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Research article

Egoroff's theorems for random sets on non-additive measure spaces

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Abstract: We present four versions of Egoroff's theorems for measurable closed-valued multifunctions on non-additive measure spaces. The conditions provided for each of these four versions are not only sufficient, but also necessary. In our discussions the continuity of non-additive measures is not required. The previous related results are improved and generalized.

Keywords: random set; non-additive measure; the Egoroff theorem
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1. Introduction

The Egoroff theorem, the well-known convergence theorem in real analysis theory, describes the relationship between the convergence almost everywhere and almost uniform for real-valued measurable functions sequence ([3]). In [16] the convergence of a sequence of measurable closed-valued multifunctions (i.e., random sets) on σ-additive measure spaces (in particular, probability measure spaces) were discussed and a version of Egoroff's theorem for closed-valued measurable multifunctions was shown.

This important convergence theorem has been widely extended in non-additive measure theory (see [4, 5, 9–12, 14, 19]. The Egoroff theorem for closed-valued measurable multifunctions, which was established in [16], was generalized from σ-additive measures space to non-additive measure spaces (see [8, 11, 14]). However, in these discussion the continuity and autocontinuity of non-additive measures are required.

In this short paper we further investigate the (pseudo-)convergence almost everywhere and (pseudo-)almost uniform of a sequence of random sets on non-additive measure spaces. By means of the *condition* [**E**] and *condition* [$\overline{\mathbf{E}}$] (see [9]) of non-additive measures, we show four versions of the Egoroff theorem — one standard version and three pseudo-versions — for random sets sequence on general non-additive measure spaces. The necessary and sufficient conditions are respectively presented for these results to remain valid for non-additive measures. The non-additive measures we considered are not necessarily continuous, and thus the previous related results ([11]) are improved and generalized.

2. Preliminaries

Let $(\Omega, \mathcal{A})$ be a measurable space, i.e., Ω is a nonempty set and $\mathcal{A}$ is a σ-algebra of subsets of Ω. We consider $\mathbb{R}^d$, the d-dimensional Euclidean (linear) space with Euclidean norm $\|\cdot\|$. Let $\mathcal{O}, \mathcal{F}$ and $\mathcal{K}$ denote the classes of all open, closed and compact sets in $\mathbb{R}^d$, respectively, and $\mathcal{B}(\mathbb{R}^d)$ denote the Borel σ-algebra on $\mathbb{R}^d$, i.e., it is the smallest σ-algebra containing $\mathcal{O}$. For $x \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$, the distance from the point x to the subset E is defined by $\rho(x, E) = \inf\{\|x - y\| : y \in E\}$. Let $B(x, \epsilon)$ and $\overline{B}(x, \epsilon)$ denote the open ball and closed ball of radius ϵ and center $x \in \mathbb{R}^d$, respectively, i.e., $B(x, \epsilon) = \{y \in \mathbb{R}^d : \|x - y\| < \epsilon\}$ and $\overline{B}(x, \epsilon) = \{y \in \mathbb{R}^d : \|x - y\| \leq \epsilon\}$.

2.1. Random sets

We recall the basic definitions dealing with set-valued maps [1], also called multifunctions, or multi-valued mapping. Let us consider a set-valued map $\Gamma : \Omega \to \mathcal{P}(\mathbb{R}^d)$ (the power set of $\mathbb{R}^d$); its *effective domain* is $\mathrm{dom}(\Gamma) = \{\omega \in \Omega : \Gamma(\omega) \neq \emptyset\}$. We denote $\Gamma^{-1}(F) \triangleq \{\omega \in \Omega : \Gamma(\omega) \cap F \neq \emptyset\}$, where $F \in \mathcal{P}(\mathbb{R}^d)$. The set-valued mapping Γ is said to be closed-valued, if its values are closed subsets of $\mathbb{R}^d$, i.e., $\Gamma : \Omega \to \mathcal{F}$. A closed-valued mapping Γ is measurable (with respect to $\mathcal{A}$), if for all closed subset F of $\mathbb{R}^d$, $\Gamma^{-1}(F) \in \mathcal{A}$ (see [1, 16]).

A measurable closed-valued mapping Γ is called a *random set (with respect to $\mathcal{A}$)*. Let $\mathcal{R}[\Omega]$ denote the class of all random sets defined on Ω (with respect to $\mathcal{A}$).

2.2. Convergence of sequence of closed sets

Definition 2.1. *([1, 16]) Let $C \in \mathcal{F}, (C_n)_{n\in\mathbb{N}} \subset \mathcal{F}$. If $\limsup_{n\to\infty} C_n = \liminf_{n\to\infty} C_n = C$, then C is called to be the set limit of the sequence $(C_n)_{n\in\mathbb{N}}$, denoted by $\lim_{n\to\infty} C_n = C$, where $\limsup_{n\to\infty} C_n \triangleq \{x \in \mathbb{R}^d : \liminf_{n\to\infty} \rho(x, C_n) = 0\}$ and $\liminf_{n\to\infty} C_n \triangleq \{x \in \mathbb{R}^d : \lim_{n\to\infty} \rho(x, C_n) = 0\}$.*

For given $\epsilon > 0$ and $A \subset \mathbb{R}^d$, let ϵA denote an open ϵ-neighborhood of the set A defined as follows: if A is nonempty then $\epsilon A = \{x \in \mathbb{R}^d : \rho(x, A) < \epsilon\}$ and $\epsilon\emptyset = \mathbb{R}^d \setminus B(0, 1/\epsilon)$.

Proposition 2.1. *([16]) Let $C \in \mathcal{F}, (C_n)_{n\in\mathbb{N}} \subset \mathcal{F}$. Then the following four conditions are equivalent:*

(i) $\lim_{n\to\infty} C_n = C$;

(ii) for all $\epsilon > 0$, $\lim_{n\to\infty}[(C \setminus \epsilon C_n) \cup (C_n \setminus \epsilon C)] = \emptyset$;

(iii) for each $\epsilon > 0$ *and each* $K \in \mathcal{K}$*, there exists* $n(\epsilon, K)$ *such that for all* $n \geq n(\epsilon, K)$

$$[(C \setminus \epsilon C_n) \cup (C_n \setminus \epsilon C)] \cap K = \emptyset;$$

(iv) for each $\epsilon > 0, r > 0$ *and* $x \in \mathbb{R}^d$*, there corresponds* $n(\epsilon, r, x)$ *such that for all* $n \geq n(\epsilon, r, x)$

$$[(C \setminus \epsilon C_n) \cup (C_n \setminus \epsilon C)] \cap \overline{B}(x, r) = \emptyset.$$

2.3. Non-additive measures

A set function $\lambda : \mathcal{A} \to [0, +\infty]$ is called a *non-additive measure* on $(\Omega, \mathcal{A})$, if it satisfies the following two conditions:

(1) $\lambda(\emptyset) = 0$; (vanishing at $\emptyset$)

(2) For any $E_1, E_2 \in \mathcal{A}$, $E_1 \subset E_2$ implies $\lambda(E_1) \leq \lambda(E_2)$. (monotonicity)

A non-additive measure is also known as "capacity", "fuzzy measure", "monotone measure" or "nonlinear probability", etc. For more information concerning non-additive measures, we recommend [2, 13, 15, 17, 19].

Let $\mathcal{M}$ denote the set of all non-additive measures defined on $(\Omega, \mathcal{A})$.

Let $\lambda \in \mathcal{M}$. λ is said to be *continuous from below* [3], if $\lim_{n\to\infty} \lambda(L_n) = \lambda(L)$ whenever $L_n \nearrow L$; *continuous from above* [3], if $\lim_{n\to\infty} \lambda(U_n) = \lambda(U)$ whenever $U_n \searrow U$ and there exists n_0 with $\lambda(U_{n_0}) < +\infty$; *continuous*, if λ is continuous both from below and from above; *strongly order continuous* [7, 9], if $\lim_{n\to\infty} \lambda(A_n) = 0$ whenever $A_n \searrow A$ and $\lambda(A) = 0$.

For a non-additive measure $\lambda \in \mathcal{M}$ with $\lambda(\Omega) < +\infty$, the non-additive measure $\overline{\lambda}$, defined by

$$\overline{\lambda}(A) = \lambda(\Omega) - \lambda(\Omega \setminus A), \quad A \in \mathcal{A},$$

is called the conjugate of λ (the conjugate $\overline{\lambda}$ is also called the dual of λ).

3. Egoroff's theorems for random sets on monotone measure spaces

Given $\lambda \in \mathcal{M}$. Let $\Gamma \in \mathcal{R}[\Omega]$, $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathcal{R}[\Omega]$, $A \in \mathcal{A}$. The following concepts come from [16] (see also [11]). We say that

(1a) $(\Gamma_n)_{n\in\mathbb{N}}$ *converges to* Γ *almost everywhere on* Ω (with respect to λ), if there exists $E_0 \in \mathcal{A}$, such that $\lambda(E_0) = 0$ and for every $\omega \in \Omega \setminus E_0$, $\lim_{n\to\infty} \Gamma_n(\omega) = \Gamma(\omega)$ (in the sense of Definition 2.1, the same below), write $\Gamma_n \xrightarrow{a.e.} \Gamma[\lambda]$;

(1b) $(\Gamma_n)_{n\in\mathbb{N}}$ *converges to* Γ *pseudo-almost everywhere on* Ω (with respect to λ), if there exists $F_0 \in \mathcal{A}$, such that $\lambda(\Omega \setminus F_0) = \lambda(\Omega)$ and for every $\omega \in \Omega \setminus F_0$, $\lim_{n\to\infty} \Gamma_n(\omega) = \Gamma(\omega)$, write $\Gamma_n \xrightarrow{p.a.e.} \Gamma[\lambda]$;

(2) $(\Gamma_n)_{n\in\mathbb{N}}$ *converges uniformly to* Γ *on* A, denoted by $\Gamma_n \xrightarrow{unif.} \Gamma$ on A, if for any $\epsilon > 0$ and any compact subset K of $\mathbb{R}^d$, there exists some positive integer $n(\epsilon, K)$, such that $(\triangle_{\epsilon n}^{-1}(K)) \cap A = \emptyset$ whenever $n \geq n(\epsilon, K)$, where

$$\triangle_{\epsilon n}^{-1}(K) \triangleq \Big\{\omega \in \Omega : [(\Gamma_n \setminus \epsilon\Gamma) \cup (\Gamma \setminus \epsilon\Gamma_n)](\omega) \cap K \neq \emptyset\Big\}; \tag{3.1}$$

(2a) $(\Gamma_n)_{n\in\mathbb{N}}$ *converges almost uniformly to* Γ *on* Ω (with respect to λ), denoted by $\Gamma_n \xrightarrow{a.u.} \Gamma[\lambda]$, if for any $\delta > 0$, there exists $A_\delta \subset \mathcal{A}$ such that $\Gamma_n \xrightarrow{unif.} \Gamma$ on A_δ and $\mu(\Omega \setminus A_\delta) < \delta$;

(2b) $(\Gamma_n)_{n\in\mathbb{N}}$ *converges pseudo-almost uniformly to* Γ *on* Ω (with respect to λ), denoted by $\Gamma_n \xrightarrow{p.a.u.} \Gamma[\lambda]$, if there is $\{F_m\}_{m\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{m\to\infty} \lambda(\Omega \setminus F_m) = \lambda(\Omega)$, and $\Gamma_n \xrightarrow{uni.} \Gamma$ on $\Omega \setminus F_m (m = 1, 2, \ldots)$.

We recall the *condition* [**E**] of non-additive measures, which plays an important role in generalizing the Egoroff theorem from classical measure theory to non-additive measure theory ([6, 9]).

Definition 3.1. *Let $\lambda \in \mathcal{M}$. If for every double sequence $(P_n^{(m)})_{(m,n)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}$ satisfying the condition: for any $m = 1, 2, \ldots$, $P_n^{(m)} \searrow P^{(m)}$ $(n \to \infty)$ with $\lambda(\bigcup_{m=1}^{+\infty} P^{(m)}) = 0$, there are increasing sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ of natural numbers, such that*

$$\lim_{k\to+\infty} \lambda\Big(\bigcup_{i=k}^{+\infty} P_{n_i}^{(m_i)}\Big) = 0 \ \ \Big(\text{resp. } \lim_{k\to+\infty} \lambda\Big(\Omega \setminus \bigcup_{i=k}^{+\infty} P_{n_i}^{(m_i)}\Big) = \lambda(\Omega)\Big), \tag{3.2}$$

then we say that λ fulfils condition [**E**] *(resp. condition* $[\overline{\mathbf{E}}]$*).*

Proposition 3.1. *([6, 12]) Let $\lambda \in \mathcal{M}$.*

(1) If λ is finite (i.e., $\lambda(\Omega) < \infty$) and continuous, then it fulfils condition [**E**].

(2) If λ fulfils condition [**E**]*, then it is strongly order continuous (i.e., $\lim_{n\to+\infty} \lambda(A_n) = 0$ whenever $A_n \searrow A$ with $\lambda(A) = 0$).*

3.1. The standard version of Egoroff's theorem for random sets

In [6] (see also [9]) it was shown that Egoroff's theorem for real-valued measurable functions holds in the case of monotone measures if and only if the monotone measures fulfill condition [**E**] (or *Egoroff condition*, see [12]). Now we show a version of the Egoroff theorem for random sets sequence on non-additive measure spaces. It only concerns convergence *a.e.* and convergence *a.u.*, and we refer to it as the standard-form of Egoroff's theorem (for random sets on non-additive measure spaces).

Theorem 3.1. *Let $\lambda \in \mathcal{M}$. Then the following are equivalent:*

(i) λ fulfils condition [**E**] *(or Egoroff condition);*

(ii) for all $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathcal{R}[\Omega]$ and all $\Gamma \in \mathcal{R}[\Omega]$, we have

$$\Gamma_n \xrightarrow{a.e.} \Gamma[\lambda] \implies \Gamma_n \xrightarrow{a.u.} \Gamma[\lambda]. \tag{3.3}$$

Proof. $(i) \Rightarrow (ii)$ Let $\Omega_0 = \{\omega \in \Omega : \lim_{n\to\infty} \Gamma_n(\omega) \neq \Gamma(\omega)\}$. Since $\Gamma_n \xrightarrow{a.e.} \Gamma[\lambda]$, we have $\lambda(\Omega_0) = 0$, and Γ_n converges to Γ everywhere on $\Omega \setminus \Omega_0$.

Denote

$$W_n(\epsilon, r, x) \triangleq \Big\{\omega \in \Omega : [(\Gamma_n \setminus \epsilon\Gamma) \cup (\Gamma \setminus \epsilon\Gamma_n)](\omega) \cap \overline{B}(x, r) \neq \emptyset\Big\}.$$

For $m, k = 1, 2, \ldots$, let $E_m^{(k)} = \bigcup_{n=m}^{\infty} W_m(\frac{1}{k}, k, 0)$, then $E_m^{(k)}$ is decreasing in m for each fixed k.

Denote $E^{(k)} = \bigcap_{m=1}^{\infty} E_m^{(k)}$, $k = 1, 2, \ldots$. From Proposition 2.1, it is not difficult to verify that $\bigcup_{k=1}^{\infty} E^{(k)} = \Omega_0$. Therefore $\lambda(\bigcup_{k=1}^{\infty} E^{(k)}) = 0$. Thus the double sequence $(E_m^{(k)})_{(m,k)\in\mathbb{N}\times\mathbb{N}} \subset \mathcal{A}$ satisfies the condition: for any fixed $k = 1, 2, \ldots$, as $m \to \infty$,

$$E_m^{(k)} \searrow E^{(k)} \ \text{ and } \ \lambda\Big(\bigcup_{k=1}^{+\infty} E^{(k)}\Big) = 0.$$

By using the condition [**E**], we get increasing sequences $(m_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$, such that

$$\lim_{j\to+\infty} \lambda\Big(\bigcup_{i=j}^{+\infty} E_{m_i}^{(k_i)}\Big) = 0.$$

For any $\delta > 0$, we take j_0 such that

$$\lambda\Big(\bigcup_{i=j_0}^{+\infty} E_{m_i}^{(k_i)}\Big) < \delta.$$

Let $A_\delta = \Omega - \bigcup_{i=j_0}^{+\infty} E_{m_i}^{(k_i)}$, then $A_\delta \in \mathcal{A}$ and

$$\lambda(\Omega - A_\delta) = \lambda\Big(\bigcup_{i=j_0}^{+\infty} E_{m_i}^{(k_i)}\Big) < \delta.$$

In the following we prove that $(\Gamma_n)_{n\in\mathbb{N}}$ converges to Γ on A_δ uniformly.

Noting that

$$A_\delta = \bigcap_{i=j_0}^{\infty}\bigcap_{n=m_i}^{\infty}\Big\{\omega \in \Omega : \Big[(\Gamma_n \setminus \frac{1}{k_i}\Gamma) \cup (\Gamma \setminus \frac{1}{k_i}\Gamma_n)\Big](\omega) \cap \overline{B}(0, k_i) = \emptyset\Big\},$$

then for any $i \ge j_0$, we have

$$A_\delta \subset \bigcap_{n=m_i}^{\infty}\Big\{\omega \in \Omega : \Big[(\Gamma_n \setminus \frac{1}{k_i}\Gamma) \cup (\Gamma \setminus \frac{1}{k_i}\Gamma_n)\Big](\omega) \cap \overline{B}(0, k_i) = \emptyset\Big\},$$

i.e.,

$$A_\delta \subset \Big\{\omega \in \Omega : \Big[(\Gamma_n \setminus \frac{1}{k_i}\Gamma) \cup (\Gamma \setminus \frac{1}{k_i}\Gamma_n)\Big](\omega) \cap \overline{B}(0, k_i) = \emptyset\Big\}$$

whenever $n \ge m_i$. Therefore

$$\Big\{\omega \in \Omega : \Big[(\Gamma_n \setminus \frac{1}{k_i}\Gamma) \cup (\Gamma \setminus \frac{1}{k_i}\Gamma_n)\Big](\omega) \cap \overline{B}(0, k_i) \neq \emptyset\Big\} \cap A_\delta = \emptyset$$

whenever $n \ge m_i$, i.e., $W_m(\frac{1}{k}, k, 0) \cap A_\delta = \emptyset$ whenever $n \ge m_i$.

On the other hand, for any $\epsilon > 0$ and any compact subset K of $\mathbb{R}^d$, we take i_0 such that $i_0 > j_0$, $1/k_{i_0} < \epsilon$ and $K \subset \overline{B}(0, k_{i_0})$. Take $n(\epsilon, K) = m_{i_0}$. Then as $n \ge n(\epsilon, K)$,

$$\Big\{\omega \in \Omega : \Big[(\Gamma_n \setminus \frac{1}{k_{i_0}}\Gamma) \cup (\Gamma \setminus \frac{1}{k_{i_0}}\Gamma_n)\Big](\omega) \cap \overline{B}(0, k_{i_0}) \neq \emptyset\Big\} \cap A_\delta = \emptyset.$$

Noting that

$$\Big[\Big(\Gamma_n \setminus \frac{1}{k_{i_0}}\Gamma\Big) \cup \Big(\Gamma \setminus \frac{1}{k_{i_0}}\Gamma_n\Big)\Big](\omega) \cap \overline{B}(0, k_{i_0}) \supset \Big[\Big(\Gamma_n \setminus \epsilon\Gamma\Big) \cup \Big(\Gamma \setminus \epsilon\Gamma_n\Big)\Big](\omega) \cap K,$$

we have

$$\Big\{\omega \in \Omega : \Big[(\Gamma_n \setminus \epsilon\Gamma) \cup (\Gamma \setminus \epsilon\Gamma_n)\Big](\omega) \cap K \neq \emptyset\Big\} \cap A_\delta = \emptyset$$

whenever $n \ge n(\epsilon, K)$. That is, $(\triangle_{\epsilon n}^{-1}(K)) \cap A_\delta = \emptyset$ whenever $n \ge n(\epsilon, K)$. This shows $\Gamma_n \xrightarrow{a.u} \Gamma[\lambda]$.

$(ii) \Rightarrow (i)$ Considering the singleton-valued functions, it is similar to the proof of Theorem 1 in [6]. □

As a direct consequence of Theorem 3.1 and Proposition 3.1(1), we obtain the following result. It is a version of Egoroff's theorem of sequence of random sets for continuous non-additive measures.

Corollary 3.1. *Let λ be a continuous non-additive measure on $(\Omega, \mathcal{A})$ and $\lambda(\Omega) < \infty$. Then, for all $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathcal{R}[\Omega]$ and all $\Gamma \in \mathcal{R}[\Omega]$,*

$$\Gamma_n \xrightarrow{a.e.} \Gamma[\lambda] \implies \Gamma_n \xrightarrow{a.u.} \Gamma[\lambda]. \tag{3.4}$$

Remark 3.1. A non-additive measure λ is called *null-additive* [13, 19], if $\lambda(P \cup Q) = \lambda(P)$ whenever $P, Q \in \mathcal{A}$ and $\lambda(Q) = 0$. In [11] the above Corollary 3.1 was obtained under the assumption of null-additivity of non-additive measures. In fact, the condition of null-additivity can be abandoned.

A non-additive measure λ is called to have *property* (S), if for any $(A_n)_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty}\lambda(A_n) = 0$, there exists a subsequence $(A_{n_i})_{i\in\mathbb{N}}$ of $(A_n)_{n\in\mathbb{N}}$ such that $\lambda\left(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_{n_i}\right) = 0$ (see [13, 18]). If λ is strong order continuous and has property (S), then λ fulfils the condition [**E**] (see [6, 7]). Thus, as a special result of Theorem 3.1, we obtain following corollary:

Corollary 3.2. *(Li et al. [8, Theorem 1]) Let $\lambda \in \mathcal{M}$. If λ is strongly order continuous and has property (S), then the formula (3.3) holds for all $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathcal{R}[\Omega]$ and all $\Gamma \in \mathcal{R}[\Omega]$.*

From Proposition 3.1(2) we get a necessary condition of the validity of formula (3.3).

Corollary 3.3. *Let $\lambda \in \mathcal{M}$. If for all $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathcal{R}[\Omega]$ and all $\Gamma \in \mathcal{R}[\Omega]$,*

$$\Gamma_n \xrightarrow{a.e.} \Gamma[\lambda] \implies \Gamma_n \xrightarrow{a.u.} \Gamma[\lambda], \tag{3.5}$$

then λ is strongly order continuous.

3.2. The pseudo-versions of Egoroff's theorems for random sets

Since non-additive measures lose additivity, the two concepts of almost everywhere convergence and almost uniform convergence have so-called "pseudo-" variants, respectively: "pseudo-almost everywhere convergence" and "pseudo-almost uniform convergence" ([19]). Thus, Egoroff's theorem is divided into four different forms in the case of non-additive measures (see [9, 19]). As we have stated, the above Theorem 3.1, which only concerns convergence *a.e.* and convergence *a.u.*, is referred to as the standard-version of Egoroff's theorem. In the following we show other three pseudo-versions of Egoroff's theorem for random sets on finite non-additive measure spaces. They were established in the context of (pseudo-)convergence.

Theorem 3.2. *Let $\lambda \in \mathcal{M}$ and $\lambda(\Omega) < \infty$. Then,*
(1) for all $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathcal{R}[\Omega]$ and all $\Gamma \in \mathcal{R}[\Omega]$,

$$\Gamma_n \xrightarrow{p.a.e.} \Gamma[\lambda] \implies \Gamma_n \xrightarrow{p.a.u.} \Gamma[\lambda] \tag{3.6}$$

if and only if $\overline{\lambda}$ fulfils condition [**E**].
(2) for all $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathcal{R}[\Omega]$ and all $\Gamma \in \mathcal{R}[\Omega]$,

$$\Gamma_n \xrightarrow{a.e.} \Gamma[\lambda] \implies \Gamma_n \xrightarrow{p.a.u.} \Gamma[\lambda] \tag{3.7}$$

if and only if λ fulfils condition $[\overline{\mathbf{E}}]$.

(3) for all $(\Gamma_n)_{n\in\mathbb{N}} \subset \mathcal{R}[\Omega]$ *and all* $\Gamma \in \mathcal{R}[\Omega]$,

$$\Gamma_n \stackrel{p.a.e.}{\longrightarrow} \Gamma[\lambda] \implies \Gamma_n \stackrel{a.u.}{\longrightarrow} \Gamma[\lambda] \tag{3.8}$$

if and only if $\overline{\lambda}$ *fulfils condition* $[\overline{\mathbf{E}}]$.

Proof. It is similar to the proof of Theorem 3.1. □

Remark 3.2. Recently, Li *et al.* ([10]) established the generalized Egoroff theorem (for real-valued measurable functions sequence) concerning a pair of non-additive measures by using type **E** of absolute continuity for non-additive measures. Similarly, the generalized Egoroff theorem in [10] can be extended to the cases relating to the sequence of random sets (i.e., measurable closed-valued mappings) in the framework involving a pair of non-additive measures.

4. Conclusions

We have shown four versions of Egoroff's theorem for measurable closed-valued multifunctions (i.e., random sets) sequence on general non-additive measure spaces (Theorem 3.1 and Theorem 3.2(1),(2) and (3)). As we have seen, the necessary and sufficient conditions under which these four kinds of Egoroff's theorem remain valid for non-additive measures are respectively presented. In our discussion the condition [**E**] and condition $[\overline{\mathbf{E}}]$ of non-additive measures play important roles and the continuity of non-additive measures is not required. Therefore the previous related results in [8, 11] are improved and generalized.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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