(t,x_q(t))}{d_{2q}}\right\}dt.$$

Also, we have

$$\frac{M_0(\cdot,x_q(\cdot))}{d_{2q}^2} \xrightarrow{L^\infty} \tfrac{1}{2}Y_0^*(\cdot)F_{0xx}(\tilde{x}_0(\cdot))Y_0(\cdot),$$

$$\frac{N_0(\cdot,x_q(\cdot))}{d_{2q}} \xrightarrow{L^\infty} F_{0ux}(\tilde{x}_0(\cdot))Y_0(\cdot),$$

both on T and, since $v_{2q} \overset{L^1}{\rightharpoonup} v_{02}$ on T,

$$\tfrac{1}{2}J_0''(z_0;(Y_0,v_{02})) = \lim_{q\to\infty}\frac{\mathcal{K}_0(z_q)}{d_{2q}^2} + \frac{1}{2}\int_{t_0}^{t_1} v_{02}^*(t)F_{0uu}(\tilde{x}_0(t))v_{02}(t)dt. \tag{11}$$

Now, for all $t \in T$ and $q \in \mathbf{N}$,

$$\frac{1}{d_{2q}^2}E_0(t,x_q(t),u_0(t),u_q(t)) = \tfrac{1}{2}v_{2q}^*(t)R_q(t)v_{2q}(t)$$

where

$$R_q(t) := 2\int_0^1 (1-\lambda)F_{0uu}(t,x_q(t),u_0(t)+\lambda[u_q(t)-u_0(t)])d\lambda.$$

Clearly,

$$R_q(\cdot) \xrightarrow{L^\infty} R_0(\cdot) := F_{0uu}(\tilde{x}_0(\cdot)) \text{ on } T.$$

Since $\|z_q - z_0\| \to 0$ as $q \to \infty$, it follows that $W_{2q} \xrightarrow{L^\infty} 1$ on T and, by condition (i) of Theorem 2.2, $R_0(t) \geq 0$ (a.e. in T). Consequently,

$$\liminf_{q\to\infty}\frac{\mathcal{E}_0(z_q)}{d_{2q}^2} \geq \frac{1}{2}\int_{t_0}^{t_1} v_{02}^*(t)R_0(t)v_{02}(t)dt. \tag{12}$$

On the other hand, since

$$\dot{\rho}(t) = -H_x^*(\tilde{x}_0(t),\rho(t),\mu(t)) \text{ (a.e. in } T), \quad H_u^*(\tilde{x}_0(t),\rho(t),\mu(t)) = 0 \ (t \in T),$$

we have that $J_0'(z_0;(y,v)) = 0$ for all $(y,v) \in \mathcal{X} \times L^2(T;\mathbf{R}^m)$. With this in mind, (10)–(12),

$$\tfrac{1}{2}J_0''(z_0;(Y_0,v_{02})) \le \lim_{q\to\infty} \frac{\mathcal{K}_0(z_q)}{d_{2q}^2} + \liminf_{q\to\infty} \frac{\mathcal{E}_0(z_q)}{d_{2q}^2} = \liminf_{q\to\infty} \frac{J_0(z_q)-J_0(z_0)}{d_{2q}^2} \le 0.$$

If $(Y_0,v_{02}) = (0,0)$, then

$$\lim_{q\to\infty} \frac{\mathcal{K}_0(z_q)}{d_{2q}^2} = 0$$

and so, by condition (iii)(a′) of Theorem 2.2,

$$\tfrac{1}{2}\delta \le \liminf_{q\to\infty} \frac{\mathcal{E}_0(z_q)}{d_{2q}^2} \le 0,$$

which contradicts the positivity of δ and this proves (i) of our claim.

Now, we also claim that

$$\dot{Y}_0(t) = f_x(\tilde{x}_0(t))Y_0(t) + f_u(\tilde{x}_0(t))v_{02}(t) \text{ (a.e. in } T), \quad Y_0(t_i) = 0 \ (i=0,1).$$

Indeed, the equalities $Y_0(t_i) = 0$ $(i = 0,1)$ follow from the definition of Y_q, the admissibility of z_q and the fact that $Y_q \overset{\mathrm{u}}{\longrightarrow} Y_0$ on T. Also, observe that since $Y_q \overset{\mathrm{u}}{\longrightarrow} Y_0$,

$$A_q(\cdot) \overset{L^\infty}{\longrightarrow} A_0(\cdot) := f_x(\tilde{x}_0(\cdot)),$$

$$B_q(\cdot) \overset{L^\infty}{\longrightarrow} B_0(\cdot) := f_u(\tilde{x}_0(\cdot)),$$

and $v_{2q} \overset{L^1}{\rightharpoonup} v_{02}$ all on T, then $\dot{Y}_q \overset{L^1}{\rightharpoonup} A_0Y_0 + B_0v_{02}$ on T. By condition (b) of Lemma 4.1, $\dot{Y}_q \overset{L^1}{\rightharpoonup} \sigma_0 = \dot{Y}_0$ on T, which accordingly implies that

$$\dot{Y}_0(t) = A_0(t)Y_0(t) + B_0(t)v_{02}(t) \quad \text{(a.e. in } T)$$

and our claim is proved.

Finally, in order to prove (iii)–(v) of our claim it is enough to copy the proofs given in [28] from Eqs (8)–(15) by replacing y_0 by Y_0, v_0 by v_{02} and Υ by T. □

5. Conclusions

In this article, we have provided sufficiency theorems for weak and strong minima in an optimal control problem of Lagrange with fixed end-points, nonlinear dynamics, inequality and equality isoperimetric restrictions and inequality and equality mixed time-state-control constraints. The sufficiency treatment studied in this paper does not need that the proposed optimal controls be continuous but only purely measurable. The sufficiency results not only provide local minima but they also measure the deviation between optimal and admissible costs by means of a functional playing a similar role of the square of the classical norm of the Banach space L^1. Additionally, all the crucial sufficiency hypotheses are included in the theorems, in contrast, with other necessary and sufficiency theories which strongly depend upon some preliminary assumptions not embedded in the corresponding theorems of optimality. Finally, our sufficiency technique is self-contained because it is independent of some classical sufficient approaches involving Hamilton-Jacobi inequalities, matrix-valued Riccati equations, generalizations of Jacobi's theory appealing to extended notions of conjugate points or insertions of the original problem in some abstract Banach spaces.

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Conflict of interest

The author declares no conflict of interest.

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