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Research article

Existence, uniqueness and Ulam's stabilities for a class of implicit impulsive Langevin equation with Hilfer fractional derivatives

Xiaoming Wang[1], Rizwan Rizwan[2,*], Jung Rey Lee[3,*], Akbar Zada[4] and Syed Omar Shah[5]

[1] School of Mathematics and Computer Science, Shangrao Normal University, Shangrao, China
[2] Department of Mathematics, University of Buner, Buner, Pakistan
[3] Department of Mathematics, Daejin University, Kyunggi 11159, Korea
[4] Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan
[5] Department of Physical and Numerical Sciences Qurtaba University of Science and Technology Peshawar, DI Khan, Pakistan

* **Correspondence:** Email: jrlee@daejin.ac.kr, rixwan4630@gmail.com; Tel: +821082544297.

Abstract: In this manuscript, a class of implicit impulsive Langevin equation with Hilfer fractional derivatives is considered. Using the techniques of nonlinear functional analysis, we establish appropriate conditions and results to discuss existence, uniqueness, Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability results of our proposed model, with the help of Banach's fixed point theorem. An example is provided at the end to illustrate our results.

Keywords: Langevin equation; Hilfer fractional derivative; instantaneous impulses; Ulam-Hyers-Rassias stability
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1. Introduction

Fractional differential equations is the generalized form of classical differential equations of integer order. Fractional calculus is now a developed area and it has many applications in porous media, electrochemistry, economics, electromagnetics, physical sciences, medicine etc., Progressively, the role of fractional differential equations is very important in viscoelasticity, statistical physics, optics, signal processing, control, defence, electrical circuits, astronomy etc. Some interesting articles provide the main theoretical tools for the qualitative analysis of this area and also shows the interconnection as well as the distinction between classical, integral models and fractional differential equations, see [1, 17, 19, 22–26, 29, 34, 35].

The Langevin equation is an excellent technique to describe some phenomena which can help physicians, engineers, economists, etc., effectively to describe processes. The Langevin equation (drafted for first by Langevin in 1908) is obtained to be an accurate tool to describe the development of physical phenomena. These equations are used to described stochastic problems in physics, defence system, image processing, chemistry, astronomy, mechanical and electrical engineering. They are also used to describe Brownian motion when the random oscillation force is supposed to be Gaussian noise. Fractional order differential equations are utilized for the removal of noise. For more details, see [2, 12, 20, 21, 28].

Recently impulsive differential equations have been considered by many authors due to their significant applications in various fields of science and technology. These equations describe the evolution processes that are subjected to abrupt changes and discontinuous jumps in their states. Many physical systems like the function of pendulum clock, the impact of mechanical systems, preservation of species by means of periodic stocking or harvesting and the heart's function, etc. naturally experience the impulsive phenomena. Similarly in many other situations, the evolutional processes have the impulsive behavior. For example, the interruptions in cellular neural networks, the damper's operation with percussive effects, electromechanical systems subject to relaxational oscillations, dynamical systems having automatic regulations, etc., have the impulsive phenomena. For detail study, see [5, 10, 13, 16, 18, 30, 38, 40, 42, 45]. Due to its large number of applications, this area has been received great importance and remarkable attention from the researchers.

At Wisconsin university, Ulam raised a question about the stability of functional equations in the year 1940. The question of Ulam was: under what conditions does there exist an additive mapping near an approximately additive mapping [36]. In 1941, Hyers was the first mathematician who gave partial answer to Ulam's question [14], over Banach space. Afterwards, stability of such form is known as Ulam-Hyers stability. In 1978, Rassias [27], provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. For more information about the topic, we refer the reader to [6, 15, 31, 33, 37, 43, 44, 46].

Recently, the existence, uniqueness and different types of fractional nonlinear differential equations with Caputo fractional derivative have received a considerable attention, see [3, 7–9, 32, 33].

Wang et al. [39], studied generalized Ulam-Hyers-Rassias stability of the following fractional differential equation:

$$\begin{cases} {}^c\mathcal{D}^{\alpha}_{0,\upsilon}x(\upsilon) = f(\upsilon, x(\upsilon)), \quad \upsilon \in (\upsilon_i, s_i], \quad i = 0, 1, \ldots, m, \quad 0 < \alpha < 1, \\ x(\upsilon) = g_i(\upsilon, x(\upsilon)), \quad \upsilon \in (s_{i-1}, \upsilon_i], \quad i = 1, 2, \ldots, m. \end{cases}$$

Zada et al. [41], studied existence, uniqueness of solutions by using Diaz-Margolis's fixed point theorem [11] and presented different types of Ulam-Hyers stability for a class of nonlinear implicit fractional differential equation with non-instantaneous integral impulses and nonlinear integral boundary conditions:

$$\begin{cases} {}^c\mathcal{D}^{\alpha}_{0,\upsilon}x(\upsilon) = f(\upsilon, x(\upsilon), {}^c\mathcal{D}^{\alpha}_{0,\upsilon}x(\upsilon)), \; \upsilon \in (\upsilon_i, s_i], \; i = 0, 1, \ldots, m, \; 0 < \alpha < 1, \; \upsilon \in (0, 1], \\ x(\upsilon) = I^{\alpha}_{s_{i-1},\upsilon_i}(\xi_i(\upsilon, x(\upsilon))), \; \upsilon \in (s_{i-1}, \upsilon_i], \; i = 1, 2, \ldots, m, \\ x(0) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^T (T - \varsigma)^{\alpha-1} \eta(\varsigma, x(\varsigma)) d\varsigma. \end{cases}$$

Motivated by the aforesaid work, in this manuscript, we investigate the existence, uniqueness, Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability results for the following nonlinear implicit impulsive Langevin equation with two Hilfer fractional derivatives:

$$\begin{cases} \mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta}+\lambda)x(\upsilon) = f(\upsilon, x(\upsilon), \mathcal{D}^{\alpha_1,\beta}x(\upsilon)), \ \ \upsilon \in J = [0,T], \ 0 < \alpha_1, \alpha_2 < 1, \ \ 0 \leq \beta \leq 1, \\ \Delta\, x(\upsilon_i) = I_i(x(\upsilon_i)), \ \ i = 1,2,\ldots,m, \\ I^{1-\gamma}x(0) = x_0, \quad \gamma = (\alpha_1+\alpha_2)(1-\beta)+\beta, \end{cases} \tag{1.1}$$

where $\mathcal{D}^{\alpha_1,\beta}$ and $\mathcal{D}^{\alpha_2,\beta}$ represents two Hilfer fractional derivatives, of order α_1 and α_2 respectively, β determines to the type of initial condition used in the problem. Further $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and $I_i : \mathbb{R} \to \mathbb{R}$ for all $i = 1,2,\ldots,m,$ represents impulsive nonlinear mapping and $\Delta x(\upsilon_i) = x(\upsilon_i^+) - x(\upsilon_i^-)$, where $x(\upsilon_i^+)$ and $x(\upsilon_i^-)$ represent the right and the lift limits, respectively, at $\upsilon = \upsilon_i$ for $i = 1,2,\ldots,m$.

In the second section of this paper, we introduce some notations, definitions and auxiliary results. In section 3, we give the existence, uniqueness results for the proposed model (1.1) obtained via the Banach's contraction. In Section 4, we investigate the Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability of our proposed model (1.1). Finally, we give an example which supports our main result.

2. Preliminaries

We recall some definitions of fractional calculus from [17, 26] as follows.

Definition 2.1. *The fractional integral of order α from 0 to x for the function f is*

$$I^{\alpha}_{0,x}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x f(\varsigma)(x-\varsigma)^{\alpha-1}d\varsigma, \quad x>0, \ \alpha>0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. *The Riemman-Liouville fractional derivative of fractional order α for f is*

$${}^{L}\mathcal{D}^{\alpha}_{0,x}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x \frac{f(\varsigma)}{(x-\varsigma)^{\alpha+1-n}}d\varsigma, \quad x>0, \ n-1<\alpha<n.$$

Definition 2.3. *The Caputo derivative of fractional order α for f is*

$${}^{c}\mathcal{D}^{\alpha}_{0,x}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\varsigma)^{n-\alpha-1}f^{(n)}(\varsigma)d\varsigma, \quad \text{where } n = [\alpha]+1.$$

Definition 2.4. *The classical Caputo derivative of order α of f is*

$${}^{c}\mathcal{D}^{\alpha}_{0,x} = {}^{L}\mathcal{D}^{\alpha}_{0,x}\Big(f(x) - \sum_{k=0}^{n-1}\frac{x^k}{k!}f^{(k)}(0)\Big), \quad x>0, \ n-1<\alpha<n.$$

Definition 2.5. *The Hilfer fractional derivative of order $0<\alpha<1$ and $0\leq\beta\leq1$ of function $f(x)$ is*

$$\mathcal{D}^{\alpha,\beta}f(x) = (I^{\beta(1-\alpha)}\mathcal{D}(I^{(1-\beta)(1-\alpha)}(f))(x).$$

The Hilfer fractional derivative is used as an interpolator between the Riemman-Liouville and Caputo derivative.

Remark 2.1. **(a)** *Operator $\mathcal{D}^{\alpha,\beta}$ also can be written as*

$$\mathcal{D}^{\alpha,\beta} f(x) = (I^{\beta(1-\alpha)}\mathcal{D}(I^{(1-\beta)(1-\alpha)})) = I^{\beta(1-\alpha)}D^{\gamma},\ \gamma = \alpha + \beta - \alpha\beta.$$

(b) *If $\beta = 0$, then $\mathcal{D}^{\alpha,\beta} = \mathcal{D}^{\alpha,0}$ is called Riemman–Liouville fractional derivative.*

(c) *If $\beta = 1$, then $\mathcal{D}^{\alpha,\beta} = I^{1-\alpha}\mathcal{D}$ is called Caputo fractional derivative.*

Remark 2.2. **(i)** *If $f(\cdot) \in C^m([0,\infty),\mathbb{R})$, then*

$${}^{c}\mathcal{D}^{\alpha}_{0,x} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^m(\varsigma)}{(x-\varsigma)^{\alpha+1-m}} d\varsigma = I^{m-\alpha}_{0,x} f^{(m)}(x), \quad x > 0,\ m-1 < \alpha < m.$$

(ii) *In Definition 2.4, the integrable function f can be discontinuous. This fact can support us to consider impulsive fractional problems in the sequel.*

Lemma 2.1. *[17] The fractional differential equation ${}^{c}D^{\alpha} f(x) = 0$ with $\alpha > 0$, involving Caputo differential operator ${}^{c}D^{\alpha}$ have a solution in the following form:*

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{m-1} x^{m-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, m-1$ *and* $m = [\alpha] + 1$.

Lemma 2.2. *[17] For arbitrary $\alpha > 0$, we have*

$$I^{\alpha}({}^{c}\mathcal{D}^{\alpha} f(x)) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{m-1} x^{m-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, m-1$ *and* $m = [\alpha] + 1$.

Lemma 2.3. *[26] Let $\alpha > 0$ and $\beta > 0$, $f \in L^1([a,b])$.*

Then $I^{\alpha} I^{\beta} f(x) = I^{\alpha+\beta} f(x), {}^{c}\mathcal{D}^{\alpha}_{0,x}({}^{c}\mathcal{D}^{\beta}_{0,x} f(x)) = {}^{c}\mathcal{D}^{\alpha+\beta}_{0,x} f(x)$ and $I^{\alpha}\mathcal{D}^{\alpha}_{0,x} f(x) = f(x), x \in [a,b]$.

Let $J = [0,T],\ J_0 = [0,\upsilon_1],\ J_1 = (\upsilon_1,\upsilon_2],\ J_2 = (\upsilon_2,\upsilon_3],\ldots,\ J_{m-1} = (\upsilon_{m-1},\upsilon_m],\ J_m = (\upsilon_m,T]$, $J' = J - \{\upsilon_0,\upsilon_1,\upsilon_2,\ldots,\upsilon_m\}$. Also for convenience use the notation $J_i = (\upsilon_i,\upsilon_{i+1}]$.

Theorem 2.1. ***[[4](Banach's fixed point theorem)].*** *Let B be a Banach space. Then any contraction mapping $N : B \to B$ has a unique fixed point.*

3. Existence and uniqueness

In this section, we investigate the existence, uniqueness of solutions to the proposed Langevin equation using two Hilfer fractional derivatives.

Lemma 3.1. *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, x(\cdot), \mathcal{D}^{\alpha_1,\beta}x(\cdot)) \in C_{1-\gamma}[0,T]$ for all $x \in C_{1-\gamma}[0,T]$. A function $x \in C^{\gamma}_{1-\gamma}[0,T]$ is equivalent to the integral equation*

$$x(\upsilon) = \begin{cases} \frac{x_0}{\Gamma(\gamma)}\upsilon^{\gamma-1} + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma \;\; \upsilon \in J_0, \\ \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} + \int_{\upsilon_1}^{\upsilon}\frac{(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma + \int_0^{\upsilon_1}\frac{(\upsilon_1-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma \\ -\frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_1-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma + I_1(x(\upsilon_1)) \quad \upsilon \in J_1, \\ \frac{x_0}{\Gamma(\gamma)}\upsilon_m^{\gamma-1} + \sum_{i=1}^{m}\int_{\upsilon_{i-1}}^{\upsilon_i}\frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma - \sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma \\ +\sum_{i=1}^{m}I_i(x(\upsilon_i)) \quad \upsilon \in J_i \quad i=1,2,\ldots,m, \end{cases} \tag{3.1}$$

is the only solution of the problem (1.1)

Proof. Let x satisfies (1.1), then for any $\upsilon \in J_0$, there exists a constant $c \in \mathbb{R}$, such that

$$x(\upsilon) = c + \int_0^{\upsilon}\frac{(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma. \tag{3.2}$$

Using the condition $I^{1-\gamma}x(0) = x_0$, Eq (3.2) yields that

$$x(\upsilon) = \frac{x_0}{\Gamma(\gamma)}\upsilon^{\gamma-1} + \int_0^{\upsilon}\frac{(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma, \quad \upsilon \in J_0.$$

Similarly for $\upsilon \in J_1$, there exists a constant $d_1 \in \mathbb{R}$, such that

$$x(\upsilon) = d_1 + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma.$$

Using the condition, we get

$$x(\upsilon_1^-) = \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} + \int_0^{\upsilon_1}\frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma,$$

$$x(\upsilon_1^+) = d_1.$$

In view of

$$\Delta\, x(\upsilon_1) = x(\upsilon_1^+) - x(\upsilon_1^-) = I_1(x(\upsilon_1)),$$

we get

$$x(\upsilon_1^+) - x(\upsilon_1^-) = d_1 - \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} - \int_0^{\upsilon_1}\frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma + \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma,$$

$$I_1(x(\upsilon_1)) = d_1 - \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} - \int_0^{\upsilon_1} \frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma + \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma,$$

$$d_1 = \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} + \int_0^{\upsilon_1} \frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\beta-1}x(\varsigma)d\varsigma + I_1(x(\upsilon_1)).$$

For this value of d_1, we have

$$\begin{aligned} x(\upsilon) &= \int_{\upsilon_1}^{\upsilon} \frac{(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma + \int_0^{\upsilon_1} \frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma \\ &\quad -\frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_1}^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma + \frac{x_0}{\Gamma(\gamma)}\upsilon_1^{\gamma-1} + I_1(x(\upsilon_1)). \end{aligned}$$

Similarly for $\upsilon \in J_i$, we get

$$\begin{aligned} x(\upsilon) &= \frac{x_0}{\Gamma(\gamma)}\upsilon_i^{\gamma-1} + \sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1} f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma \\ &\quad -\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_1}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma + \sum_{i=1}^{m} I_i(x(\upsilon_i)). \end{aligned}$$

Conversely, let that x satisfies (3.1), then it can be easily proved that the solution $x(\upsilon)$ given by (3.1) satisfies (1.1). □

Consider some assumptions as follows:

(H_1) $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is continuous.

(H_2) There exists positive constants Ł$_f$ and Ł$_g$, such that
$\left|f(w,u,m) - f(w,v,n)\right| \le Ł_f|u-v| + Ł_g|m-n|,$ for each $w \in J$ and all $u, v, m, n \in \mathbb{R}$.

(H_3) There exists $Ł_k > 0$, such that
$\left|I_i(u) - I_i(v)\right| \le Ł_k|u-v|,$ for each $\upsilon \in J_i$, $i = 1, 2, \ldots, m$, and for all $u, v \in \mathbb{R}$.

(H_4) There exists $\varphi \in PC(J, \mathbb{R}^+)$ and $\lambda_\varphi > 0 \ni I^\alpha\varphi(\upsilon) \le \lambda_\varphi\varphi(\upsilon)$ for each $\upsilon \in J$.

Theorem 3.1. *Let assumptions $(H_1) - (H_3)$ be satisfied and if*

$$\left(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}T^{\alpha_1+\alpha_2} + \frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)}T^{\alpha_1+\alpha_2} + \frac{m\lambda}{\Gamma(\alpha_1+1)}T^{\alpha_1-1} + mŁ_k\right) < 1, \tag{3.3}$$

then (1.1) *has a unique solution x in $C_{1-\gamma}[0,T]$.*

Proof. We define a mapping $N : C_{1-\gamma}[0,T] \to C_{1-\gamma}[0,T]$

$$\begin{cases} (Nx)(\upsilon) = \dfrac{x_0}{\Gamma(\gamma)}\upsilon^{\gamma-1} + \displaystyle\int_0^{\upsilon} \frac{(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma \quad \upsilon \in J_0, \\ (Nx)(\upsilon) = \dfrac{x_0}{\Gamma(\gamma)}\upsilon_m^{\gamma-1} + \displaystyle\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1} f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma \\ \qquad -\displaystyle\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma + \sum_{i=1}^{m} I_i(x(\upsilon_i)) \quad \upsilon \in J_i \quad i = 1, 2, \ldots, m. \end{cases}$$

For any $x, y \in C_{1-\gamma}[0, T]$ and $\upsilon \in J_i$, consider the following

$$\begin{aligned}
|(Nx)(\upsilon) - (Ny)(\upsilon)| &\leq \sum_{i=1}^{m} \int_{\upsilon_{i-1}}^{\upsilon_i} \frac{(\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} |f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma)) - f(\varsigma, y(\varsigma), \mathcal{D}^{\alpha_1,\beta}y(\varsigma))| d\varsigma \\
&\quad - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} |x(\varsigma) - y(\varsigma)| d\varsigma + \sum_{i=1}^{m} |I_i(x(\upsilon_i)) - I_i(y(\upsilon_i))| \\
&\leq \sum_{i=1}^{m} \frac{Ł_f}{\Gamma(\alpha_1+\alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} |x(\varsigma) - y(\varsigma)| d\varsigma \\
&\quad + \sum_{i=1}^{m} \frac{Ł_g}{\Gamma(\alpha_1+\alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} |\mathcal{D}^{\alpha_1,\beta}x(\varsigma) - \mathcal{D}^{\alpha_1,\beta}y(\varsigma)| d\varsigma \\
&\quad - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} |x(\varsigma) - y(\varsigma)| d\varsigma + Ł_k \sum_{i=1}^{m} |x(\upsilon) - y(\upsilon)| \\
&\leq \sum_{i=1}^{m} \frac{Ł_f}{\Gamma(\alpha_1+\alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} |x(\varsigma) - y(\varsigma)| d\varsigma \\
&\quad + \sum_{i=1}^{m} \frac{Ł_g}{\Gamma(\alpha_1+\alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} \mathcal{D}^{\alpha_1,\beta} |x(\varsigma) - y(\varsigma)| d\varsigma \\
&\quad - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} |x(\varsigma) - y(\varsigma)| d\varsigma + Ł_k \sum_{i=1}^{m} |x(\upsilon) - y(\upsilon)|
\end{aligned}$$

$$\begin{aligned}
&\leq \Big(\frac{mŁ_f (\upsilon_i - \upsilon_{i-1})^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{m\lambda Ł_g (\upsilon_i - \upsilon_{i-1})^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - \frac{m\lambda}{\Gamma(\alpha_1+1)} (\upsilon_i - \upsilon_{i-1})^{\alpha_1} + mŁ_k \Big) |x(\upsilon) - y(\upsilon)| \\
&\leq \Big(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)} T^{\alpha_1+\alpha_2} + \frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)} T^{\alpha_1+\alpha_2} + \frac{m\lambda}{\Gamma(\alpha_1+1)} T^{\alpha_1} + mŁ_k \Big) |x(\upsilon) - y(\upsilon)|.
\end{aligned}$$

Now since

$$\Big(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)} T^{\alpha_1+\alpha_2} + \frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)} T^{\alpha_1+\alpha_2} + \frac{m\lambda}{\Gamma(\alpha_1+1)} T^{\alpha_1-1} + mŁ_k \Big) < 1.$$

Hence x is a contraction according to Banach's contraction theorem and so it has only one fixed point, which is the only one solution of (1.1). □

4. Ulam-Hyers stability analysis

Let $\varepsilon > 0$ and $\varphi : J \to \mathbb{R}^+$ be a continuous function. Consider

$$\begin{cases} |\mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta} + \lambda)z(\upsilon) - f(\upsilon, z(\upsilon), \mathcal{D}^{\alpha_1,\beta}z(\upsilon))| \leq \varepsilon, \quad \upsilon \in J_i, \quad i = 1, 2, \ldots, m, \\ |\Delta\, z(\upsilon_i) - I_i(z(\upsilon_i))| \leq \varepsilon, \quad i = 1, 2, \ldots, m, \end{cases} \tag{4.1}$$

$$\begin{cases} |\mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta} + \lambda)z(\upsilon) - f(\upsilon, z(\upsilon), \mathcal{D}^{\alpha_1,\beta}z(\upsilon))| \leq \varphi(\upsilon), \quad \upsilon \in J_i, \quad i = 1, 2, \ldots, m, \\ |\Delta\, z(\upsilon_i) - I_i(z(\upsilon_i))| \leq \psi, \quad i = 1, 2, \ldots, m, \end{cases} \tag{4.2}$$

and

$$\begin{cases} |\mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta}+\lambda)z(\upsilon)-f(\upsilon,z(\upsilon),\mathcal{D}^{\alpha_1,\beta}z(\upsilon))| \le \varepsilon\varphi(\upsilon), \quad \upsilon\in J_i, \quad i=1,2,\dots,m, \\ |\Delta\, z(\upsilon_i)-I_i(z(\upsilon_i))| \le \varepsilon\psi, \quad i=1,2,\dots,m. \end{cases} \tag{4.3}$$

Definition 4.1. *The problem* (1.1) *is Ulam-Hyers stable if there exists a real number* $C_{f,i,q,\sigma}$ *such that for each solution* $\varepsilon > 0$ *and for each solution* $z \in C_{1-\gamma}[0,T]$ *of the inequality* (4.1)*, there exists a solution* $x \in C_{1-\gamma}[0,T]$ *of the problem* (1.1) *such that*

$$|z(\upsilon)-x(\upsilon)| \le C_{f,i,q,\sigma}\ \varepsilon \ \ \upsilon\in J. \tag{4.4}$$

Definition 4.2. *The problem* (1.1) *is generalized Ulam-Hyers stable if there exists* $\phi_{f,i,q,\sigma} \in C_{1-\gamma}[0,T]$*,* $\phi_{f,i,q,\sigma}(0) = 0$ *and* $\varepsilon > 0$ *such that for each solution* $z \in C_{1-\gamma[0,T]}$ *of the inequality* (4.1)*, there exists a solution* $x \in C_{1-\gamma}[0,T]$ *of the problem* (1.1) *such that*

$$|z(\upsilon)-x(\upsilon)| \le \phi_{f,i,q,\sigma}\ \varepsilon \ \ \upsilon\in J. \tag{4.5}$$

Remark 4.1. *Keep in mind that Definition 4.1* $\Rightarrow$ *Definition 4.2.*

Definition 4.3. *The problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to* (φ,ψ) *if there exists* $C_{f,i,q,\sigma,\varphi} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $z \in C_{1-\gamma}[0,T]$ *of inequality* (4.3) *there is a solution* $x \in C_{1-\gamma}[0,T]$ *of the problem* (1.1) *with*

$$|z(\upsilon)-x(\upsilon)| \le C_{f,i,q,\sigma,\varphi}\varepsilon(\varphi(\upsilon)+\psi)\ \varepsilon \ \ \upsilon\in J. \tag{4.6}$$

Definition 4.4. *The problem* (1.1) *is generalized Ulam-Hyers-Rassias stable with respect to* (φ,ψ) *if there exists* $C_{f,i,q,\sigma,\varphi} > 0$ *such that for each solution* $z \in C_{1-\gamma}[0,T]$ *of inequality* (4.2) *there is a solution* $x \in C_{1-\gamma}[0,T]$ *of the problem* (1.1) *with*

$$|z(\upsilon)-x(\upsilon)| \le C_{f,i,q,\sigma,\varphi}(\varphi(\upsilon)+\psi)\ \varepsilon \ \ \upsilon\in J. \tag{4.7}$$

Remark 4.2. *It should be noted that Definition 4.3 implies Definition 4.4.*

Remark 4.3. *A function* $z \in C_{1-\gamma}[0,T]$ *is a solution of the inequality* (4.1) $\Leftrightarrow$ *there exists a function* $g \in C_{1-\gamma}[0,T]$ *and a sequence* $g_i, i = 1,2,\dots,m$*, depending on* g*, such that*

(a) $|g(\upsilon)| \le \varepsilon,\ |g_i| \le \varepsilon \ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

(b) $\mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta}+\lambda)z(\upsilon) = f(\upsilon,z(\upsilon),\mathcal{D}^{\alpha_1,\beta}z(\upsilon))+g(\upsilon),\ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

(c) $\Delta\, x(\upsilon_i) = I_i(x(\upsilon_i))+g_i,\ \ \upsilon\in J_i,\ i=1,2,\dots,m.$

Remark 4.4. *A function* $z \in C_{1-\gamma}[0,T]$ *satisfies* (4.2) $\Leftrightarrow$ *there exists* $g \in C_{1-\gamma}[0,T]$ *and a sequence* $g_i, i = 1,2,\dots,m$*, depending on* g*, such that*

(a) $|g(\upsilon)| \le \varphi(\upsilon),\ |g_i| \le \psi \ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

(b) $\mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta}+\lambda)z(\upsilon) = f(\upsilon,z(\upsilon),\mathcal{D}^{\alpha_1,\beta}z(\upsilon))+g(\upsilon),\ \ \upsilon\in J_i,\ i=1,2,\dots,m,$

(c) $\Delta\, x(\upsilon_i) = I_i(x(\upsilon_i))+g_i,\ \ \upsilon\in J_i,\ i=1,2,\dots,m.$

Remark 4.5. *A function $z \in C_{1-\gamma}[0,T]$ satisfies* (4.2) $\Leftrightarrow$ *there exists $g \in C_{1-\gamma}[0,T]$ and a sequence $g_i, i = 1,2,\ldots,m$, depending on g, such that*

(a) $|g(\upsilon)| \leq \varepsilon\varphi(\upsilon),\ |g_i| \leq \varepsilon\psi\ \ \upsilon \in J_i,\ i = 1,2,\ldots,m,$

(b) $\mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta} + \lambda)z(\upsilon) = f(\upsilon, z(\upsilon), \mathcal{D}^{\alpha_1,\beta}z(\upsilon)) + g(\upsilon),\ \ \upsilon \in J_i,\ i = 1,2,\ldots,m,$

(c) $\Delta\, x(\upsilon_i) = I_i(x(\upsilon_i)) + g_i,\ \ \upsilon \in J_i,\ i = 1,2,\ldots,m.$

Theorem 4.1. *If the assumptions* $(H1) - (H3)$ *and the inequality* (3.3) *hold, then Eq* (1.1) *is Ulam–Hyers stable and consequently generalized Ulam–Hyers stable.*

Proof. Let $y \in C_{1-\gamma}[0,T]$ satisfies (4.1) and let x be the only one solution of

$$\begin{cases} \mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta} + \lambda)x(\upsilon) = f(\upsilon, x(\upsilon), \mathcal{D}^{\alpha_1,\beta}x(\upsilon))\ \ \upsilon \in J = [0,T],\ 0 < \alpha_1, \alpha_2 < 1,\ \ 0 \leq \beta \leq 1, \\ \Delta\, x(\upsilon_m) = I_m(x(\upsilon_m)),\ \ i = 1,2,\ldots,m, \\ I^{1-\gamma}x(0) = x_0,\quad \gamma = (\alpha_1 + \alpha_2)(1-\beta) + \beta. \end{cases}$$

By Lemma 3.1, we have for each $\upsilon \in J_i$

$$\begin{aligned} x(\upsilon) &= \frac{x_0}{\Gamma(\gamma)}\upsilon_m^{\gamma-1} + \sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma))d\varsigma \\ &\quad - \sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i - \varsigma)^{\alpha_1-1}x(\varsigma)d\varsigma + \sum_{i=1}^{m}I_i(x(\upsilon_i))\quad \upsilon \in J_i\quad i = 1,2,\ldots,m. \end{aligned}$$

Since y satisfies inequality (4.1), so by Remark 4.3, we get

$$\begin{cases} \mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta} + \lambda)y(\upsilon) = f(\upsilon, y(\upsilon), \mathcal{D}^{\alpha_1,\beta}y(\upsilon)) + g_i\ \ \upsilon \in J = [0,T],\ 0 < \alpha_1, \alpha_2 < 1,\ \ 0 \leq \beta \leq 1, \\ \Delta\, x(\upsilon_m) = I_m(y(\upsilon_m)) + g_i,\ \ i = 1,2,\ldots,m, \\ I^{1-\gamma}y(0) = y_0,\quad \gamma = (\alpha_1 + \alpha_2)(1-\beta) + \beta. \end{cases} \tag{4.8}$$

Obviously the solution of (4.8), will be

$$y(\upsilon) = \begin{cases} \frac{y_0}{\Gamma(\gamma)}\upsilon^{\gamma-1} + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}f(\varsigma, y(\varsigma), \mathcal{D}^{\alpha_1,\beta}y(\varsigma))d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}y(\varsigma)d\varsigma \\ + \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1+\alpha_2-1}g_i(\varsigma)d\varsigma - \frac{\lambda}{\Gamma(\alpha_1)}\int_0^{\upsilon}(\upsilon-\varsigma)^{\alpha_1-1}g_i(\varsigma)d\varsigma\quad \upsilon \in J_0, \\ \frac{x_0}{\Gamma(\gamma)}\upsilon_m^{\gamma-1} + \sum_{i=1}^{m}\int_{\upsilon_{i-1}}^{\upsilon_i}\frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}f(\varsigma, y(\varsigma), \mathcal{D}^{\alpha_1,\beta}y(\varsigma))d\varsigma - \sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}y(\varsigma)d\varsigma \\ + \sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}g_i(\varsigma)d\varsigma - \sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}g_i(\varsigma)d\varsigma \\ + \sum_{i=1}^{m}I_i(x(\upsilon_i)) + \sum_{i=1}^{m}g_i,\quad \upsilon \in J_i,\quad i = 1,2,\ldots,m. \end{cases}$$

Therefore, for each $\upsilon \in J_i$, we have the following

$$|x(\upsilon) - y(\upsilon)| \leq \sum_{i=1}^{m}\int_{\upsilon_{i-1}}^{\upsilon_i}\frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}|f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma)) - f(\varsigma, y(\varsigma), \mathcal{D}^{\alpha_1,\beta}y(\varsigma))|d\varsigma$$

$$
\begin{aligned}
&-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}|x(\varsigma)-y(\varsigma)|d\varsigma+\sum_{i=1}^{m}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}g_i(\varsigma)d\varsigma\\
&-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}g_i(\varsigma)d\varsigma+\sum_{i=1}^{m}|I_i(x(\upsilon_i))-I_i(y(\upsilon_i))|+\sum_{i=1}^{m}g_i\\
\leq\;&\sum_{i=1}^{m}Ł_f\int_{\upsilon_{i-1}}^{\upsilon_i}\frac{(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}|x(\varsigma)-y(\varsigma)|d\varsigma-\sum_{i=1}^{m}\frac{\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}|x(\varsigma)-y(\varsigma)|d\varsigma\\
&+\sum_{i=1}^{m}\frac{Ł_g}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}\mathcal{D}^{\alpha_1,\beta}|x(\varsigma)-y(\varsigma)|d\varsigma+Ł_k\sum_{i=1}^{m}|x(\upsilon)-y(\upsilon)|\\
&+\sum_{i=1}^{m}\frac{\varepsilon}{\Gamma(\alpha_1+\alpha_2)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1+\alpha_2-1}d\varsigma-\sum_{i=1}^{m}\frac{\varepsilon\lambda}{\Gamma(\alpha_1)}\int_{\upsilon_{i-1}}^{\upsilon_i}(\upsilon_i-\varsigma)^{\alpha_1-1}d\varsigma+\sum_{i=1}^{m}\varepsilon\\
\leq\;&\Big(\frac{mŁ_f(T)^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma(\alpha_1+1)}(T)^{\alpha_1}+mŁ_k\Big)|x(\upsilon)-y(\upsilon)|\\
&+\frac{m\varepsilon}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\varepsilon\lambda}{\Gamma(\alpha_1+1)}(T)^{\alpha_1}+m\varepsilon,
\end{aligned}
$$

which implies that

$$
|x(\upsilon)-y(\upsilon)|\;\leq\;\varepsilon\Bigg(\frac{\frac{m}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma(\alpha_1+1)}(T)^{\alpha_1}+m}{1-(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}+\frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma(\alpha_1+1)}(T)^{\alpha_1}+mŁ_k)}\Bigg).
$$

Thus

$$
|x(\upsilon)-y(\upsilon)|\leq\varepsilon C_{f,g,\alpha_1,\alpha_2},
$$

where

$$
C_{f,g,\alpha_1,\alpha_2}=\frac{\frac{m}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma(\alpha_1+1)}(T)^{\alpha_1}+m}{1-(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}+\frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)}(T)^{\alpha_1+\alpha_2}-\frac{m\lambda}{\Gamma(\alpha_1+1)}(T)^{\alpha_1}+mŁ_k)}.
$$

So Eq (1.1) is Ulam-Hyers stable and if we set $\phi(\varepsilon)=\varepsilon C_{f,g,\alpha_1,\alpha_2}$, $\phi(0)=0$, then Eq (1.1) is generalized Ulam-Hyers stable. □

Theorem 4.2. *If the assumptions* $(H_1)-(H_4)$ *and the inequality* (3.3) *are satisfied, then the problem* (1.1) *is Ulam-Hyers-Rassias stable with respect to* (φ,ψ)*, consequently generalized Ulam-Hyers-Rassias stable.*

Proof. Let $y\in C_{1-\gamma}[0,T]$ be a solution of the inequality (4.3) and let x be the only one solution of the following problem

$$
\begin{cases}
\mathcal{D}^{\alpha_1,\beta}(\mathcal{D}^{\alpha_2,\beta}+\lambda)x(\upsilon)=f(\upsilon,x(\upsilon),\mathcal{D}^{\alpha_1,\beta}x(\upsilon))\;\;\upsilon\in J=[0,T],\;0<\alpha_1,\alpha_2<1,\;\;0\leq\beta\leq1,\\
\Delta\, x(\upsilon_m)=I_m(x(\upsilon_m)),\;\;i=1,2,\ldots,m,\\
I^{1-\gamma}x(0)=x_0,\quad\gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta.
\end{cases}
$$

From Theorem 4.1, $\forall\, \upsilon \in J_i$, we get

$$
\begin{aligned}
|x(\upsilon) - y(\upsilon)| \;\leq\; & \sum_{i=1}^{m} \int_{\upsilon_{i-1}}^{\upsilon_i} \frac{(\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1 + \alpha_2)} |f(\varsigma, x(\varsigma), \mathcal{D}^{\alpha_1,\beta}x(\varsigma)) - f(\varsigma, y(\varsigma), \mathcal{D}^{\alpha_1,\beta}y(\varsigma))| d\varsigma \\
& - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} |x(\varsigma) - y(\varsigma)| d\varsigma + \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha_1 + \alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} g_i(\varsigma) d\varsigma \\
& - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} g_i(\varsigma) d\varsigma + \sum_{i=1}^{m} |I_i(x(\upsilon_i)) - I_i(y(\upsilon_i))| + \sum_{i=1}^{m} g_i \\
\leq\; & \sum_{i=1}^{m} \frac{Ł_f}{\Gamma(\alpha_1 + \alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} |x(\varsigma) - y(\varsigma)| d\varsigma \\
& + \sum_{i=1}^{m} \frac{Ł_g}{\Gamma(\alpha_1 + \alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} \mathcal{D}^{\alpha_1,\beta} |x(\varsigma) - y(\varsigma)| d\varsigma \\
& - \sum_{i=1}^{m} \frac{\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} |x(\varsigma) - y(\varsigma)| d\varsigma + \sum_{i=1}^{m} \frac{\varepsilon}{\Gamma(\alpha_1 + \alpha_2)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1+\alpha_2-1} \varphi(\varsigma) d\varsigma \\
& - \sum_{i=1}^{m} \frac{\varepsilon\lambda}{\Gamma(\alpha_1)} \int_{\upsilon_{i-1}}^{\upsilon_i} (\upsilon_i - \varsigma)^{\alpha_1-1} \varphi(\varsigma) d\varsigma + Ł_k \sum_{i=1}^{m} |x(\upsilon) - y(\upsilon)| + \sum_{i=1}^{m} \psi \\
\leq\; & \Big(\frac{mŁ_f(\upsilon_i - \upsilon_{i-1})^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} + \frac{m\lambda Ł_g(\upsilon_i - \upsilon_{i-1})^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1 + \alpha_2 + 1)} - \frac{m\lambda(\upsilon_i - \upsilon_{i-1})^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + mŁ_k \Big) |x(\upsilon) - y(\upsilon)| \\
& + \frac{m\varepsilon\lambda_\varphi\varphi(\upsilon)}{\Gamma(\alpha_1 + \alpha_2 + 1)} (\upsilon_i - \upsilon_{i-1})^{\alpha_1+\alpha_2} - \frac{m\varepsilon\lambda_\varphi\varphi(\upsilon)\lambda}{\Gamma(\alpha_1 + 1)} (\upsilon_i - \upsilon_{i-1})^{\alpha_1} + m\varepsilon\psi,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
|x(\upsilon) - y(\upsilon)| \;\leq\; & \varepsilon \Bigg(\frac{\frac{m\lambda_\varphi\varphi(\upsilon)}{\Gamma(\alpha_1+\alpha_2+1)} (\upsilon_i - \upsilon_{i-1})^{\alpha_1+\alpha_2} - \frac{m\lambda_\varphi\varphi(\upsilon)\lambda}{\Gamma(\alpha_1+1)} (\upsilon_i - \upsilon_{i-1})^{\alpha_1} + m\psi}{1 - (\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)} (\upsilon_i - \upsilon_{i-1})^{\alpha_1+\alpha_2} + \frac{m\lambda Ł_g(\upsilon_i-\upsilon_{i-1})^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - \frac{m\lambda}{\Gamma(\alpha_1+1)} (\upsilon_i - \upsilon_{i-1})^{\alpha_1} + mŁ_k)} \Bigg) \\
\leq\; & \Bigg(\frac{\frac{m\lambda_\varphi}{\Gamma(\alpha_1+\alpha_2+1)} (T)^{\alpha_1+\alpha_2} - \frac{m\lambda_\varphi\lambda}{\Gamma(\alpha_1+1)} (T)^{\alpha_1} + m}{1 - (\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)} (T)^{\alpha_1+\alpha_2} + \frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)} (T)^{\alpha_1+\alpha_2} - \frac{m\lambda}{\Gamma(\alpha_1+1)} (T)^{\alpha_1} + mŁ_k)} \Bigg) \varepsilon(\varphi(\upsilon) + \psi).
\end{aligned}
$$

Thus

$$
|x(\upsilon) - y(\upsilon)| \leq C_{f,g,\alpha_1,\alpha_2,\varphi,\psi}\, \varepsilon(\varphi(\upsilon) + \psi),
$$

where

$$
C_{f,g,\alpha_1,\alpha_2,\varphi,\psi} = \Bigg(\frac{\frac{m\lambda_\varphi}{\Gamma(\alpha_1+\alpha_2+1)} (T)^{\alpha_1+\alpha_2} - \frac{m\lambda_\varphi\lambda}{\Gamma(\alpha_1+1)} (T)^{\alpha_1} + m}{1 - (\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)} (T)^{\alpha_1+\alpha_2} + \frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)} (T)^{\alpha_1+\alpha_2} - \frac{m\lambda}{\Gamma(\alpha_1+1)} (T)^{\alpha_1} + mŁ_k)} \Bigg).
$$

Hence (1.1) is Ulam-Hyers-Rassias stable and is obviously generalized Ulam-Hyers-Rassias stable. □

Finally we give an example to illustrate our main result.

Example 4.1.

$$\begin{cases} \mathcal{D}^{(\frac{1}{2},\frac{1}{2})}(\mathcal{D}^{(\frac{1}{3},\frac{1}{2})}+\frac{1}{2})x(\upsilon)=\dfrac{|x(\upsilon)+\mathcal{D}^{(\frac{1}{2},\frac{1}{2})}x(\upsilon)|}{8+e^{\upsilon}+\upsilon^2}, & \upsilon\neq\dfrac{1}{2}\in J=[0,1] \\ I_i x(\dfrac{1}{2})=\dfrac{x|(\frac{1}{2})|}{70+|x(\frac{1}{2})|}, \\ I^{1-\gamma}x(0)=0, \quad \gamma=(\alpha_1+\alpha_2)(1-\beta)+\beta, \end{cases} \tag{4.9}$$

Let $J_0=[0,\frac{1}{2}]$, $J_1=[\frac{1}{2},1]$ $\alpha_1=\frac{1}{2}$, $\alpha_2=\frac{1}{3}$, $\lambda=\lambda_\varphi=\frac{1}{2}$, $Ł_f=Ł_k=\frac{1}{90e^2}$ *and* $m=T=1$.
Obviously

$$\Big(\frac{mŁ_f}{\Gamma(\alpha_1+\alpha_2+1)}T^{\alpha_1+\alpha_2}+\frac{m\lambda Ł_g}{\Gamma(\alpha_1+\alpha_2+1)}T^{\alpha_1+\alpha_2}+\frac{m\lambda}{\Gamma(\alpha_1+1)}T^{\alpha_1-1}+mŁ_k\Big)<1.$$

Thus, thanks to Theorem 3.1, the given problem (4.9) *has a unique solution. Further the conditions of Theorem 4.1 are satisfied so the solution of the given problem* (4.9) *is Ulam-Hyers stable and generalized Ulam-Hyers stable. Further it is also easy to check the conditions of Theorem 4.2 hold and thus the problem* (4.9) *is Ulam-Hyers-Rassias stable and consequently generalized Ulam-Hyers-Rassias stable.*

5. Conclusions

In this article, we consider a class of implicit impulsive Langevin equation with Hilfer fractional derivative. Some conditions are made to beat the hurdles to investigate the existence, uniqueness and to discuss different types of Ulam-Hyers stability of our considered model, using Banach's fixed point theorem. We give an example which supports our main result.

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Conflict of interest

The authors declare no conflict of interest in this paper.

References

1. R. P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.*, **109** (2010), 973–1033.
2. B. Ahmad, J. J. Nieto, A. Alsaedi, M. El-Shahed, A study of nonlinear Langevin equation involving two fractional orders in different intervals, *Nonlinear Anal. Real*, **13** (2012), 599–606.

3. B. Ahmad, J. J. Nieto, Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions, *Comput. Math. Appl.*, **58** (2009), 1838–1843.

4. Z. Ali, F. Rabiei, K. Shah, On Ulam's type stability for a class of impulsive fractional differential equations with nonlinear integral boundary conditions, *J. Nonlinear Sci. Appl.*, **10** (2017), 4760–4775.

5. Z. Ali, A. Zada, K. Shah, Ulam satbility to a toppled systems of nonlinear implicit fractional order boundary value problem, *Bound. Value Probl.*, **2018** (2018), 1–16.

6. Z. Ali, A. Zada, K. Shah, On Ulam's stability for a coupled systems of nonlinear implicit fractional differential equations, *B. Malays. Math. Sci. So.*, **42** (2019), 2681–2699.

7. Z. Bai, On positive solutions of a non-local fractional boundary value problem, *Nonlinear Anal. Theor.*, **72** (2010), 916–924.

8. D. Baleanu, H. Khan, H. Jafari, R. A. Khan, M. Alipure, On existence results for solutions of a coupled system of hybrid boundary value problems with hybrid conditions, *Adv. Differ. Equ.*, **2015** (2015), 1–14.

9. M. Benchohra, J. R. Graef, S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.*, **87** (2008), 851–863.

10. M. Benchohra, D. Seba, Impulsive fractional differential equations in Banach spaces, *Electron. J. Qual. Theo.*, **8** (2009), 1–14.

11. J. B. Diaz, B. Margolis, A fixesd point theorem of the alternative, for contractions on a generalized complete matric space, *Bull. Amer. Math. Soc.*, **74** (1968), 305–309.

12. K. S. Fa, Generalized Langevin equation with fractional derivative and long-time correlation function, *Phys. Rev. E*, **73** (2006), 061104.

13. M. Feckan, Y. Zhou, J. Wang, On the concept and existence of solution for impulsive fractional differential equations, *Commun. Nonlinear Sci.*, **17** (2012), 3050–3060.

14. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, **27** (1941), 222–224.

15. A. Khan, J. F. Gomez–Aguilar, T. S. Khan, H. Khan, Stability analysis and numerical solutions of fractional order HIV/AIDS model, *Chaos, Soliton. Fract.*, **122** (2019), 119–128.

16. Z. G. Hu, W. B. Liu, T. V. Chen, Existence of solutions for a coupled system of fractional differential equations at resonance, *Bound. Value Probl.*, **98** (2012), 1–13.

17. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equation*, Elsevier Science Inc., 2006.

18. N. Kosmatov, Initial value problems of fractional order with fractional impulsive conditions, *Results Math.*, **63** (2013), 1289–1310.

19. V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of fractional dynamic systems*, Cambridge Scientific Publishers, 2009.

20. S. C. Lim, M. Li, L. P. Teo, Langevin equation with two fractional orders, *Phys. Lett. A*, **372** (2008), 6309–6320.

21. F. Mainardi, P. Pironi, The fractional Langevin equation: Brownian motion revisited, *Extracta Math.*, **11** (1996), 140–154.

22. P. A. Naik, Global dynamics of a fractional-order SIR epidemic model with memory, *Int. J. Biomath.*, **13** (2020), 2050071.

23. P. A. Naik, K. M. Owolabi, M. Yavuz, J. Zu, Chaotic dynamics of a fractional order HIV-1 model involving AIDS-related cancer cells, *Chaos, Soliton. Fract.*, **140** (2020), 110272.

24. P. A. Naik, M. Yavuz, J. Zu, The role of prostitution on HIV transmission with memory: A modeling approach, *Alex. Eng. J.*, **59** (2020), 2513–2531.

25. P. A. Naik, J. Zu, K. M. Owolabi, Modeling the mechanics of viral kinetics under immune control during primary infection of HIV-1 with treatment in fractional order, *Physica A*, **545** (2020), 123816.

26. I. Podlubny, *Fractional differential equations*, Academic Press, 1999.

27. T. M. Rassias, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math.*, **72** (1978), 297–300.

28. R. Rizwan, Existence theory and stability snalysis of fractional Langevin equation, *Int. J. Nonlin. Sci. Num.*, **20** (2019).

29. R. Rizwan, A. Zada, X. Wang, Stability analysis of non linear implicit fractional Langevin equation with non-instantaneous impulses, *Adv. Differ. Equ.*, **2019** (2019), 1–31.

30. R. Rizwan, A. Zada, Nonlinear impulsive Langevin equation with mixed derivatives, *Math. Method. Appl. Sci.*, **43** (2020), 427–442.

31. I. A. Rus, Ulam stability of ordinary differential equations, *Stud. Univ. Babes Bolyai Math.*, **54** (2009), 125–133.

32. R. Shah, A. Zada, A fixed point approach to the stability of a nonlinear volterra integro diferential equation with delay, *Hacet. J. Math. Stat.*, **47** (2018), 615–623.

33. S. O. Shah, A. Zada, A. E. Hamza, Stability analysis of the first order non–linear impulsive time varying delay dynamic system on time scales, *Qual. Theor. Dyn. Syst.*, **18** (2019), 825–840.

34. S. Tang, A. Zada, S. Faisal, M. M. A. El-Sheikh, T. Li, Stability of higher–order nonlinear impulsive differential equations, *J. Nonlinear Sci. Appl.*, **9** (2016), 4713–4721.

35. V. E. Tarasov, *Fractional dynamics: Application of fractional calculus to dynamics of particles, fields and media*, Springer-Verlag Berlin Heidelberg, 2010.

36. S. M. Ulam, *A collection of mathematical problems*, Interscience Publishers Inc., 1968.

37. J. Wang, M. Feckan, Y. Zhou, Ulams type stability of impulsive ordinary differential equation, *J. Math. Anal. Appl.*, **35** (2012), 258–264.

38. J. Wang, Y. Zhou, M. Feckan, Nonlinear impulsive problems for fractional differential equations and Ulam stability, *Comput. Math. Appl.*, **64** (2012), 3389–3405.

39. J. Wang, Y. Zhou, Z. Lin, On a new class of impulsive fractional differential equations, *Appl. Math. Comput.*, **242** (2014), 649–657.

40. A. Zada, S. Ali, Stability Analysis of Multi-point Boundary Value Problem for Sequential Fractional Differential Equations with Non-instantaneous Impulses, *Int. J. Nonlin. Sci. Num.*, **19** (2018), 763–774

41. A. Zada, S. Ali, Y. Li, Ulam-type stability for a class of implicit fractional differential equations with non-instantaneous integral impulses and boundary condition, *Adv. Differ. Equ.*, **2017** (2017), 1–26.

42. A. Zada, W. Ali, S. Farina, Hyers–Ulam stability of nonlinear differential equations with fractional integrable impulses, *Math. Method. Appl. Sci.*, **40** (2017), 5502–5514.

43. A. Zada, W. Ali, C. Park, Ulam's type stability of higher order nonlinear delay differential equations via integral inequality of Grönwall-Bellman-Bihari's type, *Appl. Math. Comput.*, **350** (2019), 60–65.

44. A. Zada, R. Rizwan, J. Xu, Z. Fu, On implicit impulsive Langevin equation involving mixed order derivatives, *Adv. Differ. Equ.*, **2019** (2019), 1–26.

45. A. Zada, S. O. Shah, Hyers-Ulam stability of first-order non-linear delay dierential equations with fractional integrable impulses, *Hacet. J. Math. Stat.*, **47** (2018), 1196–1205.

46. A. Zada, O. Shah, R. Shah, Hyers-Ulam stability of non-autonomous systems in terms of boundedness of Cauchy problems, *Appl. Math. Comput.*, **271** (2015), 512–518.

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