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Research article

Modified Tseng's splitting algorithms for the sum of two monotone operators in Banach spaces

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Abstract: In this work, we introduce two modified Tseng's splitting algorithms with a new non-monotone adaptive step size for solving monotone inclusion problem in the framework of Banach spaces. Under some mild assumptions, we establish the weak and strong convergence results of the proposed algorithms. Moreover, we also apply our results to variational inequality problem, convex minimization problem and signal recovery, and provide several numerical experiments including comparisons with other related algorithms.

Keywords: maximal monotone operator; Banach space; strong convergence; self adaptive method
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1. Introduction

Let E be a real Banach space with its dual space E^*. In this paper, we study the so-called *monotone inclusion problem*:

$$\text{find } z \in E \text{ such that } 0 \in (A + B)z, \tag{1.1}$$

where $A : E \to E^*$ is a single mapping and $B : E \to 2^{E^*}$ is a multi-valued mapping. The set of solutions of the problem (1.1) is denoted by $(A + B)^{-1}0 := \{x \in E : 0 \in (A + B)x\}$. This problem draws much attention since it stands at the core of many mathematical problems, such as: variational inequalities, split feasibility problem and minimization problem with applications in machine learning, statistical regression, image processing and signal recovery (see [17, 33, 44]).

A classical method for solving the problem (1.1) in Hilbert space H, is known as *forward-backward splitting algorithm* (FBSA) [15, 29] which generates iterative sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = J_\lambda^B (I - \lambda A) x_n, \quad \forall n \geq 1, \end{cases} \tag{1.2}$$

where $J_\lambda^B := (I + \lambda B)^{-1}$ is the resolvent operator of an operator B. Here, I denotes the identity operator on H. It was proved that the sequence generated by (1.2) converges weakly to an element in $(A+B)^{-1}0$ under the assumption of the α-cocoercivity of the operator A, that is,

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \ \forall x, y \in H$$

and λ is chosen in $(0, 2\alpha)$. In fact, FBSA includes, as special cases, the proximal point algorithm (when $A = 0$) [11, 20, 34] and the gradient method [18].

In order to get strong convergence result, Takashashi et al. [41] introduced the following algorithm:

$$\begin{cases} x_1, u \in H, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\lambda_n}^B (x_n - \lambda_n A x_n), \quad \forall n \geq 1, \end{cases} \tag{1.3}$$

where A is an α-cocoercive mapping on H. It was shown that if $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy the following assumptions:

$$0 < a \leq \lambda_n \leq b < 2\alpha, \ \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

$$\lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

then the sequence $\{x_n\}$ defined by (1.3) converges strongly to an element in $(A + B)^{-1}0$.

In 2016, Cholamjiak [12] introduced the following FBSA in a uniformly convex and q-uniformly smooth Banach space E:

$$\begin{cases} x_1, u \in E, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n J_{\lambda_n}^B (x_n - \lambda_n A x_n), \ \forall n \geq 1, \end{cases} \tag{1.4}$$

where $J_{\lambda_n}^B := (I + \lambda_n B)^{-1}$ is the resolvent operator of an m-accretive operator B and A is an α-cocoercive mapping. He proved that the sequence generated by (1.4) converges strongly to a solution of the problem (1.1) under the following assumptions:

$$\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1) \text{ with } \alpha_n + \beta_n + \gamma_n = 1,$$

$$\lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty \text{ and } \liminf_{n \to \infty} \gamma_n > 0,$$

$$0 < \liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < (\frac{\alpha q}{\kappa_q})^{\frac{1}{q-1}},$$

where κ_q is the q-uniform smoothness coefficient of E.

In recent years, the FBSA for solving the monotone inclusion problem (1.1), when A is α-cocoercive, was studied and modified by many authors in various settings (see, *e.g.*, [1, 9, 10, 13, 26, 27, 32, 37, 38, 46]). It is important to remark that the α-cocoercivity of the operator A is a strong assumption. To relax this assumption, Tseng [45] introduced the following so-called *Tseng's splitting method*:

$$\begin{cases} x_1 \in H, \\ y_n = J^B_{\lambda_n}(x_n - \lambda_n A x_n), \\ x_{n+1} = y_n - \lambda_n(Ay_n - Ax_n), \quad \forall n \ge 1, \end{cases} \tag{1.5}$$

where A is monotone and L-Lipschitz continuous with $L > 0$. It was proved that the sequence $\{x_n\}$ generated by (1.5) converges weakly to an element in $(A+B)^{-1}0$ provided the step size λ_n is chosen in $\left(0, \frac{1}{L}\right)$. It is worth noting that Tseng's splitting method is a requirement to know Lipschitz constant of the mapping. Unfortunately, Lipschitz constants are often unknown or difficult to approximate.

Very recently, Shehu [37] extended Tseng's result to Banach spaces. He proposed the following iterative process for approximating a solution of the problem (1.1) in a 2-uniformly convex Banach space E which is also uniformly smooth:

$$\begin{cases} x_1 \in E, \\ y_n = J^B_{\lambda_n} J^{-1}(Jx_n - \lambda_n A x_n), \\ x_{n+1} = Jy_n - \lambda_n(Ay_n - Ax_n), \quad \forall n \ge 1, \end{cases} \tag{1.6}$$

where $A : E \to E^*$ is monotone and L-Lipschitz continuous, $J^B_{\lambda_n} := (J + \lambda_n B)^{-1} J$ is the resolvent of B and J is the duality mapping from E into E^*. He obtain weak convergence theorem to the solution of the problem (1.1) provided the step size λ_n is chosen in $\left(0, \frac{1}{\sqrt{2\mu\kappa}L}\right)$, where μ is the 2-uniform convexity constant of E and κ is the 2-uniform smoothness constant of E^*. At the same time, he also proposed a variant of (1.6) with a linesearch for solving the problem (1.1). It is known that any algorithm with a linesearch needs an inner loop with some stopping criterion over iteration.

In this paper, motivated by Shehu [37], we propose two modifications of Tseng's splitting method with non-monotone adaptive step sizes for solving the problem (1.1) in the framework of Banach spaces. The step size of our methods does not require the prior knowledge of the Lipschitz constant of operator and without any linesearch procedure. The remainder of this paper is organized as follows: We recall some definitions and lemmas in Section 2. Our methods are presented and analyzed in Section 3. Theoretical applications to variational inequality problem and convex minimization problem are considered in Section 4 and finally, in Section 5, we provide some numerical experiments to illustrate the behaviour of our methods.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the set of real numbers and the set of positive integers, respectively. Let E be a real Banach space with its dual space E^*. We denote $\langle x, f\rangle$ by the value of a functional f in E^* at x in E, that is, $\langle x, f\rangle = f(x)$. For a sequence $\{x_n\}$ in E, the strong convergence and the weak convergence of $\{x_n\}$ to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let $S_E = \{x \in E : \|x\| = 1\}$. The space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. The space E is said to be *uniformly smooth* if the limit (2.1) converges uniformly in $x, y \in S_E$. It is said to be *strictly convex* if $\|(x + y)/2\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. The space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, where δ_E is the modulus of convexity of E defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \tfrac{\|x+y\|}{2} : x, y \in S_E, \|x - y\| \geq \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. Let $p \geq 2$. The space E is said to be *p-uniformly convex* if there is a $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0, 2]$. Let $1 < q \leq 2$. The space E is said to be *q-uniformly smooth* if there exists a $\kappa > 0$ such that $\rho_E(t) \leq \kappa t^q$ for all $t > 0$, where ρ_E is the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\tfrac{\|x+ty\|+\|x-ty\|}{2} - 1 : x, y \in S_E\right\}$$

for all $t \geq 0$. Let $1 < q \leq 2 < p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. It is observed that every p-uniformly convex (q-uniformly smooth) space is uniformly convex (uniformly smooth) space. It is known that E is p-uniformly convex (q-uniformly smooth) if and only if its dual E^* is q-uniformly smooth (p-uniformly convex) (see [2]). If E is uniformly convex then E is reflexive and strictly convex and if E is uniformly smooth then E is reflexive and smooth (see [14]). Moreover, we know that for every $p > 1$, L_p and ℓ_p are $\min\{p, 2\}$-uniformly smooth and $\max\{p, 2\}$-uniformly convex, while Hilbert space is 2-uniformly smooth and 2-uniformly convex (see [4, 23, 47] for more details).

Definition 2.1. The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \ \forall x \in E,$$

where $\langle\cdot, \cdot\rangle$ denotes the duality pairing between E and E^*.

If E is a Hilbert space, then $J = I$ is the identity mapping on E. It is known that E is smooth if and only if J is single-valued from E into E^* and if E is a reflexive, smooth and strictly convex, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E. Moreover, if E is uniformly smooth, then J is norm-to-norm uniformly continuous on bounded subsets of E (see [2, 14] for more details). A duality mapping J from a smooth Banach space E into E^* is said to be *weakly sequentially continuous* if for any sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x$ implies that $Jx_n \rightharpoonup^* Jx$.

Lemma 2.2. [39] *Let E be a smooth Banach space and J be the duality mapping on E. Then $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Definition 2.3. A mapping $A : E \to E^*$ is said to be:

• *α-cocoercive* if there exists a constant $\alpha > 0$ such that $\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in E$;

• *monotone* if $\langle x - y, Ax - Ay\rangle \geq 0$ for all $x, y \in E$;

• *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $||Ax - Ay|| \leq L||x - y||$ for all $x, y \in E$;

• *hemicontinuous* if for each $x, y \in E$, the mapping $f : [0, 1] \to E^*$ defined by $f(t) = A(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*.

Remark 2.4. It is easy to see that if A is cocoercive, then A is monotone and Lipschitz continuous but the converse is not true in general.

The next lemma can be found in [49] (see also [47]).

Lemma 2.5. (i) *Let E be a* 2*-uniformly smooth Banach space. Then there exists a constant $\kappa > 0$ such that*

$$||x - y||^2 \leq ||x||^2 - 2\langle y, Jx\rangle + \kappa||y||^2, \ \forall x, y \in E.$$

(ii) *Let E be a* 2*-uniformly convex Banach space. Then there exists a constant $c > 0$ such that*

$$||x - y||^2 \geq ||x||^2 - 2\langle y, Jx\rangle + c||y||^2, \ \forall x, y \in E.$$

Remark 2.6. It is well-known that $\kappa = c = 1$ whenever E is a Hilbert space. Hence these inequalities reduce to the following well-known polarization identity:

$$||x - y||^2 = ||x||^2 - 2\langle x, y\rangle + ||y||^2.$$

Moreover, we refer to [49] for the exact values of constants κ and c.

Next, we recall the following Lyapunov function which was introduced in [3]:

Definition 2.7. Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \ \forall x, y \in E. \tag{2.2}$$

If E is a Hilbert space, then $\phi(x, y) = ||x - y||^2$ for all $x, y \in E$. In addition, the Lyapunov function ϕ has the following properties:

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2, \ \forall x, y \in E. \tag{2.3}$$

$$\phi(x, J^{-1}(\alpha Jy + (1 - \alpha)Jz) \leq \alpha\phi(x, y) + (1 - \alpha)\phi(x, z), \ \forall x, y, z \in E, \ \alpha \in [0, 1]. \tag{2.4}$$

$$\phi(x, y) = \phi(x, z) - \phi(y, z) + 2\langle y - x, Jy - Jz\rangle, \ \forall x, y, z \in E. \tag{2.5}$$

Lemma 2.8. [6] *Let E be a* 2*-uniformly convex Banach space, then there exists a constant $c > 0$ such that*

$$c||x - y||^2 \leq \phi(x, y),$$

where c is a constant in Lemma 2.5 (ii).

We make use of the following functional $V : E \times E^* \to \mathbb{R}$ studied in [3]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \ \forall x \in E, \ x^* \in E^*. \tag{2.6}$$

Obviously, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$.

Lemma 2.9. [3] *Let E be a reflexive, strictly convex and smooth Banach space. Then the following statement holds:*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*), \ \forall x \in E, \ x^*, y^* \in E^*.$$

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a closed and convex subset of E. Then for any $x \in E$, there exists a unique element $z \in C$ such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x).$$

Such a mapping $\Pi_C : E \to C$ defined by $z = \Pi_C(x)$ is called the *generalized projection* of E onto C. If E is a Hilbert space, then Π_C is coincident with the metric projection denoted by P_C.

Lemma 2.10. [3] *Let E be a reflexive, strictly convex and smooth Banach space and C be a closed and convex subset of E. Let $x \in E$ and $z \in C$. Then the following statements hold:*

(i) $z = \Pi_C(x)$ *if and only if* $\langle y - z, Jx - Jz\rangle \leq 0, \ \forall y \in C$.
(ii) $\phi(y, \Pi_C(x)) + \phi(\Pi_C(x), x) \leq \phi(y, x), \ \forall y \in C$.

Lemma 2.11. [25] *Let C be a closed and convex subset of a smooth and uniformly convex Banach space E. Let $\{x_n\}$ be a sequence in E such that $\phi(p, x_{n+1}) \leq \phi(p, x_n)$ for all $p \in C$ and $n \geq 1$. Then the sequence $\{\Pi_C(x_n)\}$ converges strongly to some element $x^* \in C$.*

Let $B : E \to 2^{E^*}$ be a multi-valued mapping. The effective domain of B is denoted by $D(B) = \{x \in E : Bx \neq \emptyset\}$ and the range of B is also denoted by $R(B) = \bigcup\{Bx : x \in D(B)\}$. The set of zeros of B is denoted by $B^{-1}0 = \{x \in D(B) : 0 \in Bx\}$. A multi-valued mapping B is said to be *monotone* if

$$\langle x - y, u - v\rangle \geq 0, \quad \forall x, y \in D(B), \ u \in Bx \text{ and } v \in By.$$

A monotone operator B on E is said to be *maximal* if its graph $G(B) = \{(x, y) \in E \times E^* : x \in D(B), y \in Bx\}$ is not properly contained in the graph of any other monotone operator on E. In other words, the maximality of B is equivalent to $R(J + \lambda B) = E^*$ for all $\lambda > 0$ (see [5, Theorem 1.2]). It is known that if B is maximal monotone, then $B^{-1}0$ is closed and convex (see [39]).

For a maximal monotone operator B, we define the resolvent of B by $J^B_\lambda(x) = (J + \lambda B)^{-1}Jx$ for $x \in E$ and $\lambda > 0$. It is also known that $B^{-1}0 = F(J^B_\lambda)$.

Lemma 2.12. [5] *Let E be a reflexive Banach space. Let $A : E \to E^*$ be a monotone, hemicontinuous and bounded operator and $B : E \to 2^{E^*}$ be a maximal monotone operator. Then $A + B$ is maximal monotone.*

Lemma 2.13. ([48]) *Assume that $\{a_n\}$ is a sequence of nonnegative real sequences such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n, \ \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence of real sequences such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.14. ([30]) *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence* $\{\Gamma_{n_i}\}$ *of* $\{\Gamma_n\}$ *which satisfies* $\Gamma_{n_i} < \Gamma_{n_i+1}$ *for all* $\ell \in \mathbb{N}$. *Define the sequence* $\{\sigma(n)\}$ *of integers as follows:*

$$\sigma(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

for all $n \geq n_0$ *(for some* n_0 *large enough). Then* $\{\sigma(n)\}_{n\geq n_0}$ *is a non-decreasing sequence such that* $\lim_{n\to\infty} \sigma(n) = \infty$, *and it holds that*

$$\Gamma_{\sigma(n)} \leq \Gamma_{\sigma(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\sigma(n)+1}.$$

Lemma 2.15. ([42]) *Assume that* $\{\lambda_n\}$ *and* $\{\theta_n\}$ *are two nonnegative real sequences such that*

$$\lambda_{n+1} \leq \lambda_n + \theta_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \theta_n < \infty$, *then* $\lim\limits_{n\to\infty} \lambda_n$ *exists.*

3. Main results

In this section, we introduce two modified Tseng's splitting algorithms for solving the monotone inclusion problem in Banach spaces. In order to prove the convergence results of these algorithms, we need make the following assumptions:

Assumption 3.1. *(A1) The Banach space* E *is a real* 2*-uniformly convex and uniformly smooth.*
(A2) The mappings $A : E \to E^*$ *is monotone and L-Lipschitz continuous, and* $B : E \to 2^{E^*}$ *is maximal monotone.*
(A3) The solution set of the problem (1.1) *is nonempty, that is,* $(A + B)^{-1}0 \neq \emptyset$.

Algorithm 1: Tseng type splitting algorithm for monotone inclusion problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J_{\lambda_n}^B J^{-1}(Jx_n - \lambda_n Ax_n). \tag{3.1}$$

If $x_n = y_n$, then stop and x_n is a solution of the problem (1.1). Otherwise, go to **Step 2**.

Step 2. Compute

$$x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \tag{3.2}$$

where the sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $n := n + 1$ and go to **Step 1**.

Lemma 3.2. *Assume that Assumption 3.1 holds. Let $\{x_n\}$, $\{y_n\}$ and $\{\lambda_n\}$ be sequences generated by Algorithm 1. Then the following statements hold:*

(*i*) *If $x_n = y_n$ for all $n \in \mathbb{N}$, then $x_n \in (A + B)^{-1}0$.*
(*ii*) $\lim_{n\to\infty} \lambda_n = \lambda \in \left[\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \theta\right]$, *where $\theta = \sum_{n=1}^{\infty} \theta_n$. Moreover*

$$\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|x_n - y_n\|, \ \forall n \geq 1.$$

Proof. (*i*) If $x_n = y_n$, then $x_n = J_{\lambda_n}^B J^{-1}(Jx_n - \lambda_n Ax_n)$. It follows that $x_n = (J + \lambda_n B)^{-1} J \circ J^{-1}(J - \lambda_n A)x_n$, that is, $Jx_n - \lambda_n Ax_n \in Jx_n + \lambda_n Bx_n$, which implies that $0 \in (A + B)x_n$. Hence $x_n \in (A + B)^{-1}0$.

(*ii*) In the case $Ax_n - Ay_n \neq 0$, using the Lipschitz continuity of A, we have

$$\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|} \geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}.$$

From (3.3) and mathematical induction, we have the sequence $\{\lambda_n\}$ has upper bound $\lambda_1 + \theta$ and lower bound $\min\{\frac{\mu}{L}, \lambda_1\}$. From Lemma 2.15, we have $\lim_{n\to\infty} \lambda_n$ exists and we denote $\lambda = \lim_{n\to\infty} \lambda_n$. It is obvious that $\lambda \in \left[\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + \theta\right]$. By the definition of λ_n, we have

$$\lambda_{n+1} = \min\left\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} \leq \frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}.$$

This implies that

$$\|Ax_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|x_n - y_n\|, \ \forall n \geq 1. \tag{3.4}$$

□

Lemma 3.3. *Assume that Assumption 3.1 holds. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Hence*

$$\phi(z, x_{n+1}) \leq \phi(z, x_n) - \Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, x_n), \quad \forall z \in (A+B)^{-1}0, \tag{3.5}$$

where c and κ are constants in Lemma 2.5.

Proof. Let $z \in (A + B)^{-1}0$. From Lemma 2.5 (i) and (2.5), we have

$$\begin{aligned}
\phi(z, x_{n+1}) &= \phi(z, J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)))\\
&= V(z, Jy_n - \lambda_n(Ay_n - Ax_n))\\
&= \|z\|^2 - 2\langle z, Jy_n - \lambda_n(Ay_n - Ax_n)\rangle + \|Jy_n - \lambda_n(Ay_n - Ax_n)\|^2\\
&\leq \|z\|^2 - 2\langle z, Jy_n\rangle + 2\lambda_n\langle z, Ay_n - Ax_n\rangle + \|Jy_n\|^2 - 2\lambda_n\langle y_n, Ay_n - Ax_n\rangle + \kappa\|\lambda_n(Ay_n - Ax_n)\|^2\\
&= \|z\|^2 - 2\langle z, Jy_n\rangle + \|y_n\|^2 - 2\lambda_n\langle y_n - z, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2\\
&= \phi(z, y_n) - 2\lambda_n\langle y_n - z, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2\\
&= \phi(z, x_n) - \phi(y_n, x_n) + 2\langle y_n - z, Jy_n - Jx_n\rangle - 2\lambda_n\langle y_n - z, Ay_n - Ax_n\rangle + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2\\
&= \phi(z, x_n) - \phi(y_n, x_n) + \kappa\lambda_n^2\|Ay_n - Ax_n\|^2 - 2\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.6)
\end{aligned}$$

Combining (3.4) and (3.6), we have

$$\begin{aligned}
\phi(z, x_{n+1}) \leq\ & \phi(z, x_n) - \phi(y_n, x_n) + \kappa\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\|y_n - x_n\|^2\\
& -2\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.7)
\end{aligned}$$

By Lemma 2.8, we have

$$\begin{aligned}
\phi(z, x_{n+1}) \leq\ & \phi(z, x_n) - \Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, x_n)\\
& -2\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle. \qquad (3.8)
\end{aligned}$$

Now, we will show that

$$\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \geq 0.$$

From the definition of y_n, we note that $Jx_n - \lambda_n Ax_n \in Jy_n + \lambda_n By_n$. Since B is maximal monotone, there exists $v_n \in By_n$ such that $Jx_n - \lambda_n Ax_n = Jy_n + \lambda_n v_n$, we have

$$v_n = \frac{1}{\lambda_n}(Jx_n - Jy_n - \lambda_n Ax_n). \tag{3.9}$$

Since $0 \in (A + B)z$ and $Ay_n + v_n \in (A + B)y_n$, it follows from Lemma 2.12 that $A + B$ is maximal monotone. Hence

$$\langle y_n - z, Ay_n + v_n\rangle \geq 0. \tag{3.10}$$

Substituting (3.9) into (3.10), we have

$$\frac{1}{\lambda_n}\langle y_n - z, Jx_n - Jy_n - \lambda_n Ax_n + \lambda_n Ay_n\rangle \geq 0.$$

Hence

$$\langle y_n - z, Jx_n - Jy_n - \lambda_n(Ax_n - Ay_n)\rangle \geq 0. \tag{3.11}$$

Combining (3.8) and (3.11), thus this lemma is proved. □

Theorem 3.4. *Assume that Assumption 3.1 holds. Suppose, in addition, that J is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to an element in $(A+B)^{-1}0$.*

Proof. Since $\lim_{n\to\infty}\lambda_n$ exists and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$, it follows that $\lim_{n\to\infty}\left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \frac{\kappa\mu^2}{c} > 0$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \quad \forall n \geq n_0. \tag{3.12}$$

Combining (3.5) and (3.12), we have

$$\phi(z, x_{n+1}) \leq \phi(z, x_n), \quad \forall n \geq n_0.$$

This show that $\lim_{n\to\infty}\phi(z, x_n)$ exists and hence $\{\phi(z, x_n)\}$ is bounded. Applying Lemma 2.8, we also have $\{x_n\}$ is bounded. From (3.5), we have

$$\left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n, x_n) \leq \phi(z, x_n) - \phi(z, x_{n+1}). \tag{3.13}$$

Thus we have

$$\lim_{n\to\infty}\phi(y_n, x_n) = 0.$$

Applying Lemma 2.8, we also have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.14}$$

Since J is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{3.15}$$

Using the fact that A is Lipschitz continuous, we have

$$\lim_{n\to\infty}\|Ax_n - Ay_n\| = 0. \tag{3.16}$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in E$. From (3.14), we have $y_{n_k} \rightharpoonup x^*$. We will show that $x^* \in (A+B)^{-1}0$. Let $(v, w) \in G(A+B)$, we have $w - Av \in Bv$. From the definition of y_{n_k}, we note that

$$Jx_{n_k} - \lambda_{n_k} Ax_{n_k} \in Jy_{n_k} + \lambda_{n_k} By_{n_k},$$

which implies that

$$\frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k}) \in By_{n_k}.$$

By the maximal monotonicity of B, we have

$$\left\langle v - y_{n_k}, w - Av - \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k})\right\rangle \geq 0$$

and by the monotonicity of A, we have

$$\begin{aligned}
\langle v - y_{n_k}, w\rangle &\geq \left\langle v - y_{n_k}, Av + \frac{1}{\lambda_{n_k}}(Jx_{n_k} - Jy_{n_k} - \lambda_{n_k} Ax_{n_k})\right\rangle \\
&= \langle v - y_{n_k}, Av - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle \\
&= \langle v - y_{n_k}, Av - Ay_{n_k}\rangle + \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle \\
&\geq \langle v - y_{n_k}, Ay_{n_k} - Ax_{n_k}\rangle + \frac{1}{\lambda_{n_k}}\langle v - y_{n_k}, Jx_{n_k} - Jy_{n_k}\rangle.
\end{aligned}$$

Since $\lim_{k\to\infty} \lambda_{n_k} = \lambda > 0$ and $y_{n_k} \rightharpoonup x^*$, it follows from (3.15) and (3.16) that

$$\langle v - x^*, w\rangle \geq 0.$$

By the monotonicity of $A+B$, we get $0 \in (A+B)x^*$, that is, $x^* \in (A+B)^{-1}0$. Hence $x^* \in (A+B)^{-1}0$. Note that $(A+B)^{-1}0$ is closed and convex. Put $u_n = \Pi_{(A+B)^{-1}0}(x_n)$. It follows from Lemma 2.11 that there exists $x^* \in (A+B)^{-1}0$ such that $u_n \to x^*$. Finally, we show that $x_n \rightharpoonup x^*$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in (A+B)^{-1}0$. Then we have

$$\langle \hat{x} - u_{n_k}, Jx_{n_k} - Ju_{n_k}\rangle \leq 0$$

for all $k \in \mathbb{N}$. Since $u_n \to x^*$, $x_{n_k} \rightharpoonup \hat{x}$ and J is weakly sequentially continuous, we have

$$\langle \hat{x} - x^*, J\hat{x} - Jx^*\rangle \leq 0.$$

By the strict monotonicity of J, we obtain $x^* = \hat{x}$. In summary, we have shown that every subsequence of $\{x_n\}$ has a further subsequence which converges weakly to x^*. We conclude that $x_n \rightharpoonup x^* = \lim_{n\to\infty} \Pi_{(A+B)^{-1}0}(x_n)$. This completes the proof. □

Algorithm 2: Halpern-Tseng type splitting algorithm for monotone inclusion problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.
Step 1. Compute

$$y_n = J_{\lambda_n}^B J^{-1}(Jx_n - \lambda_n Ax_n). \tag{3.17}$$

If $x_n = y_n$, then stop and x_n is a solution of the problem (1.1). Otherwise, go to **Step 2**.
Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{3.18}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n), \tag{3.19}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{3.20}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 3.5. *Assume that Assumption 3.1 holds. If* $\{\alpha_n\} \subset (0, 1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *then the sequence* $\{x_n\}$ *generated by Algorithm 2 converges strongly to* $x^* \in (A + B)^{-1}0$.

Proof. We will show that $\{x_n\}$ is bounded. Let $z \in (A + B)^{-1}0$. Using the same arguments as in the proof of Lemma 3.3, we can show that

$$\phi(z, z_n) \leq \phi(z, x_n) - \left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n, x_n). \tag{3.21}$$

Since $\lim_{n\to\infty} \lambda_n$ exists and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$, it follows that $\lim_{n\to\infty}\left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \frac{\kappa\mu^2}{c} > 0$. Thus there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \quad \forall n \geq n_0. \tag{3.22}$$

Combining (3.21) and (3.22), we have

$$\phi(z, z_n) \leq \phi(z, x_n), \quad \forall n \geq n_0. \tag{3.23}$$

By (2.4), we have

$$\phi(z, x_{n+1}) = \phi(z, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n))$$

$$
\begin{aligned}
&\le \alpha_n\phi(z,u) + (1-\alpha_n)\phi(z,z_n) \\
&\le \alpha_n\phi(z,u) + (1-\alpha_n)\phi(z,x_n) \\
&\le \max\{\phi(z,u), \phi(z,x_n)\} \\
&\vdots \\
&\le \max\{\phi(z,u), \phi(z,x_{n_0})\}.
\end{aligned}
$$

This implies that $\{\phi(z,x_n)\}$ is bounded. Applying Lemma 2.8, we also have $\{x_n\}$ is bounded.

Let $x^* = \Pi_{(A+B)^{-1}0}(u)$. From (3.21), we have

$$
\begin{aligned}
\phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n)) \\
&\le \alpha_n\phi(x^*,u) + (1-\alpha_n)\phi(x^*,z_n) \\
&\le \alpha_n\phi(x^*,u) + (1-\alpha_n)\phi(x^*,x_n) - (1-\alpha_n)\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,x_n).
\end{aligned}
$$

This implies that

$$
(1-\alpha_n)\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,x_n) \le \phi(x^*,x_n) - \phi(x^*,x_{n+1}) + \alpha_n K, \tag{3.24}
$$

where $K = \sup_{n\in\mathbb{N}}\{|\phi(x^*,u) - \phi(x^*,x_n)|\}$.

Now, we will divide the rest of the proof into two cases.

Case 1. Suppose that there exists $N \in \mathbb{N}$ such that $\phi(x^*,x_{n+1}) \le \phi(x^*,x_n)$ for all $n \ge N$. Hence $\lim_{n\to\infty}\phi(x^*,x_n)$ exists. By our assumptions, we have from (3.24) that

$$
\lim_{n\to\infty}\phi(y_n,x_n) = 0
$$

and hence

$$
\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.25}
$$

Since J is norm-to-norm uniformly continuous on bounded subsets of E, we have

$$
\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{3.26}
$$

Using the fact that A is Lipschitz continuous, we have

$$
\lim_{n\to\infty}\|Ax_n - Ay_n\| = 0.
$$

Then from (3.18), we have

$$
\|Jz_n - Jy_n\| = \lambda_n\|Ax_n - Ay_n\| \to 0. \tag{3.27}
$$

Moreover from (3.26) and (3.27), we obtain

$$
\|Jx_{n+1} - Jx_n\| \le \|Jx_{n+1} - Jz_n\| + \|Jz_n - Jy_n\| + \|Jy_n - Jx_n\|
$$

$$\begin{aligned} &= \alpha_n\|Ju - Jz_n\| + \|Jz_n - Jy_n\| + \|Jy_n - Jx_n\| \\ &\to 0. \end{aligned}$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subset of E^*, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.28}$$

By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in E$ and

$$\limsup_{n\to\infty}\langle x_n - x^*, Ju - Jx^*\rangle = \lim_{k\to\infty}\langle x_{n_k} - x^*, Ju - Jx^*\rangle,$$

where $x^* = \Pi_{(A+B)^{-1}0}(u)$. By a similar argument to that of Theorem 3.4, we can show that $\hat{x} \in (A+B)^{-1}0$. Thus we have

$$\limsup_{n\to\infty}\langle x_n - x^*, Ju - Jx^*\rangle = \langle \hat{x} - x^*, Ju - Jx^*\rangle \leq 0.$$

From (3.28), we also have

$$\limsup_{n\to\infty}\langle x_{n+1} - x^*, Ju - Jx^*\rangle \leq 0. \tag{3.29}$$

Finally, we show that $x_n \to x^*$. From Lemma 2.9, we have

$$\begin{aligned} \phi(x^*, x_{n+1}) &= \phi(x^*, J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n)) \\ &= V(x^*, \alpha_n Ju + (1-\alpha_n)Jz_n) \\ &\leq V(x^*, \alpha_n Ju + (1-\alpha_n)Jz_n - \alpha_n(Ju - Jx^*)) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\ &= V(x^*, \alpha_n Jx^* + (1-\alpha_n)Jz_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\ &= \alpha_n\phi(x^*, x^*) + (1-\alpha_n)\phi(x^*, z_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle \\ &\leq (1-\alpha_n)\phi(x^*, x_n) + 2\alpha_n\langle x_{n+1} - x^*, Ju - Jx^*\rangle. \end{aligned} \tag{3.30}$$

This together with (3.29) and (3.30), so we can conclude by Lemma 2.13 that $\phi(x^*, x_n) \to 0$. Therefore $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\sigma : \mathbb{N} \to \mathbb{N}$ by

$$\sigma(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}$$

for all $n \geq n_0$ (for some n_0 large enough). From Lemma 2.14, we have $\sigma(n)$ is non-decreasing such that $\lim_{n\to\infty}\sigma(n) = \infty$ and the following inequalities hold for all $n \geq n_0$:

$$\Gamma_{\sigma(n)} < \Gamma_{\sigma(n)+1} \ \text{ and } \ \Gamma_n \leq \Gamma_{\sigma(n)+1}. \tag{3.31}$$

Put $\Gamma_n = \phi(x^*, x_n)$ for all $n \in \mathbb{N}$. As proved in the **Case 1**, we obtain

$$(1-\alpha_{\sigma(n)})\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_{\sigma(n)}^2}{\lambda_{\sigma(n)+1}^2}\Big)\phi(y_{\sigma(n)}, x_{\sigma(n)}) \leq \phi(x^*, x_{\sigma(n)}) - \phi(x^*, x_{\sigma(n)+1}) + \alpha_{\sigma(n)}K$$

$$\leq \quad \alpha_{\sigma(n)}K,$$

where $K = \sup_{n\in\mathbb{N}}\{|\phi(x^*, u) - \phi(x^*, x_{\sigma(n)})|\}$. By our assumptions, we have

$$\lim_{n\to\infty} \phi(y_{\sigma(n)}, x_{\sigma(n)}) = 0$$

and hence

$$\lim_{n\to\infty} \|x_{\sigma(n)} - y_{\sigma(n)}\| = 0.$$

Using the same arguments as in the proof of **Case 1**, we can show that

$$\lim_{n\to\infty} \|x_{\sigma(n)+1} - x_{\sigma(n)}\| = 0$$

and

$$\limsup_{n\to\infty} \langle x_{\sigma(n)+1} - x^*, Ju - Jx^*\rangle \leq 0.$$

From (3.30) and (3.31), we have

$$\begin{aligned} \phi(x^*, x_{\sigma(n)+1}) &\leq (1 - \alpha_{\sigma(n)})\phi(x^*, x_{\sigma(n)}) + \alpha_{\sigma(n)}\langle x_{\sigma(n)+1} - x^*, Ju - Jx^*\rangle \\ &\leq (1 - \alpha_{\sigma(n)})\phi(x^*, x_{\sigma(n)+1}) + \alpha_{\sigma(n)}\langle x_{\sigma(n)+1} - x^*, Ju - Jx^*\rangle. \end{aligned}$$

This implies that

$$\phi(x^*, x_n) \leq \phi(x^*, x_{\sigma(n)+1}) \leq \langle x_{\sigma(n)+1} - x^*, Ju - Jx^*\rangle.$$

Hence $\limsup_{n\to\infty} \phi(x^*, x_n) = 0$ and so $\lim_{n\to\infty} \phi(x^*, x_n) = 0$. Therefore $x_n \to x^*$. This completes the proof. □

4. Theoretical applications

4.1. The case of variational inequality problem

Let C be a nonempty, closed and convex subset of E. Let $A : C \to E^*$ be a mapping. The *variational inequality problem* is to find $x^* \in C$ such that

$$\langle y - x^*, Ax^*\rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

The set of solutions of the problem (4.1) is denoted by $VI(C, A)$. In particular, if A is a continuous and monotone mapping, then $VI(C, A)$ is closed and convex (see [7, 24]). Recall that the indicator function of C given by

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

It is known that i_C is proper convex, lower semicontinuous and convex function with its subdifferential ∂i_C is maximal monotone (see [35]). From [2], we know that

$$\partial i_C(v) = N_C(v) = \{u \in E^* : \langle y - v, u\rangle \leq 0, \ \forall y \in C\},$$

where N_C is the normal cone for C at a point v. Thus we can define the resolvent of ∂i_C for $\lambda > 0$ by

$$J_\lambda^{\partial i_C}(x) = (J + \lambda \partial i_C)^{-1} Jx, \quad \forall x \in E.$$

As shown in [40], for any $x \in E$ and $z \in C$, $z = J_\lambda^{\partial i_C}(x) \iff z = \Pi_C(x)$, where Π_C is the generalized projection from E onto C.

Lemma 4.1. [36] *Let C be a nonempty, closed convex subset of a Banach space E. Let $A : C \to E^*$ be a monotone and hemicontinuous operator and $T : E \to 2^{E^*}$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v) & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

If we set $B = \partial i_C$, then we obtain the following results regarding the problem (4.1).

Assumption 4.2. *(A1) The feasible set C is a nonempty, closed and convex subset of a real 2-uniformly convex and uniformly smooth Banach space E.*

(A2) The mapping $A : E \to E^*$ is monotone and L-Lipschitz continuous.

(A3) The solution set of the problem (4.1) *is nonempty, that is, $VI(C, A) \neq \emptyset$.*

Algorithm 3: Tseng type splitting algorithm for variational inequality problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $x_1 \in C$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n). \tag{4.2}$$

Step 2. Compute

$$x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \tag{4.3}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{4.4}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 4.3. *Assume that Assumption 4.2 holds. Suppose, in addition, that J is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 3 converges weakly to an element in $(A + B)^{-1}0$.*

Algorithm 4: Halpern-Tseng type splitting algorithm for variational inequality problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $u, x_1 \in C$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n). \tag{4.5}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)). \tag{4.6}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1-\alpha_n)Jz_n), \tag{4.7}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n + \theta_n\right\} & \text{if } Ax_n - Ay_n \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{4.8}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 4.4. *Assume that Assumption 4.2 holds. If* $\{\alpha_n\} \subset (0,1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *then the sequence* $\{x_n\}$ *generated by Algorithm 4 converges strongly to* $x^* \in VI(C, A)$.

4.2. The case of convex minimization problem

Let $f : E \to \mathbb{R}$ be a convex function and $g : E \to \mathbb{R}$ be a convex, lower semicontinuous and non-smooth function. We consider the following *convex minimization problem*:

$$\min_{x\in E} f(x) + g(x). \tag{4.9}$$

By Fermat's rule, we know that above problem is equivalent to the problem of finding $x \in E$ such that

$$0 \in \nabla f(x) + \partial g(x), \tag{4.10}$$

where ∇f is the gradient of f and ∂g is the subdifferential of g. In this situation, we assume that f is a convex and differentiable function with its gradient ∇f is L-Lipschitz continuous. Further, ∇f is cocoercive with a constant $1/L$ (see [31, Theorem 3.13]). This implies that ∇f is monotone and Lipschitz continuous. Moreover, ∂g is maximal monotone (see [35, Theorem A]). In this point of view, we set $A = \nabla f$ and $B = \partial g$, then we obtain the following results regarding the problem (4.9).

Assumption 4.5. *(A1) The Banach space E is real* 2*-uniformly convex and uniformly smooth Banach space.*

(A2) The functions $f : E \to \mathbb{R}$ *is convex and differentiable and its gradient* ∇f *is L-Lipschitz continuous and* $g : E \to \mathbb{R}$ *is convex and lower semicontinuous which* $f + g$ *attains a minimizer.*

Algorithm 5: Tseng type splitting algorithm for convex minimization problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J_{\lambda_n}^{\partial g} J^{-1}(Jx_n - \lambda_n \nabla f(x_n)). \tag{4.11}$$

Step 2. Compute

$$x_{n+1} = J^{-1}(Jy_n - \lambda_n(\nabla f(y_n) - \nabla f(x_n))), \tag{4.12}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|\nabla f(y_n) - \nabla f(x_n)\|}, \lambda_n + \theta_n\right\} & \text{if } \nabla f(y_n) - \nabla f(x_n) \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{4.13}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 4.6. *Assume that Assumption 4.5 holds. Suppose, in addition, that J is weakly sequentially continuous on E. Then the sequence $\{x_n\}$ generated by Algorithm 5 converges weakly to a minimizer of $f + g$.*

Algorithm 6: Halpern-Tseng type splitting algorithm for convex minimization problem

Step 0. Given $\lambda_1 > 0$ and $\mu \in (0, \sqrt{\frac{c}{\kappa}})$. Choose a nonnegative real sequence $\{\theta_n\}$ such that $\sum_{n=1}^{\infty} \theta_n < \infty$. Let $u, x_1 \in E$ be arbitrary. Set $n = 1$.

Step 1. Compute

$$y_n = J_{\lambda_n}^{\partial g} J^{-1}(Jx_n - \lambda_n \nabla f(x_n)). \tag{4.14}$$

Step 2. Compute

$$z_n = J^{-1}(Jy_n - \lambda_n(\nabla f(y_n) - \nabla f(x_n))). \tag{4.15}$$

Step 3. Compute

$$x_{n+1} = J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jz_n), \tag{4.16}$$

where the step sizes are adaptively updated as follows:

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu\|x_n - y_n\|}{\|\nabla f(y_n) - \nabla f(x_n)\|}, \lambda_n + \theta_n\right\} & \text{if } \nabla f(y_n) - \nabla f(x_n) \neq 0, \\ \lambda_n + \theta_n & \text{otherwise.} \end{cases} \tag{4.17}$$

Set $n := n + 1$ and go to **Step 1**.

Theorem 4.7. *Assume that Assumption 4.5 holds. If* $\{\alpha_n\} \subset (0,1)$ *with* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, *then the sequence* $\{x_n\}$ *generated by Algorithm 6 converges strongly to a minimizer of* $f + g$.

5. Numerical experiments

In this section, we provide some numerical experiments to illustrate the behaviour of our methods and compare them with some existing methods.

Example 5.1. *We consider the HpHard problem which is taken from [22]. Let* $A : \mathbb{R}^m \to \mathbb{R}^m$ *be an operator defined by* $Ax = Mx + q$ *with* $q \in \mathbb{R}^m$ *and*

$$M = NN^T + S + D,$$

where N *is an* $m \times m$ *matrix,* S *is an* $m \times m$ *skew-symmetric matrix and* D *is an* $m \times m$ *positive definite diagonal matrix. The feasible set is* $C = \mathbb{R}^+_m$. *It is clear that* A *is monotone and Lipschitz continuous with* $L = \|M\|$. *In this experiments, we compare our Algorithm 3 and Algorithm 4 with the extragradient method (EGM) proposed in [28] and the subgradient extragradient method (SEGM) proposed in [8]. The parameters are chosen as follows:*

- *Algorithm 3:* $\lambda_1 = 0.4/\|M\|$ *and* $\mu = 0.9$;
- *Algorithm 4:* $\lambda_1 = 0.4/\|M\|$, $\mu = 0.9$, $\alpha_n = \frac{1}{10000(n+2)}$ *and* $u = x_1$;
- *EGM and SEGM:* $\lambda = 0.4/\|M\|$.

All entries of N *and* S *are generated randomly in* $(-5, 5)$, *of* D *are in* $(0, 0.3)$, *of* q *uniformly generated from* $(-500, 0)$. *For every* m, *we have generated two random samples with different choices of* M *and* q. *We perform the numerical experiments with three different cases of* m $(m = 100, 500, 1000)$. *We take the starting point* $x_1 = (1, 1, 1, \dots, 1)^T \in \mathbb{R}^m$ *and use stopping criterion* $\|x_n - y_n\| \le \varepsilon = 10^{-6}$. *The numerical results are reported in Table 1.*

Table 1. Numerical results for Example 5.1.

m	Algorithm 3 ($\theta_n = 0$)		Algorithm 3 ($\theta_n = 100/n^{1.1}$)		Algorithm 4 ($\theta_n = 0$)		Algorithm 4 ($\theta_n = 100/n^{1.1}$)		EGM		SEGM	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
100	2454	0.02	1162	0.01	35112	1.31	25204	0.65	2454	0.03	2454	0.04
	1920	0.04	917	0.02	35072	1.48	25203	0.66	1920	0.03	1920	0.05
500	2275	0.95	1104	0.29	35010	7.28	25201	5.12	2275	0.50	2275	0.65
	2291	0.93	1107	0.43	34989	7.20	25198	5.06	2291	0.47	2291	0.59
1000	2027	8.08	996	4.25	34993	113.2	25200	78.2	2027	7.83	2027	7.96
	2017	7.80	987	3.87	35003	109.8	25200	78.0	2017	7.01	2017	7.16

Example 5.2. *We consider the problem* (4.1) *in* $L_2([0, 2\pi])$ *with the inner product* $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt$ *and the norm* $\|x\| = \left(\int_0^{2\pi} x^2(t)dt\right)^{1/2}$ *for all* $x, y \in L_2([0, 2\pi])$. *Let* $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ *be an operator defined by*

$$(Ax)(t) = \frac{1}{2}\max\{0, x(t)\}$$

for all $x \in L_2([0, 2\pi])$ and $t \in [0, 2\pi]$. It can be easily verified that A is monotone and Lipschitz continuous with $L = 1$ (see [50, 51]). The feasible set is $C = \{x \in L_2([0, 2\pi]) : \int_0^{2\pi}(t^2 + 1)x(t)dt \le 1\}$. Observe that $0 \in VI(C, A)$ and so $VI(C, A) \ne \emptyset$. In this numerical experiment, we take all parameters α_n, λ_n and μ are the same as in Example 5.1. We perform numerical experiments with three different cases of starting point x_1 and use stopping criterion $\|x_n - y_n\| \le \varepsilon = 10^{-3}$. The numerical results are reported in Table 2.

Table 2. Numerical results for Example 5.2.

x_1	Algorithm 3 ($\theta_n = 0$)		Algorithm 3 ($\theta_n = 0.001/(1.01)^n$)		Algorithm 4 ($\theta_n = 0$)		Algorithm 4 ($\theta_n = 0.001/(1.01)^n$)	
	iter.	time	iter.	time	iter.	time	iter.	time
$\frac{1}{100}\sin(t)$	7	9.9	7	8.9	7	9.8	7	10.1
$\frac{1}{3}t^2e^{-4t}$	5	0.4	5	0.3	5	0.3	5	0.3
$\frac{1}{70}(1 - t^2)$	6	3.2	6	2.5	6	2.7	6	2.7

Example 5.3. *Consider the minimization problem:*

$$\min_{x \in \mathbb{R}^3} \|x\|_1 + 2\|x\|_2^2 + (-1, 2, 5)x + 1,$$

where $x = (w_1, w_2, w_3)^T \in \mathbb{R}^3$. Let $f(x) = 2\|x\|_2^2 + (-1, 2, 5)x + 1$ and $g(x) = \|x\|_1$. Thus we have $\nabla f(x) = 4x + (-1, 2, 5)^T$. It is easy to check that f is a convex and differentiable function and its gradient ∇f is Lipschitz continuous with $L = 4$. Moreover, g is a convex and lower semicontinuous function but not differentiable on $\mathbb{R}^3$. From [21], we know that

$$\begin{aligned} J_\lambda^{\partial g}(x) &= (I + \lambda\partial g)^{-1}(x) \\ &= \left(\max\{|w_1| - \lambda, 0\}sgn(w_1), \max\{|w_2| - \lambda, 0\}sgn(w_2), \max\{|w_3| - \lambda, 0\}sgn(w_3)\right)^T \end{aligned}$$

for $\lambda > 0$. In this experiments, we compare our Algorithm 5 and Algorithm 6 with Algorithm (1.4) of Cholamjiak [12]. The parameters are chosen as follows:

- *Algorithm 5: $\lambda_1 = 0.1$ and $\mu = 0.9$;*
- *Algorithm 6: $\lambda_1 = 0.1$, $\mu = 0.9$, $\alpha_n = \frac{1}{10000(n+1)}$ and $u = x_1$;*
- *Algorithm (1.4): all parameters α_n, λ_n, δ_n, r_n and e_n are the same as Example 4.2 in [12], and $u = x_1$.*

We perform the numerical experiments with four different cases of starting point x_1 and use stopping criterion $\|x_{n+1} - x_n\| \le \varepsilon = 10^{-12}$. The numerical results are reported in Table 3.

Table 3. Numerical results for Example 5.3.

x_1	Algorithm 5 ($\theta_n = 0$)		Algorithm 5 ($\theta_n = 100/n^{1.1}$)		Algorithm 6 ($\theta_n = 0$)		Algorithm 6 ($\theta_n = 100/n^{1.1}$)		Algorithm (1.4)	
	iter.	time	iter.	time	iter.	time	iter.	time	iter.	time
$(1, 2, 4)^T$	101	0.003	284	0.003	27818	0.10	25263	0.08	263957	0.33
$(1, -7, 3)^T$	103	0.002	288	0.003	27809	0.12	25264	0.08	314417	0.38
$(-100, 100, 50)^T$	111	0.004	315	0.004	27802	0.11	25252	0.09	1313442	1.58
$(-1000, -5000, -800)^T$	127	0.005	356	0.01	27787	0.11	25241	0.07	8004199	9.4

Example 5.4. *In signal processing, compressed sensing can be modeled as the following under-determinated linear equation system:*

$$y = Dx + \varepsilon, \tag{5.1}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε, and $D : \mathbb{R}^N \to \mathbb{R}^M (M < N)$ is a bounded linear operator. It is known that to solve (5.1) *can be seen as solving the LASSO problem:*

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|Dx - y\|_2^2 + \lambda\|x\|_1, \tag{5.2}$$

where $\lambda > 0$. Following [19], we define $Ax := \nabla\Big(\frac{1}{2}\|Dx - y\|_2^2\Big) = D^T(Dx - y)$ and $Bx := \partial(\lambda\|x\|_1)$. It is known that A is $\|D\|^2$-Lipschitz continuous and monotone. Moreover, B is maximal monotone (see [35]).

In this experiment, the sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m nonzero elements. The matrix $D \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and one invariance. The observation y is generated by white Gaussian noise with signal-to-noise ratio (SNR)=40. The restoration accuracy is measured by the mean squared error (MSE) as follows:

$$E_n = \frac{1}{N}\|x_n - x\|_2^2 < 10^{-4}, \tag{5.3}$$

where x_n is an estimated signal of x.

We compare our proposed Algorithm 1 and Algorithm 2 with the forward-backward splitting algorithm (FBSA) (1.2), the Tseng's splitting algorithm (TSA) (1.5) and the contraction forward-backward splitting algorithm (CFBSA) proposed in ([43, Algorithm 3.1]). The parameters are chosen as follows:

- Algorithm 1: $\theta_n = 0$, $\lambda_1 = 0.0013$ and $\mu = 0.5$;
- Algorithm 2: $\theta_n = 0$, $\lambda_1 = 0.0013$, $\mu = 0.5$, $\alpha_n = \frac{1}{200n+5}$ and $u = (1, 1, \ldots, 1)^T$;
- CFBSA: $\alpha_n = \frac{1}{200n+5}$, $\mu = 0.5$, $\delta = 0.5$, $l = 0.5$, $\gamma = 0.45$ and $f(x) = \frac{x}{5}$;
- TSA: $\lambda_n = \frac{0.2}{\|D\|^2}$;
- FBSA: $\lambda = 2 \times 10^{-5}$.

The starting points x_1 of all methods are randomly chosen in $\mathbb{R}^N$. We perform the numerical test with the following three cases:

Case 1: $N = 512$, $M = 256$ and $m = 20$;
Case 2: $N = 1024$, $M = 512$ and $m = 30$;
Case 3: $N = 2048$, $M = 1024$ and $m = 60$;
The numerical results for all test are reported in Figures 1–6.

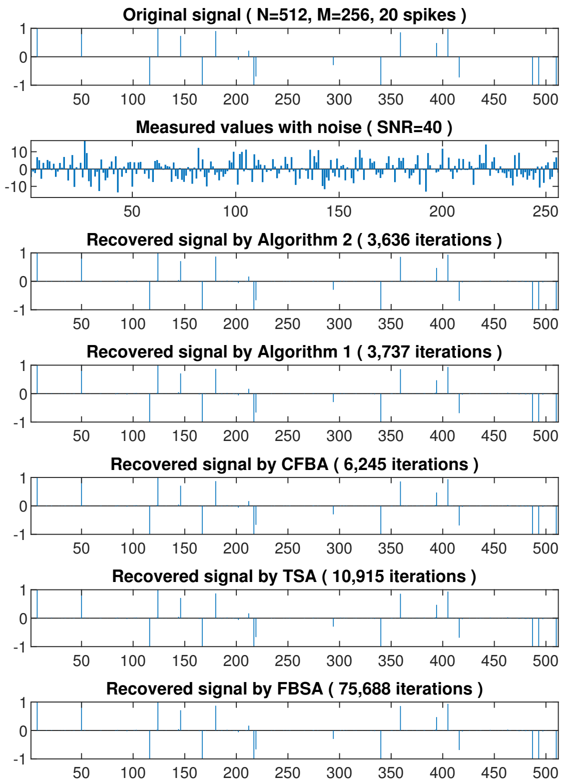

Figure 1. Comparison of recovered signal by using different algorithms in Case 1.

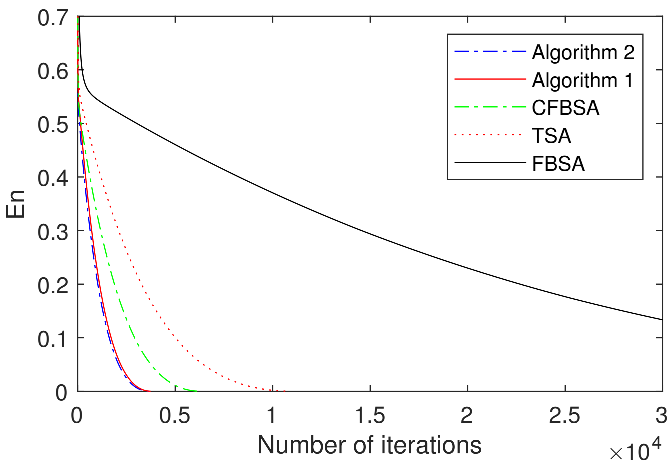

Figure 2. The plotting of MSE versus number of iterations in Case 1.

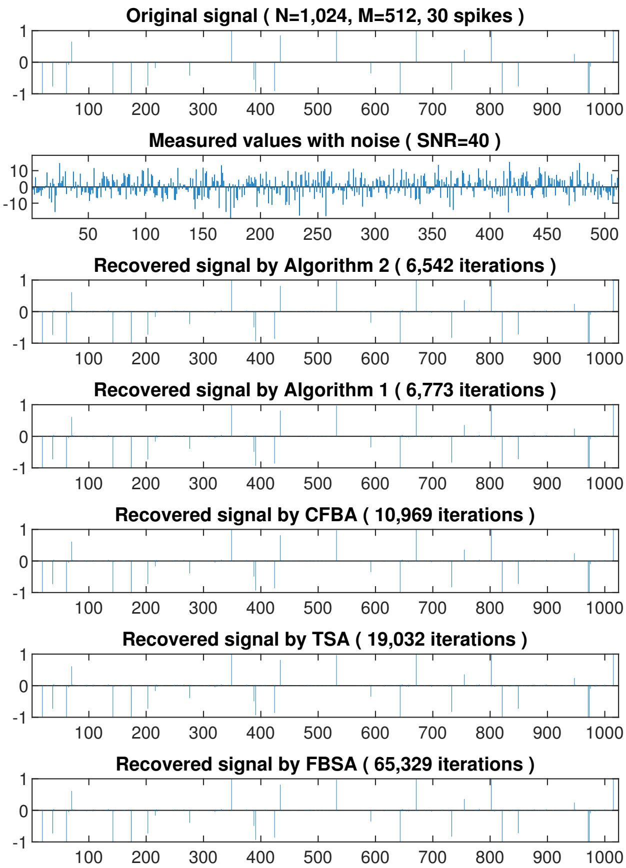

Figure 3. Comparison of recovered signal by using different algorithms in Case 2.

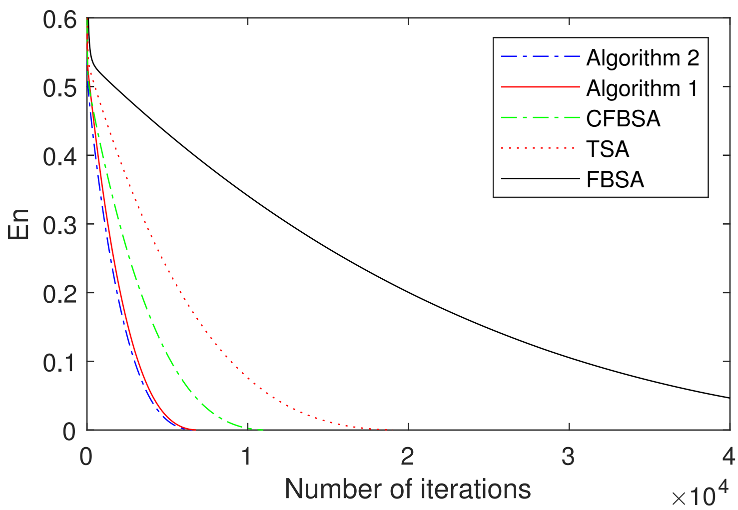

Figure 4. The plotting of MSE versus number of iterations in Case 2.

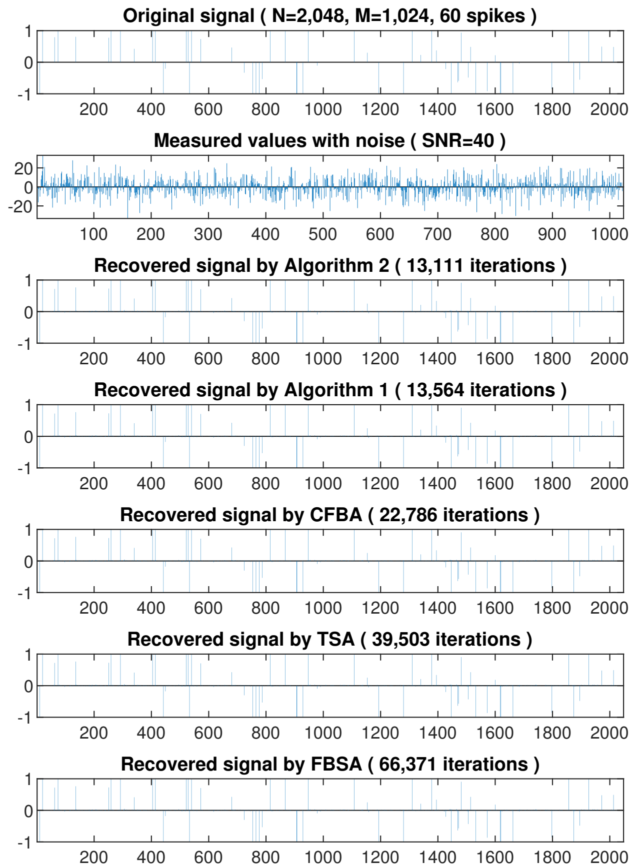

Figure 5. Comparison of recovered signal by using different algorithms in Case 3.

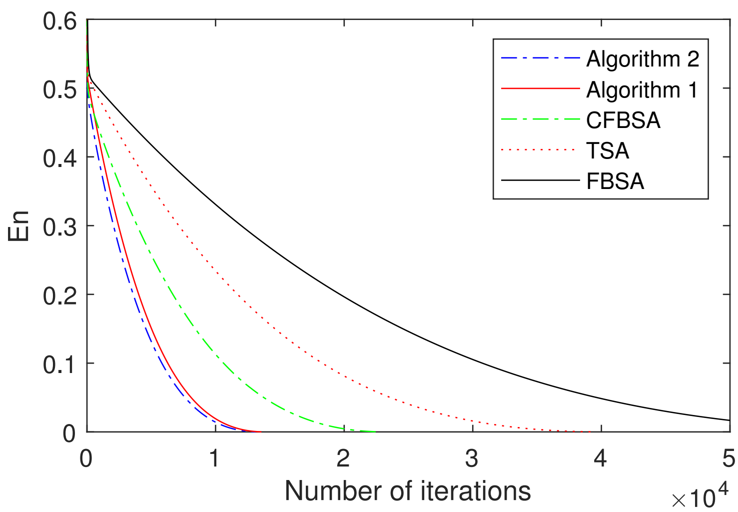

Figure 6. The plotting of MSE versus number of iterations in Case 3.

6. Conclusions

In this paper, we propose Tseng's splitting algorithms with non-monotone adaptive step sizes for finding zeros of the sum of two monotone operators in the framework of Banach space. Under some suitable conditions, we prove the weak and strong convergence results of the algorithms without the knowledge of the Lipschitz constant of the mapping. Some applications related to the obtained results are presented. Finally, several numerical experiments are performed to illustrate the convergence of our algorithms and compared with many known methods.

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Conflict of interest

The authors declare no conflict of interest.

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