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*Research article*

# On the integral solutions of the Egyptian fraction equation $\frac{a}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

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**Abstract:** It is an interesting question to investigate the integral solutions for the Egyptian fraction equation $\frac{a}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, which is known as Erdős-Straus equation when $a = 4$. Recently, Lazar proved that this equation has not integral solutions with $xy < \sqrt{z/2}$ and $\gcd(x, y) = 1$ when $a = 4$. But his method is difficult to get an analogous result for arbitrary $\frac{a}{p}$, especially when $p$ and $a$ are lager numbers. In this paper, we extend Lazar's result to arbitrary integer $a$ with $4 \le a \le \frac{1+\sqrt{1+6p^3}}{p}$, and release the condition $\gcd(x, y) = 1$. We show that $\frac{a}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ has no integral solutions satisfying that $xy < \sqrt{lz}$, where $l \le \frac{(3p+a)p}{a^2}$ when $p \nmid y$ and $l \le \frac{3p^2+a}{pa^2}$ when $p \mid y$. Besides, we extend Monks and Velingker's result to the case $4 \le a < p$.

**Keywords:** Egyptian fraction; Erdős-Straus conjecture; integral solution
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## 1. Introduction

Let $\mathbb{Z}_+$ denote the set of positive integers. Egyptian fractions are rational numbers which can be represented as the sum of positive unit fractions:

$$\frac{a}{n} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k},$$

where $a, n, x_1, \ldots, x_k \in \mathbb{Z}_+$, and they appeared in one of the oldest written mathematics, the Rhind papyrus [3]. Since then numerous problems on Egyptian fractions have been introduced, and unfortunately, many of them remain unsolved.

For $a, n \in \mathbb{Z}_+$, let $f(n, a)$ stand for the number of ways to write $\frac{a}{n}$ as the sum of three positive unit fractions. Formally, $f(n, a)$ is the number of positive integral solutions $(x, y, z) \in \mathbb{Z}_+^3$ of the Diophantine equation

$$\frac{a}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}. \tag{1.1}$$

It is clear that $f(n, 1) > 0$, $f(n, 2) > 0$ and $f(n, 3) > 0$. However, whether $f(n, 4) > 0$ holds is a well-known open problem [3].

**Conjecture 1** (Erdős-Straus conjecture)**.** *It holds that $f(n, 4) > 0$ for all integers $n$ with $n \geq 2$.*

Since Erdős and Straus proposed Conjecture 1, numerous number theorists studied it and have already made progress to confirm its correctness for $n$ being one of some (infinitely many) particular integers. Straus [2], Chao Ko et al. [5], Jollenstein [4] and Salez [11], respectively, showed that this conjecture holds for some range of $n$. A profound result in Mordell's book [8] shows that Conjecture 1 is true if $(n \bmod 840) \notin \{1^2, 11^2, 13^2, 17^2, 19^2, 23^2\}$.

Furthermore, the Erdős-Straus conjecture has also stimulated number theorists to investigate its variants. Substituting 5 for 4, Sierpiński [12] proposed a conjecture on $f(n, 5)$ similar to Conjecture 1, and his conjecture has already been proved to be true for $0 < n < 922321$ [9] and also for $\{0 < n < 1057438801 : n \not\equiv 1 \bmod 278460\}$ [10]. Recently, Elsholtz and Planitzer [1] proved that for any $a, n \in \mathbb{Z}_+$ and $\epsilon > 0$, there exists a constant $c(\epsilon) > 0$ such that $f(n, a) \leq c(\epsilon) n^{\epsilon} (n^3/a^2)^{1/5}$. Note that Conjecture 1 holds if and only if $f(n, 4) > 0$ for all primes $p$, i.e.,

$$\frac{4}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{1.2}$$

is always solvable. Therefore, in the sequel we only consider the denominator in (1.1) as a prime $p$ and concentrate on the equation

$$\frac{a}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}. \tag{1.3}$$

Additionally, in the rest of this paper, we always assume that $x \leq y \leq z$ due to symmetry.

Since a complete answer to Conjecture 1 remains challenging, it is natural to study its solutions under certain restrictions. On the one hand, via continued fraction Lazar [6] showed that (1.2) has no integral solutions satisfying both $\gcd(x, y) = 1$ and $xy < \sqrt{z/2}$. Naturally, Lazar expected a similar result on (1.3), which is anyhow beyond the techniques in [6] and is still open. On the other hand, Monks and Velingker [7] investigated (1.2) where $y$ and $z$ are of a special $p$-adic discrete valuation. Let $a \in \mathbb{Z}_+$. For any given $r, s \in \mathbb{Z}_+$, let

$$\alpha_a(r, s) := \frac{ars - r - s}{\gcd(r, s) \gcd\left(\frac{ars - r - s}{\gcd(r, s)}, \gcd(r, s)\right)}.$$

As shown in [7], for given $j, k \in \mathbb{Z}_+$, there exists at most one prime $p$ such that the Diophantine equation

$$\frac{4}{p} = \frac{1}{x} + \frac{1}{pj} + \frac{1}{pk} \tag{1.4}$$

holds. Particularly, such a prime $p$ exists if and only if $\alpha_4(j, k)$ is prime, and in that case it was proved $p = \alpha_4(j, k)$. Also it was proved that the sequence $\{\alpha_4(j, k)\}_{k=1}^{+\infty}$ contains infinitely many primes [7].

Because the integral solution for (1.3) is trivial when $a = 1, 2, 3$, and therefore in this paper, we study Egyptian fractions of the form (1.3), where $4 \leq a < p$. Indeed, we have the following result.

**Theorem 1.** *Let p be a prime, and a be an integer with $4 \leq a < p$. Then for any positive real number $l \leq G(a, p)$, (1.3) has no positive integral solutions satisfying $xy < \sqrt{lz}$, where $G(a, p) = \frac{3p^2}{a^2} + \frac{p}{a}$ if $p|y$ and $G(a, p) = \frac{3p}{a^2} + \frac{1}{ap}$ if $p \nmid y$.*

The corollary below follows from Theorem 1.

**Corollary 1.** *Let p be any prime number. If a is an integer satisfying $4 \leq a \leq \frac{1+\sqrt{1+6p^3}}{p}$, then there are no positive integral solutions to the Diophantine equation (1.3) satisfying $xy < \sqrt{z/2}$.*

Corollary 1 extends the results in [6] in two aspects. One is that applying $a = 4$ in Corollary 1 derives Lazar's result [6] since $4 < \frac{1+\sqrt{1+6p^3}}{p}$ for any prime $p \geq 3$. The other is that our proof of Lazar's result [6] in this way releases the restricted condition $\gcd(x, y) = 1$. Thus, we find an analog of the Lazar's main result for $\frac{a}{p}$ instead of $\frac{4}{p}$ under the condition $a \leq \frac{1+\sqrt{1+6p^3}}{p}$, and give a partial answer to Lazar's question proposed in [6].

In this paper, we also give a generalized result for the fraction $\frac{a}{p}$ by Monks and Velingker [7].

**Theorem 2.** *For any $a, j, k \in \mathbb{Z}_+$, and $a \geq 4$, the following two statements on the Diophantine equation*

$$\frac{a}{p} = \frac{1}{x} + \frac{1}{pj} + \frac{1}{pk} \tag{1.5}$$

*hold:*

*(i). There exists a prime p such that (1.5) is solvable if and only if $\alpha_a(j, k)$ is prime.*
*(ii). There exists at most one prime p such that (1.5) holds for some $x \in \mathbb{Z}_+$. Additionally, for such a prime p, $\alpha_a(j, k) = p$.*

**Theorem 3.** *Let $a \geq 4$ and j be positive integers. Then there exist infinitely many primes in the set $\{\alpha_a(j, k) : k \in \mathbb{Z}_+\}$.*

Therefore, by Theorems 2 and 3 we come to the conclusion that for any $a \in \mathbb{Z}_+$ there are infinitely many primes satisfying (1.3).

The rest of this paper is organized as follows. In Section 2, we present the proof of Theorem 1. Consequently, the proofs of Theorems 2 and 3 are given in Section 3.

## 2. Proof of Theorem 1

In this section, we give the proof of Theorem 1. Given a prime $p$ and a positive integer $m$, there exist unique integers $j$ and $n$, with $p \nmid j$ and $n \geq 0$, such that $m = p^n j$. The number $n$ is called the *p-adic valuation* of $m$, denoted by $n = v_p(m)$. To prove Theorem 1, we need the following two lemmas.

**Lemma 1.** *( [7]) Let p be a prime, and a be an integer with $4 \leq a < p$. Let $(x, y, z) \in \mathbb{Z}_+^3$ be a positive integral solution to (1.3) with $x \leq y \leq z$. Then the following statements hold:*

*(i). Let $q \in \mathbb{Z}_+$ and $p \nmid q$. If q divides one of x, y or z, then q divides the product of the remaining two.*
*(ii). $p \nmid x$ and $p \mid z$ and $x < p$.*
*(iii). If $\max\{v_p(y), v_p(z)\} > 1$, then $v_p(y) = v_p(z)$.*

Lemma 1 comes from the Theorem 2.1 of [7], where (i) and (ii) are part (b) and (c), part (iii) comes from the proof of part (d).

**Lemma 2.** *Let $p$ be a prime, and $a$ be an integer with $4 \le a < p$. Let $(x, y, z)$ be a positive integral solution to (1.3) with $x \le y \le z$. Then*

$$\lambda > \frac{3p^2}{a^2}\frac{xy}{z} + \frac{p}{a}\frac{x^2y^2}{z^2}, \tag{2.1}$$

*where* $\lambda := \frac{(xy)^2}{z}$.

*Proof.* Since $(x, y, z)$ be a positive integral solution to (1.3) with $x \le y \le z$, one has $\frac{a}{p} > \frac{1}{x}$ and $\frac{a}{p} > \frac{2}{y}$. So one obtains that

$$x > \frac{p}{a}, \quad y > \frac{2p}{a}. \tag{2.2}$$

It then follows from (1.3) and (2.2) that

$$\frac{a}{p} - \frac{1}{z} = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} > \frac{3p}{a}\frac{1}{\sqrt{\lambda z}} = \frac{3p}{a}\frac{xy}{\lambda z}. \tag{2.3}$$

By (2.3), we get that

$$\lambda a^2 - pa\frac{x^2y^2}{z^2} - 3p^2\frac{xy}{z} > 0,$$

which implies that (2.1) holds. □

In what follows, we present the proof of Theorem 1.

*Proof of Theorem 1.* Assume that the Diophantine equation (1.3) has a positive integral solution $(x, y, z)$ with $x \le y \le z$. Let $\lambda = \frac{(xy)^2}{z}$. By Lemma 1 (ii), we have $p \mid z$. Let $z = p^{v_p(z)}s$. So $v_p(z) \ge 1$ and $\gcd(p, s) = 1$. Thus by Lemma 1 (i), we know that

$$s \mid xy. \tag{2.4}$$

Now we consider the following two cases.

Case i: $p \mid y$. In this case, we let $y = p^{v_p(y)}t$. Then $v_p(y) \ge 1$ and $\gcd(p, t) = 1$. We claim that $v_p(y) = v_p(z)$. Note that $\max\{v_p(y), v_p(z)\} \ge 1$. Clearly, if $v_p(y) = v_p(z) = 1$, then the Claim is true. If $v_p(y) > 1$ or $v_p(z) > 1$, then by Statement (iii) of Lemma 1 we obtain that $v_p(y) = v_p(z)$. The Claim is proved.

From $\gcd(p, s) = 1$, $xy = xp^{v_p(y)}t$ and (2.4), we deduce that $s|xt$. This implies that $s \le xt$. It then follows from the Claim that

$$z = p^{v_p(z)}s \le xp^{v_p(z)}t = xy. \tag{2.5}$$

Applying (2.5) in (2.1), we have

$$\lambda > \frac{3p^2}{a^2}\frac{xy}{z} + \frac{p}{a}\frac{x^2y^2}{z^2} \ge \frac{3p^2}{a^2} + \frac{p}{a}. \tag{2.6}$$

Thus by (2.6), we conclude that if $p \mid y$ and

$$l \leq \frac{3p^2}{a^2} + \frac{p}{a},$$

then (1.3) has no positive integral solution satisfying $\lambda < l$ (i.e. $xy < \sqrt{lz}$). Case I is proved.

Case ii: $p \nmid y$. In this case, we know that $v_p(y) = 0$. So by the Lemma 1 (iii), we get $v_p(z) \leq 1$. But $v_p(z) \geq 1$. Then $v_p(z) = 1$ and so $z = ps$. By (2.4), we have $s \leq xy$. It then follows that

$$z = ps \leq pxy. \tag{2.7}$$

Using (2.7) and (2.1), we deduce that

$$\lambda > \frac{3p^2}{a^2}\frac{xy}{z} + \frac{p}{a}\frac{x^2y^2}{z^2} \geq \frac{3p}{a^2} + \frac{1}{ap}. \tag{2.8}$$

By (2.8), we derive that if $p \nmid y$ and

$$l \leq \frac{3p}{a^2} + \frac{1}{ap},$$

then (1.3) has no positive integral solution satisfying $xy < \sqrt{lz}$. Case II is proved.

This completes the proof of Theorem 1. □

*Proof of Corollary 1.* Since $a \leq \frac{1+\sqrt{1+6p^3}}{p}$, we have $ap - 1 \leq \sqrt{1 + 6p^3}$ and so $pa^2 - 2a \leq 6p^2$. It then follows that

$$\frac{1}{2} \leq \frac{3p^2 + a}{pa^2}$$

Clearly $\frac{3p^2+a}{pa^2} < \frac{(3p+a)p}{a^2}$. This implies that $\frac{1}{2} \leq G(a, p)$, where $G(a, p)$ is defined as in Theorem 1. Using Theorem 1 with $l = \frac{1}{2}$, we derive that (1.3) has no positive integral solutions satisfying $xy < \sqrt{\frac{z}{2}}$ as desired.

Corollary 1 is proved. □

## 3. Proof of Theorem 2 and Theorem 3

In this section we need the following three lemmas.

**Lemma 3.** *Let $a, j, k \in \mathbb{Z}_+$ and $a \geq 4$. Then $(ajk - j - k) \nmid \gcd(j, k)^2$ and $\alpha_a(j, k) \neq 1$.*

*Proof.* Since $a \geq 4$, we have

$$ajk - j - k \geq 4jk - j - k = 2jk + j(k - 1) + k(j - 1) \geq 2jk > \gcd(j, k)^2.$$

The last inequality is true since $\gcd(j, k)^2 \mid jk$. This means that $(ajk - j - k) \nmid \gcd(j, k)^2$.

Suppose that $\alpha_a(j, k) = 1$, that is

$$\alpha_a(j, k) = \frac{ajk - j - k}{\gcd(j, k)\gcd(\frac{ajk-j-k}{\gcd(j,k)}, \gcd(j, k))} = 1.$$

This means $\gcd(\frac{ajk-j-k}{\gcd(j,k)}, \gcd(j, k)) = \frac{ajk-j-k}{\gcd(j,k)}$, and therefore we have $\frac{ajk-j-k}{\gcd(j,k)} \mid \gcd(j, k)$. It then follows that $(ajk - j - k) \mid \gcd(j, k)^2$. This contradicts to $(ajk - j - k) \nmid \gcd(j, k)^2$. Then we obtain that $\alpha_a(j, k) \neq 1$. Lemma 3 is proved. □

**Lemma 4.** *Let $a, p, j, k \in \mathbb{Z}_+$ and $a \geq 4$. Then (1.5) holds for some $x \in \mathbb{Z}_+$ if and only if $(ajk - j - k) \mid p\gcd(j,k)^2$.*

*Proof.* Note that the Diophantine equation (1.5) is solvable if and only if

$$x = \frac{1}{\frac{a}{p} - \frac{1}{pj} - \frac{1}{pk}} = \frac{pjk}{ajk - j - k} \tag{3.1}$$

is an integer. Write $g = \gcd(j,k)$.

On one hand, if $(ajk - j - k) \mid pg^2$, then $(ajk - j - k) \mid pjk$ and hence the right hand of (3.1) is an integer.

On the other hand, suppose $x$ in (3.1) is an integer, i.e.,

$$(ajk - j - k) \mid pjk. \tag{3.2}$$

Let $j = j'g$ and $k = k'g$. Substituting $j'g$ (resp. $k'g$) for $j$ (resp. $k$) in (3.2), we have

$$(agj'k' - j' - k') \mid pgj'k'. \tag{3.3}$$

Note that $\gcd(j', k') = 1$, we get

$$\gcd(j', agj'k' - j' - k') = \gcd(k', agj'k' - j' - k') = 1. \tag{3.4}$$

Then by (3.3) and (3.4), we obtain that $(agj'k' - j' - k') \mid pg$, i.e., $(akj - k - j) \mid pg^2$.

Summing up the above two aspects finishes the proof of Lemma 4. □

**Lemma 5.** *Let $a, j, k \in \mathbb{Z}_+$ and $a \geq 4$. Let $p$ be a prime. Then $(ajk - j - k) \mid p\gcd(j,k)^2$ if and only if $\alpha_a(j,k) = p$.*

*Proof.* Denote $g = \gcd(j,k)$. By the definition of $\alpha_a(j,k)$, we get

$$ajk - j - k = \alpha_a(j,k)\, g \gcd(\frac{ajk - j - k}{g}, g). \tag{3.5}$$

This implies that $\gcd(\alpha_a(j,k), g) = \gcd(\frac{\frac{ajk-j-k}{g}}{\gcd(\frac{ajk-j-k}{g}, g)}, g) = 1$. It then follows from (3.5) that $(ajk - j - k) \mid pg^2$ if and only if $\alpha_a(j,k) | p$. Furthermore, as $\alpha_a(j,k) \neq 1$ by Lemma 3, $\alpha_a(j,k) \mid p$ holds if and only if $\alpha_a(j,k) = p$.

The proof of lemma 5 is completed. □

*Proof of Theorem 2.* First, statement (i) of Theorem 2 follows directly from Lemmas 4 and 5.

Now we prove statement (ii) of Theorem 2 by *reductio ad absurdum*.

Assume that there exist two different primes $p_1$ and $p_2$ such that $\frac{a}{p_1} = \frac{1}{x} + \frac{1}{p_1 j} + \frac{1}{p_1 k}$ and $\frac{a}{p_2} = \frac{1}{x} + \frac{1}{p_2 j} + \frac{1}{p_2 k}$ for some $x \in \mathbb{Z}_+$. Then by Lemma 4, we have

$$(ajk - j - k) \mid p_i \gcd(j,k)^2, i = 1, 2. \tag{3.6}$$

Because $\gcd(p_1, p_2) = 1$, (3.6) derives $(ajk - j - k) \mid \gcd(j,k)^2$, which is ridiculous by Lemma 3.

Therefore, our assumption above is not true and statement (ii) is proved.

□

*Proof of Theorem 3.* Consider the arithmetic sequence $\{(aj-1)k-j\}_{k=1}^{+\infty}$. First of all, since $\gcd(aj-1, j) = 1$, by the Dirichlet's theorem, the sequence $\{(aj-1)k-j\}_{k=1}^{+\infty}$ contains infinitely many primes.

Now, we claim that a prime number $(aj-1)k-j$ for some $k \in \mathbb{Z}_+$ should also occur as $\alpha_a(j,k)$ in $\{\alpha_a(j,k) : k \in \mathbb{Z}_+\}$. Note that $\alpha_a(j,k)$ is a divisor of $ajk-j-k$. If $ajk-j-k$ is prime, then either $\alpha_a(j,k) = ajk-j-k$ or $\alpha_a(j,k) = 1$. However, $\alpha_a(j,k) \neq 1$ by Lemma 3. Thus, we have $\alpha_a(j,k) = ajk-j-k$ if $ajk-j-k$ is prime. The claim holds.

To sum up the above two points, the set $\{\alpha_a(j,k) : k \in \mathbb{Z}_+\}$ contains infinitely many primes. □

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## Conflict of interest

Authors declare no conflict of interest in this paper.

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