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Research article

Asymptotic for a second order evolution equation with damping and regularizing terms

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Abstract: Let $\mathcal{H}$ be a real Hilbert space. We investigate the long time behavior of the trajectories $x(.)$ of the vanishing damped nonlinear dynamical system with regularizing term

$$x''(t) + \gamma(t)x'(t) + \nabla\Phi(x(t)) + \varepsilon(t)\nabla U(x(t)) = 0, \qquad (\mathrm{GAVD}_{\gamma,\varepsilon})$$

where $\Phi, U : \mathcal{H} \to \mathbb{R}$ are two convex continuously differentiable functions, $\varepsilon(.)$ is a decreasing function satisfying $\lim_{t\to+\infty} \varepsilon(t) = 0$, and $\gamma(.)$ is a nonnegative function which behaves, for t large enough, like $\frac{K}{t^\theta}$ where $K > 0$ and $0 \leq \theta \leq 1$. The main contribution of this paper is the following control result: If $\int_0^{+\infty} \frac{\varepsilon(t)}{\gamma(t)} dt = +\infty$, U is strongly convex and its unique minimizer x^* is also a minimizer of Φ then every trajectory $x(.)$ of $(\mathrm{GAVD}_{\gamma,\varepsilon})$ converges strongly to x^* and the rate of convergence to 0 of its energy function

$$W(t) = \frac{1}{2}\|x'(t)\|^2 + \Phi(x(t)) - \Phi^* + \varepsilon(t)(U(x(t)) - U^*)$$

is of order to $\circ(1/t^{1+\theta})$. Moreover, we prove a new result concerning the weak convergence of the trajectories of $(\mathrm{GAVD}_{\gamma,\varepsilon})$ to a common minimizer of Φ and U (if one exists) under a simple condition on the speed of decay of the regularizing factor $\varepsilon(t)$ to 0.

Keywords: convex optimization; ordinary differential equations; asymptotic behavior; regularization method
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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with the inner product $\langle .,. \rangle$ and the associated norm $\|.\|$. Let $\Phi, U : \mathcal{H} \to \mathbb{R}$ be two convex continuously differentiable functions and γ, ε two real positive functions defined on a fixed time interval $[t_0, +\infty)$ for some $t_0 > 0$. Motivated by the work [3] of Attouch, Chbani, and Riahi on the asymptotic behavior of the trajectories of the asymptotic vanishing damping dynamical system with regularizing regularizing term

$$x''(t) + \frac{\alpha}{t}x'(t) + \nabla\Phi(x(t)) + \varepsilon(t)x(t) = 0, \qquad (\mathrm{AVD}_{\gamma,\varepsilon})$$

we investigate in this paper the long time behavior, as $t \to +\infty$, of the trajectories of the following generalized version of the $(\mathrm{AVD}_{\alpha,\varepsilon})$ dynamical system

$$x''(t) + \gamma(t)x'(t) + \nabla\Phi(x(t)) + \varepsilon(t)\nabla U(x(t)) = 0. \qquad (\mathrm{GAVD}_{\gamma,\varepsilon})$$

For the importance and the applications of these two dynamical systems and many other related dynamical systems in Mechanics and Optimization, we refer the reader to [2, 5, 6, 14] and references therein.

Throughout this paper, we assume the following general hypothesis:

(H_1) The functions $\Phi, U : \mathcal{H} \to \mathbb{R}$ are convex, differentiable, and bounded from below. We set $\Phi^* = \inf_{x\in\mathcal{H}} \Phi(x)$ and $U^* = \inf_{x\in\mathcal{H}} U(x)$.

(H_2) The set $S_\Phi := \mathrm{argmin}\Phi = \{z \in \mathcal{H} : \Phi(z) = \Phi^*\}$ is nonempty.

(H_3) The gradient functions $\nabla\Phi$ and ∇U of Φ and U are Lipschitz on bounded subsets of $\mathcal{H}$.

(H_4) The function $\gamma : [t_0, +\infty) \to (0, +\infty)$ is absolutely continuous and satisfies the following property: there exist $t_1 \geq t_0$ and two real constants $K_1, K_2 > 0$ such that

$$\gamma(t) \geq \frac{K_1}{t} \text{ and } \gamma'(t) \leq \frac{K_2}{t^2}$$

for almost every $t \geq t_1$.

(H_5) The function $\varepsilon : [t_0, +\infty) \to (0, +\infty)$ is absolutely continuous, nonincreasing and satisfies

$$\lim_{t\to+\infty} \varepsilon(t) = 0.$$

Proceeding as in the proof of [Theorem 3.1, [6]] and using the classical Cauchy-Lipschitz theorem and the energy function

$$W(t) = \frac{1}{2}\|x'(t)\|^2 + \Phi(x(t)) - \Phi^* + \varepsilon(t)(U(x(t)) - U^*), \qquad (1.1)$$

one can easily prove that for every initial data $(x_0, v_0) \in \mathcal{H} \times \mathcal{H}$, the dynamical system $(\mathrm{GAVD}_{\gamma,\varepsilon})$ has a unique solution $x(.) \in C^2(t_0, +\infty; \mathcal{H})$ which satisfies $x(t_0) = x_0$ and $x'(t_0) = v_0$. Therefore, we assume in what follows that $x(.)$ is a global solution of $(\mathrm{GAVD}_{\gamma,\varepsilon})$ and focus our attention on the study of the long time behavior of $x(t)$ as t goes to infinity. Before starting the presentation of the main contributions of this work in this direction, let us first recall some well known results on the asymptotic behavior of solutions of a variant dynamical systems related to $(\mathrm{GAVD}_{\gamma,\varepsilon})$. In the pioneer work [1], Alvarez considered the case where $\gamma(.)$ is constant and $\varepsilon = 0$. He established that the trajectory $x(t)$ converges weakly to some element $\bar{x}$ of S_Φ. He also proved that the rate of convergence of $\Phi(x(t))$ to Φ^* is of order $\circ(1/t)$ (see [2]). To overcome the drawback of the weak convergence to a non identified minimizer of Φ, Attouch and Cazerniki [5] proved that adding a regularizing term $\varepsilon(t)x(t)$ forces any trajectory $x(t)$ of the system

$$x''(t) + \gamma x'(t) + \nabla\Phi(x(t)) + \varepsilon(t)x(t) = 0 \tag{1.2}$$

to converge strongly to the element x^* of minimum norm of the set S_Φ provided that $\displaystyle\int_{t_0}^{+\infty} \varepsilon(t)dt = +\infty$. Using a different approach, Jendoubi and May [10] proved that this strong convergence result remains true even if a perturbation integrable term $g(t)$ is added to the equation (1.2). In an other direction, in order to improve the rate of convergence of $\Phi(x(t))$ to Φ^*, Su, Boyd, and Candes [14] introduced the following dynamical system which is the continuous version of the Nestrov's accelerated minimization method [12]:

$$x''(t) + \frac{\alpha}{t}x'(t) + \nabla\Phi(x(t)) = 0. \tag{1.3}$$

They proved that if $\alpha \geq 3$ then

$$\Phi(x(t)) - \Phi^* = O(1/t^2).$$

This result was later improved in [4] and [11]. In fact it was proved that if $\alpha > 3$ then $x(t)$ converges weakly to some element $\bar{x}$ of S_Φ and that

$$\Phi(x(t)) - \Phi^* = \circ(1/t^2).$$

In order to benefit at the same time of the quick minimization property $\Phi(x(t)) - \Phi^* = \circ(1/t^2)$ due to the presence of the vanishing damping term $\gamma(t) = \dfrac{\alpha}{t}$ in (1.3) and the strong convergence of the trajectories of (1.2) to a particular minimizer of Φ which is a consequence of the regularizing term $\varepsilon(t)x(t)$, Attouch, Chbani, and Riahi [3] have considered the dynamical system ($\mathrm{AVD}_{\alpha,\varepsilon}$) and have established some properties of the asymptotic behavior of its trajectories which are summarized in the following theorem.

Theorem 1.1 (Attouch, Chbani and Riahi). *Let $x \in C^2(t_0, +\infty; \mathcal{H})$ be a solution of ($AVD_{\alpha,\varepsilon}$). The following assertions hold:*

(A) If $\alpha > 1$ and $\displaystyle\int_{t_0}^{+\infty} \frac{\varepsilon(t)}{t}dt < +\infty$, then $\displaystyle\int_{t_0}^{+\infty} \frac{\|x'(t)\|^2}{t}dt < +\infty$, $\lim_{t\to+\infty} x'(t) = 0$ and $\lim_{t\to+\infty} \Phi(x(t)) = \Phi^*$.

(B) If $\alpha > 3$ and $\displaystyle\int_{t_0}^{+\infty} t\varepsilon(t)dt < +\infty$, then $x(t)$ converges weakly to some element of S_Φ.

Furthermore, the associated energy function $E(t) = \dfrac{1}{2}\|x'(t)\|^2 + \Phi(x(t)) - \Phi^*$ satisfies $E(t) = \circ(1/t^2)$ and $\displaystyle\int_{t_0}^{+\infty} tE(t)dt < +\infty$.

(C) If the function ε satisfies moreover one of the following hypothesis

(H_{5a}) $\lim_{t\to+\infty} t^2\varepsilon(t) = +\infty$ if $\alpha = 3$

(H_{5b}) $t^2\varepsilon(t) \geq c > \frac{4}{9}\alpha(\alpha-3)$ *if* $\alpha > 3$

(H_{5c}) $\displaystyle\int_{t_0}^{+\infty} \frac{\varepsilon(t)}{t} dt = +\infty$

then $\liminf_{t\to+\infty} \|x(t) - x^*\| = 0$ *where* x^* *is the element of minimal norm of the set* S_Φ.

In our present paper, we improve and extend these results to the general dynamical system ($\mathrm{GAVD}_{\gamma,\varepsilon}$). Moreover, we highlight some new asymptotic properties of the trajectories of ($\mathrm{GAVD}_{\gamma,\varepsilon}$).

Our first main result is a general minimization property of ($\mathrm{GAVD}_{\gamma,\varepsilon}$) which improves the assertion (A) in the previous theorem.

Theorem 1.2 (A general minimization property of ($\mathrm{GAVD}_{\gamma,\varepsilon}$))**.** *Let* $x(.)$ *be a solution of (*$\mathrm{GAVD}_{\gamma,\varepsilon}$*). Then* $\displaystyle\int_{t_0}^{+\infty} \gamma(t)\,\|x'(t)\|^2\,dt < +\infty$, *and the energy function* $W(t)$, *defined by (1.1) decreases and converges to* 0 *as* $t \to +\infty$. *In particular* $\lim_{t\to+\infty} x'(t) = 0$ *and* $\lim_{t\to+\infty} \Phi(x(t)) = \Phi^*$.

The second result concerns the weak convergence properties of the trajectories of ($\mathrm{GAVD}_{\gamma,\varepsilon}$). The first part of this result is similar to the assertion (B) in Theorem 1.1. Our proof, which is different from the arguments given by Attouch, Chbani, and Riahi [Theorem 3.1, [3]], provides an other confirmation of the fact, noticed recently in many works as [2, 3, 11] and [14], that the value $\alpha = 3$ in the the system (1.3) is critical and somehow mysterious. The second part of the theorem is a simple result on the weak convergence to a common minimizer of the two convex functions Φ and U. At our knowledge, this result was not known even in the case where the damping term γ is constant. A comparable result has been proved by Cabot(see [Proposition 2.5, [7]]) for the first order system $x'(t) + \nabla\Phi(x(t)) + \varepsilon(t)\nabla U(x(t)) = 0$.

Theorem 1.3 (Weak convergence properties of ($\mathrm{GAVD}_{\gamma,\varepsilon}$))**.** *Assume that there exist* $t_1 \geq t_0$, $0 \leq \theta \leq 1, \alpha > 0$ *with* $\alpha > 3$ *if* $\theta = 1$ *such that*

$$\gamma(t) \geq \frac{\alpha}{t^\theta} \quad \forall t \geq t_1 \text{ and } \int_{t_0}^{+\infty} \left[\left(t^\theta\gamma(t)\right)'\right]^+ dt < +\infty \tag{1.4}$$

where $\left[\left(t^\theta\gamma(t)\right)'\right]^+ = \max\{0, \left(t^\theta\gamma(t)\right)'\}$. *Let* $x(.)$ *be a solution of (*$\mathrm{GAVD}_{\gamma,\varepsilon}$*). Then the two following properties hold:*

(P_1) If $\displaystyle\int_{t_0}^{+\infty} t^\theta\varepsilon(t)dt < +\infty$ *then* $x(t)$ *converges weakly to some element of* S_Φ.

(P_2) If $S_\Phi \cap S_U \neq \emptyset$ *and* $\liminf_{t\to+\infty} t^{1+\theta}\varepsilon(t) > 0$ *then* $x(t)$ *converges weakly to some element of* $S_\Phi \cap S_U$.

Moreover, in both case, the energy function W satisfies the following asymptotical behavior

$$W(t) = \circ(1/t^{1+\theta}) \text{ and } \int_{t_0}^{+\infty} t^\theta W(t)dt < +\infty. \tag{1.5}$$

The last result deals with the strong convergence of the trajectories of ($\mathrm{GAVD}_{\gamma,\varepsilon}$) to a minimizer of the function U on the set of minimizers of Φ.

Theorem 1.4 (Strong convergence properties of ($\text{GAVD}_{\gamma,\varepsilon}$))**.** *Assume that U is strongly convex and $\gamma(t) = \dfrac{\alpha}{t^\theta}$ such that $\alpha > 0$ and $0 \leq \theta < 1$ or $\alpha > 3$ if $\theta = 1$. Suppose in addition that $\displaystyle\int_{t_0}^{+\infty} t^\theta \varepsilon(t)dt = +\infty$. Let $x(.)$ be a solution of ($\text{GAVD}_{\gamma,\varepsilon}$). Then the two following assertions hold true:*

(Q_1) If $x'(t) = \circ(1/t^\theta)$ and $\displaystyle\int_{t_0}^{+\infty} t^\theta \|x'(t)\|^2 dt < +\infty$ then $x(t)$ converges strongly to the unique minimizer p^* of U on S_Φ.

(Q_2) If the unique minimizer x^* of U on $\mathcal{H}$ belongs to S_Φ then $x(t)$ converges strongly to x^* and the energy function W satisfies the asymptotic properties (1.5).

Combining Theorem 1.2 and Theorem 1.4 provides a new proof of following important result due to Attouch and Cazernicki (see [Theorem 2.3, [5]]).

Theorem 1.5 (Attouch and Cazernicki)**.** *Let $\alpha > 0$. If $\displaystyle\int_{t_0}^{+\infty} \varepsilon(t)dt = +\infty$, then any trajectory $x(.)$ to the dynamical system*

$$x''(t) + \alpha x'(t) + \nabla\Phi(x(t)) + \varepsilon(t)x(t) = 0, \tag{1.6}$$

converges strongly to the projection of zero on the closed and convex subset S_Φ.

2. A general minimization property of ($\text{GAVD}_{\gamma,\varepsilon}$)

This section is devoted to the proof of Theorem 1.2. The main idea of the proof is inspired by the paper [9].

Proof. Differentiating the energy function W defined by (1.1) and using the equation ($\text{GAVD}_{\gamma,\varepsilon}$), we obtain

$$\begin{aligned} W'(t) &= -\gamma(t)\|x'(t)\|^2 + \varepsilon'(t)(U(x(t)) - U^*) \\ &\leq -\gamma(t)\|x'(t)\|^2. \end{aligned} \tag{2.1}$$

Hence

$$\int_{t_0}^{+\infty} \gamma(t)\|x'(t)\|^2 dt < \infty, \tag{2.2}$$

and the function $W(t)$ decreases and converges to some positive real number W_∞ as $t \to +\infty$. Therefore, to conclude, we just have to show that $W_\infty \leq 0$. Let v be an arbitrarily element of $\mathcal{H}$. Consider the function

$$h_v(t) \equiv \frac{1}{2}\|x(t) - v\|^2.$$

Using the equation ($\text{GAVD}_{\gamma,\varepsilon}$) and the convexity of Φ and U, one can easily check that

$$\begin{aligned} h_v''(t) + \gamma(t)h_v'(t) &= \|x'(t)\|^2 + \langle\nabla\Phi(x(t)), v - x(t)\rangle + \varepsilon(t)\langle\nabla U(x(t)), v - x(t)\rangle \\ &\leq \|x'(t)\|^2 + \Phi(v) - \Phi(x(t)) + \varepsilon(t)(U(v) - U(x(t))) \\ &= \frac{3}{2}\|x'(t)\|^2 - W(t) + \Phi(v) - \Phi^* + \varepsilon(t)(U(v) - U^*). \end{aligned} \tag{2.3}$$

Recalling that $W(t) \geq W_\infty$, we get

$$A_\infty \leq -h_v''(t) - \gamma(t)h_v'(t) + \frac{3}{2}\|x'(t)\|^2 + \varepsilon(t)(U(v) - U^*)$$

where $A_\infty = W_\infty + \Phi^* - \Phi(v)$.
Integrating the last inequality from t_0 to $t > t_0$ and using the fact that $\gamma h_v \geq 0$ and the assumption $\gamma'(t) \leq \dfrac{K_2}{t^2}$, we find

$$(t - t_0)A_\infty \leq h_v'(t_0) + \gamma(t_0)h_v(t_0) - h_v'(t) + \frac{3}{2}\int_{t_0}^{t}\|x'(s)\|^2\, ds + \int_{t_0}^{t} f_v(s)ds, \tag{2.4}$$

where

$$f_v(s) = \varepsilon(s)(U(v) - U^*) + \frac{K_2}{s^2}h_v(s). \tag{2.5}$$

Since $\gamma(t) \geq \frac{K_1}{t}$, we deduce from (2.2) that $\displaystyle\int_{t_0}^{+\infty}\frac{\|x'(s)\|^2}{s}ds < \infty$ which, thanks to [Lemma 3.2, [9]], implies that

$$\int_{t_0}^{t}\|x'(s)\|^2\, ds = \circ(t). \tag{2.6}$$

Using now the Cauchy-Schwarz inequality, we infer

$$\begin{aligned}\|x(t)\| &\leq \|x(t_0)\| + \sqrt{t - t_0}\left(\int_{t_0}^{t}\|x'(s)\|^2\, ds\right)^{\frac{1}{2}} \\ &= \circ(t), \end{aligned} \tag{2.7}$$

which implies that $\lim\limits_{t\to+\infty} f_v(t) = 0$. Hence, we deduce that

$$\int_{t_0}^{t} f_v(s)ds = \circ(t). \tag{2.8}$$

Recalling that since W is bounded, x' is also bounded. Thus, from (2.7), we infer that

$$h_v'(t) = 2\langle x'(t), x(t) - v\rangle = \circ(t). \tag{2.9}$$

Finally, dividing the inequality (2.4) by t, using the estimates (2.6), (2.8), (2.9) and letting $t \to +\infty$, we obtain $A_\infty \leq 0$, which implies that $W_\infty \leq \Phi(v) - \Phi^*$. Since this holds for every $v \in \mathcal{H}$, the required result $W_\infty \leq 0$ follows. □

Remark 2.1. *Let us notice that in the proof of Theorem 1.2, we did not use the hypothesis (H_2). Moreover, we can prove that if S_Φ is empty, then any solution $x(.)$ of the ($GAVD_{\gamma,\varepsilon}$) system satisfies $\|x(t)\| \to +\infty$ as $t \to +\infty$. Indeed, otherwise there exists a sequence $(t_n)_n$ tending to $+\infty$ so that $(x(t_n))_n$ converges weakly to an element $\bar{x} \in \mathcal{H}$. From the lower semi-continuity property it follows that*

$$\Phi(\bar{x}) \leq \liminf_{n\to+\infty}\Phi(x(t_n)),$$

which, thanks to the fact $\lim_{t\to+\infty}\Phi(x(t)) = \Phi^*$*, implies that $\Phi(\bar{x}) \leq \Phi^*$ and contradicts the assumption $S_\Phi = \emptyset$.*

3. Weak convergence properties of (GAVD$_{\gamma,\varepsilon}$)

In this section, we prove Theorem 1.3. The proof relies on the classical Opial's lemma and the following technical lemma which will be also useful in the study of the strong convergence properties of the trajectories of (GAVD$_{\gamma,\varepsilon}$) in the next section.

Lemma 3.1. *Assume that the function $\gamma(.)$ satisfies the assumption (1.4) in Theorem 1.3. Let $x(.)$ be a solution of (GAVD$_{\gamma,\varepsilon}$) and let $v \in S_\Phi$ such that the positive function $t^\theta r_v(t)$ belongs to $L^1(t_0, +\infty; \mathbb{R})$ where $r_v(t) = \varepsilon(t)(U(v) - U^*)$. Then the function $h_v(t) = \frac{1}{2}\|x(t) - v\|^2$ converges as $t \to +\infty$ and the energy function W satisfies the asymptotic property (1.5).*

Proof. First, we notice that up to take t_1 large enough we can assume that

$$\gamma(t) \geq \frac{K}{t} \text{ for every } t \geq t_1$$

with $K > 3$ and $K = \alpha$ if $\theta = 1$.

Let $\lambda(t) = t^{1+\theta}$. Using (2.1) and the above inequality, we find

$$\begin{aligned}(\lambda W)' &\leq \lambda' W - \lambda\gamma \|x'\|^2 \\ &\leq \lambda' W - \frac{K}{1+\theta}\lambda' \|x'\|^2 \\ &\leq \lambda' W - \frac{K}{2}\lambda' \|x'\|^2 .\end{aligned} \tag{3.1}$$

Therefore,

$$\frac{3}{2}\lambda' \|x'\|^2 \leq \frac{3}{K}\lambda' W - \frac{3}{K}(\lambda W)'.$$

Multiplying (2.3) by $\lambda'(t)$ (we recall that, since $v \in S_\Phi$, $\Phi(v) = \Phi^*$) and using the above inequality, we obtain

$$(1 - \frac{3}{K})\lambda' W + \frac{3}{K}(\lambda W)' \leq -\lambda' h_v'' - \lambda'\gamma h_v' + \lambda' r_v.$$

Integrating this inequality from t_1 to $t > t_1$ leads after some simple computations to the following inequality

$$(1 - \frac{3}{K})\int_{t_1}^t \lambda'(s)W(s)ds + \frac{3}{K}\lambda(t)W(t) \leq C_0 - \lambda'(t)h_v'(t) + f_\theta(t)h_v(t) + \int_{t_1}^t g_\theta(s)h_v(s)ds, \tag{3.2}$$

where

$$f_\theta(t) = \lambda''(t) - (\lambda'\gamma)(t), \tag{3.3}$$

$$g_\theta(t) = [(\lambda'\gamma)']^+(t) - \lambda'''(t), \tag{3.4}$$

and

$$C_0 = \lambda'(t_1)h_v'(t_1) - \lambda''(t_1)h_v(t_1) + \frac{3}{K}\lambda(t_1)W(t_1) + \lambda'(t_1)h_v'(t_1) + \int_{t_1}^{+\infty} \lambda'(s)r_v(s)ds$$

which is a finite real constant thanks to the assumption on the function r_v. Let $A(\theta)$ and $\mu(\theta)$ be two strictly positive constants such that $A(\theta) + \mu(\theta) < \alpha(\theta + 1)$ if $\theta < 1$ and $A(\theta) + \mu(\theta) = 2(\alpha - 1)$ if $\theta = 1$. Since $\gamma(t) \geq \frac{\alpha}{t^\theta}$, we have

$$f_\theta(t) \leq (1 + \theta)(\theta t^{\theta-1} - \alpha).$$

Therefore, up to take t_1 large enough in the case $\theta < 1$, we can assume that

$$f_\theta(t) \leq -A(\theta) - \mu(\theta) \; \forall t \geq t_1. \tag{3.5}$$

Using now the fact that

$$\begin{aligned} \left| h_v'(t) \right| &\leq \|x'(t)\| \, \|x(t) - v\| \\ &\leq 2\sqrt{W(t)}\sqrt{h_v(t)}, \end{aligned}$$

it follows, from the estimate (3.5), that

$$-\lambda'(t)h_v'(t) + f_\theta(t)h_v(t) \leq -A(\theta)h_v(t) + 2\lambda'(t)\sqrt{W(t)}\sqrt{h_v(t)} - \mu(\theta)h_v(t).$$

Applying now the elementary inequality

$$bx - ax^2 \leq \frac{b^2}{4a} \quad \forall a > 0, \; (x, b) \in \mathbb{R}^2$$

with $x = \sqrt{h_v(t)}$, we deduce that for every $t \geq t_1$

$$\begin{aligned} -\lambda'(t)h_v'(t) + f_\theta(t)h_v(t) &\leq \frac{(\lambda'(t))^2 W(t)}{A(\theta)} - \mu(\theta)h_v(t) \\ &= B(\theta, t)\lambda(t)W(t) - \mu(\theta)h_v(t), \end{aligned} \tag{3.6}$$

where

$$B(\theta, t) = \frac{(\theta+1)^2 t^{\theta-1}}{A(\theta)}.$$

Inserting (3.6) in the inequality (3.2), we obtain

$$(1 - \frac{3}{K})\int_{t_1}^{t} \lambda'(s)W(s)ds + (\frac{3}{K} - B(\theta, t))\lambda(t)W(t) + \mu(\theta)h_v(t) \leq C_0 + \int_{t_1}^{t} g_\theta(s)h_v(s)ds \tag{3.7}$$

Let us notice that if $0 \leq \theta < 1$ then $\lim\limits_{t \to +\infty} B(\theta, t) = 0$ and in the case where $\theta = 1$, since $\alpha > 3$, one can choose $0 < \mu(1) < \frac{2}{3}(\alpha - 3)$ to get

$$\frac{3}{K} - B(1, t) \quad = \quad \frac{3}{\alpha} - \frac{4}{A(1)} > 0.$$

Hence, up to take t_1 large enough, we can assume that, for every $0 \leq \theta \leq 1$, there exists a constant $\nu(\theta) > 0$ such that

$$\frac{3}{K} - B(\theta, t) \geq \nu(\theta), \text{ for all } t \geq t_1.$$

In particular, the inequality (3.7) implies

$$\mu(\theta)h_v(t) \leq C_0 + \int_{t_1}^{t} g_\theta(s)h_v(s)ds.$$

Recalling that the function g_θ is integrable over $[t_1, +\infty)$ and applying the Gronwall's lemma, we deduce that the function h_v is bounded. Hence, by going back to the inequality (3.7), we infer that

$$\sup_{t \geq t_1} \lambda(t) W(t) < +\infty$$

and

$$\int_{t_1}^{+\infty} \lambda'(s) W(s) ds < +\infty. \tag{3.8}$$

Now, using the fact that the energy function W is decreasing, we deduce from (3.8) that $t^{1+\theta} W(t) \to 0$ as $t \to +\infty$; in fact, for every $t \geq 2t_1$, we have

$$(1+\theta)\left(\frac{t}{2}\right)^{1+\theta} W(t) \leq \int_{\frac{t}{2}}^{t} \lambda'(s) W(s) ds.$$

To conclude, it remains to prove that $\lim_{t \to +\infty} h_v(t)$ exists. From (2.3), the function h_v satisfies the differential inequality

$$h_v''(t) + \gamma(t) h_v'(t) \leq \zeta(t)$$

where

$$\zeta(t) = \frac{3}{2} \|x'(t)\|^2 + r_v(t).$$

The assumption on the function r_v and the estimate (3.8) imply that $t^\theta \zeta(t) \in L^1(t_0, +\infty; \mathbb{R}^+)$, thus the existence of $\lim_{t \to +\infty} h_v(t)$ follows from the following lemma. □

Lemma 3.2. *Let $a > 0$ and $w : [a, +\infty) \to \mathbb{R}^+$ be a continuous function satisfying*

$$w(t) \geq \frac{\alpha}{t^\theta} \ \forall t \geq a$$

where α and θ are nonnegative constants with $0 \leq \theta \leq 1$ and $\alpha > 1$ if $\theta = 1$. Let $\varphi \in C^2(a, +\infty; \mathbb{R}^+)$ satisfy a differential inequality

$$\varphi''(t) + w(t)\varphi'(t) \leq \psi(t) \tag{3.9}$$

with $t^\theta \psi(t) \in L^1(a, +\infty; \mathbb{R}^+)$. Then $\lim_{t \to +\infty} \varphi(t)$ exists.

Proof. Multiplying each member of the differential inequality (3.9) by $e^{\Gamma(t,a)}$, where

$$\Gamma(t, s) = \int_s^t w(\tau) d\tau,$$

and integrating on $[t_0, t]$, we get

$$\varphi'(t) \leq e^{-\Gamma(t,a)} \varphi'(a) + \int_a^t e^{-\Gamma(t,s)} \psi(s) ds. \tag{3.10}$$

Applying Fubini's Theorem and using [Lemma 3.14, [8]], we deduce that there is a real constant $M > 0$ such that

$$\int_a^{+\infty} \int_a^t e^{-\Gamma(t,s)} \psi(s) ds dt \leq M \int_a^{+\infty} s^\theta \psi(s) ds.$$

We therefore infer from (3.10) that the positive part $[\varphi']^+$ of φ' belongs to $L^1(a, +\infty; \mathbb{R}^+)$ which implies that $\lim_{t \to +\infty} \varphi(t)$ exists. □

Before starting the proof of Theorem 1.3, let us recall the classical Opial's lemma.

Lemma 3.3 (Opial's lemma). *Let $x : [t_0, +\infty) \to \mathcal{H}$. Assume that there exists a nonempty subset S of $\mathcal{H}$ such that:*

i) if $t_n \to +\infty$ and $x(t_n) \rightharpoonup x$ weakly in $\mathcal{H}$, then $x \in S$,
ii) for every $z \in S$, $\lim\limits_{t\to+\infty} \|x(t) - z\|$ exists.

Then there exists $z_\infty \in S$ such that $x(t) \rightharpoonup z_\infty$ weakly in $\mathcal{H}$ as $t \to +\infty$.

For a simple proof of Opial's lemma, we refer the reader to [13].

Proof of Theorem 1.3. :
Step 1: Proof of the property (P_1).
According to Lemma 3.1, $\lim\limits_{t\to+\infty} h_v(t)$ exists for every $v \in S_\Phi$ and the energy function W satisfies (1.5). Let $t_n \to +\infty$ such that $x(t_n)$ converges weakly in $\mathcal{H}$ to some $\bar{x}$. Since $\Phi(x(t)) \to \Phi^*$ as $t \to +\infty$, the weak lower semi-continuity of the convex function Φ implies that $\Phi(\bar{x}) \le \Phi^*$ which means that $\bar{x} \in S_\Phi$. By Opial's lemma, we deduce that $x(t)$ converges weakly in $\mathcal{H}$ as $t \to +\infty$ to some element of S_Φ.
Step2: Proof of the property (P_2).
Let $v \in S = S_\Phi \cap S_U$. Since $r_v = 0$, Lemma 3.1 implies that $\lim\limits_{t\to+\infty} h_v(t)$ exists and W satisfies (1.5). Thus, in view of the assumption $\liminf\limits_{t\to+\infty} t^{\theta+1}\varepsilon(t) > 0$, we have $U(x(t)) \to U^*$ as $t \to +\infty$. Therefore, as in the above step, the weak lower semi-continuity of the convex functions Φ and U yields that every sequential weak cluster point of $x(t)$, as $t \to +\infty$, belongs to the subset S. This completes the proof of the property (P_2) thanks to Opial's lemma. □

4. Strong convergence properties of (GAVD$_{\gamma,\varepsilon}$)

This section is devoted to the proof of Theorem 1.4. Before proving separately the two properties (Q_1) and (Q_2), let us first recall some general facts about strongly convex functions and a Tikhonov approximation method [15]. Since the function U is strongly convex, there exists a positive real m such that $U(x) - \frac{m}{2}\|x\|^2$ is convex (we say that U is $m-$strongly convex). Moreover, for every nonempty, convex and closed subset C of $\mathcal{H}$, the function U has a unique minimizer x^*_C on C. Let x^* be the minimizer of U on $\mathcal{H}$ and p^* its minimizer on S_Φ. For every $t \ge t_0$, we consider the function Φ_t defined on $\mathcal{H}$ by

$$\Phi_t(x) = \Phi(x) + \varepsilon(t)U(x).$$

Clearly, Φ_t is $\varepsilon(t)m$-strongly convex. Therefore, Φ_t satisfies the convex inequality

$$\Phi_t(z) \ge \Phi_t(y) + \langle \nabla\Phi_t(y), z - y\rangle + \frac{m}{2}\varepsilon(t)\|z - y\|^2, \tag{4.1}$$

and has a unique global minimizer which we denote by $x_{\varepsilon(t)}$. Adopting the Tikhonov method, we can prove that $x_{\varepsilon(t)}$ converges strongly to p^* as $t \to +\infty$. Indeed, since

$$\Phi_t(x_{\varepsilon(t)}) \le \Phi_t(p^*) \tag{4.2}$$

and

$$\Phi(p^*) \le \Phi(x_{\varepsilon(t)}),$$

then

$$U(x_{\varepsilon(t)}) \leq U(p^*). \tag{4.3}$$

Furthermore, since U is strongly convex, $U(x) \to +\infty$ as $\|x\| \to +\infty$; hence from the inequality 4.3 we deduce that $(x_{\varepsilon(t)})_{t \geq t_0}$ is bounded. So, let $\tilde{x} \in \mathcal{H}$ be a weak limit of a sequence $(x_{\varepsilon(t_n)})$ where $t_n \to +\infty$. Using the weak lower semi-continuity of the two convex functions Φ and U and letting $t = t_n \to +\infty$ in the inequalities (4.2) and (4.3), we deduce that $\Phi(\tilde{x}) \leq \Phi(p^*)$ and $U(\tilde{x}) \leq U(p^*)$ which is, from the definition of p^*, is equivalent to $\tilde{x} = p^*$. Consequently, we infer that $x_{\varepsilon(t)}$ converges weakly to p^* as $t_n \to +\infty$. Now, since U is $m-$strongly convex, we have

$$U(x_{\varepsilon(t)}) \geq U(p^*) + \langle \nabla U(p^*), x_{\varepsilon(t)} - p^* \rangle + \frac{m}{2} \left\| x_{\varepsilon(t)} - p^* \right\|^2 .$$

Hence, by (4.3), we conclude that $\lim_{t \to +\infty} \left\| x_{\varepsilon(t)} - p^* \right\| = 0$ which completes the proof of the claim.

Proof of Theorem 1.4. Let us first prove the assertion (Q_1). We consider the function $h(t) = h_{p^*}(t) = \frac{1}{2} \|x(t) - p^*\|^2$. Using the equation (GAVDγ, ε) and the convex inequality (4.1), we obtain

$$\begin{aligned} h''(t) + \gamma(t)h'(t) &= \|x'(t)\|^2 + \langle \nabla \Phi_t(x(t)), p^* - x(t) \rangle \\ &\leq \|x'(t)\|^2 + \Phi_t(p^*) - \Phi_t(x(t)) - m\, \varepsilon(t)h(t) \\ &\leq \|x'(t)\|^2 + \Phi_t(p^*) - \Phi_t(x_{\varepsilon(t)}) - m\, \varepsilon(t)h(t) \\ &\leq \|x'(t)\|^2 + \varepsilon(t)(U(p^*) - U(x_{\varepsilon(t)})) - m\, \varepsilon(t)h(t). \end{aligned} \tag{4.4}$$

In the last inequality we have used the fact that p^* is also a minimizer of Φ. Set

$$\sigma(t) \equiv U(x_{\varepsilon(t)}) - U(p^*) + m\, h(t).$$

The inequality (4.4) becomes

$$h''(t) + \gamma(t)h'(t) + \varepsilon(t)\sigma(t) \leq \|x'(t)\|^2 . \tag{4.5}$$

Let us prove that $\liminf_{t \to +\infty} h(t) = 0$. We argue by contradiction. As consequence of

$$\lim_{t \to +\infty} U(x_{\varepsilon(t)}) - U(p^*) = 0,$$

there exists $t_2 \geq t_0$ large enough and $\mu > 0$ such that $\sigma(t) \geq \mu$ for every $t \geq t_2$. Therefore the differential inequality (4.5) implies that, for every $t \geq t_2$, we have

$$h(t) + \mu \int_{t_2}^{t} \int_{t_2}^{\tau} e^{-\Gamma(\tau,s)} \varepsilon(s) ds d\tau \leq h(t_2) + \int_{t_2}^{t} e^{-\Gamma(\tau,t_2)} d\tau h'(t_2) + \int_{t_2}^{t} \int_{t_2}^{\tau} e^{-\Gamma(\tau,s)} \|x'(s)\|^2 ds d\tau,$$

where

$$\Gamma(t,s) = \int_s^t \gamma(\tau) d\tau.$$

Applying Fubini's theorem, we then infer that

$$\mu \int_{t_2}^{+\infty} \varepsilon(s) \int_s^{+\infty} e^{-\Gamma(\tau,s)} d\tau ds \leq h(t_2) + |h'(t_2)| \int_{t_2}^{+\infty} e^{-\Gamma(\tau,t_2)} d\tau + \int_{t_2}^{+\infty} \|x'(s)\|^2 \int_s^{+\infty} e^{-\Gamma(\tau,s)} d\tau ds. \tag{4.6}$$

Since $\gamma(t) = \frac{\alpha}{t^\theta}$, a simple integration by parts ensures the existence of two real constants $B_\theta > A_\theta > 0$ so that

$$A_\theta\, s^\theta \leq \int_s^{+\infty} e^{-\Gamma(\tau,s)} d\tau \leq B_\theta\, s^\theta, \text{ for every } s \geq t_0.$$

Hence, by combining the inequality (4.6) and the assumption

$$\int_{t_0}^{+\infty} s^\theta \, \|x'(s)\|^2 \, ds < +\infty,$$

we get

$$\int_{t_0}^{+\infty} s^\theta \varepsilon(s) ds < +\infty,$$

which contradicts our assumption on the function $\varepsilon(.)$. We therefore deduce that

$$\liminf_{t\to+\infty} h(t) = 0. \tag{4.7}$$

Now let us suppose that

$$\limsup_{t\to+\infty} h(t) > 0. \tag{4.8}$$

The continuity of the function h combined with (4.7) and (4.8) ensures the existence of two real numbers $\lambda < \delta$ and two positive real sequences $(s_n)_n$ and $(t_n)_n$ such that for every $n \in \mathbb{N}$ we have

$$\begin{aligned} \max\{t_*, n\} < s_n < t_n, \\ h(t_n) = \delta, \\ h(s_n) = \lambda, \\ h(s) \in [\lambda, \delta] \text{ on } [s_n, t_n], \end{aligned}$$

where $t_* > t_2$ is a fixed positive number such that for every $t \geq t_*$

$$U(x_{\varepsilon(t)}) - U(p^*) \geq -m\, \lambda.$$

We deduce from (4.5) that for every $n \in \mathbb{N}$ and for all $t \in [s_n, t_n]$

$$h''(t) + \frac{\alpha}{t^\theta} h'(t) \leq \|x'(t)\|^2 .$$

Multiplying the last differential inequality by t^θ and integrating over $[s_n, t_n]$, we obtain

$$t_n^\theta h'(t_n) - s_n^\theta h'(s_n) + \theta s_n^{\theta-1}\lambda - \theta t_n^{\theta-1}\delta + \alpha(\delta - \lambda) + \theta(\theta - 1) \int_{s_n}^{t_n} t^{\theta-2} h(t) dt \leq \int_{s_n}^{t_n} t^\theta \, \|x'(t)\|^2 . \tag{4.9}$$

Using now the three facts

$$\begin{aligned} |h'(t_n)| &\leq \|x'(t_n)\| \, \sqrt{2h(t_n)} = \|x'(t_n)\| \, \sqrt{2\delta}, \\ |h'(s_n)| &\leq \|x'(s_n)\| \, \sqrt{2\lambda}, \\ \int_{s_n}^{t_n} t^{\theta-2} h(t) dt &\leq \delta \frac{s_n^{\theta-1}}{1-\theta} \text{ if } 0 \leq \theta < 1, \end{aligned}$$

and letting n goes to $+\infty$ in the the inequality (4.9), we get

$$(\alpha - 1)(\delta - \lambda) \leq 0 \text{ if } \theta = 1,$$
$$\alpha(\delta - \lambda) \leq 0 \text{ if } 0 \leq \theta < 1.$$

This contradicts the assumption $\delta > \lambda$. We therefore conclude that $\lim\limits_{t\to+\infty} h(t) = 0$, which completes the proof of the assertion (Q_1).
In order to prove the assertion (Q_2), we first apply 3.1 with $v = x^\star$ to deduce that the energy function W satisfies the asymptotic property (1.5) which implies in particular the solution $x(.)$ fulfils the assumption in the previous assertion (Q_1). Therefore, we conclude that $x(t)$ converges strongly to $p^\star$ which is, in the present case, equal to $x^\star$. This ends the proof of our main theorem. □

Let us now prove the theorem 1.5 of Attouch and Cazernicki.

Proof. The dynamical system (1.6) is the particular case of the general system $(\mathrm{GAVD}_{\gamma,\varepsilon})$ corresponding to $\theta = 0$ and $U(x) = \frac{1}{2}\|x\|^2$. It follows from Theorem 1.2 that any solution $x(.)$ of the dynamical system (1.6) satisfies the assumptions of the assertion (Q_1) of Theorem 1.4. Therefore, we deduce that $x(t)$ converges strongly to the unique minimizer p^* of the function of U. To conclude we notice that, in this case, p^* is the projection of zero on S_Φ. □

Remark 4.1. *Without the regularizing term $\epsilon(t)\nabla U(x(t))$ and under appropriate assumptions on the damping term, the trajectories of the dynamical system ($GAVD_{\gamma,\varepsilon}$) weakly converge to a non-specified minimizer of the objective functional Φ (See Theorem 1.3 with $\epsilon(t) = 0$). As it is shown in the main theorem (Theorem 1.4), if $\epsilon(t)$ vanishes slowly at infinity, then te regularizing term $\epsilon(t)\nabla U(x(t))$ forces the trajectories of the differential system ($GAVD_{\gamma,\varepsilon}$) to converges strongly to a particular minimizer of Φ. In this sense, The added term $\epsilon(t)\nabla U(x(t))$ may be considered as a stabilizer factor for our dynamical system.*

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Conflict of interest

The authors declare that they have no conflict of interest.

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