AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(5): 4786–4802.
DOI:10.3934/math.2021281
Received: 25 November 2020
Accepted: 18 February 2021
Published: 25 February 2021

Research article

Existence of two homoclinic solutions for a nonperiodic difference equation with a perturbation

Yuhua Long[1,2,*]

[1] School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, 510006, PRC

[2] Center for Applied Mathematics, Guangzhou University, Guangzhou, 510006, PRC

* **Correspondence:** Email: sxlongyuhua@gzhu.edu.cn.

Abstract: In the present paper, with the combination of a compactness lemma and variational techniques, we establish a sufficient condition to guarantee the existence of two nontrivial homoclinic solutions for a nonperiodic fourth-order difference equation with a perturbation. Our result generalizes and improves some known results.

Keywords: nonperiodic fourth-order difference equation; homoclinic solution; mountain pass theorem; variational technique
Mathematics Subject Classification: 39A12, 39A23

1. Introduction

It is well-known that homoclinic solutions play an important role in analyzing the chaos of dynamical systems. Hence, it is significative to deal with homoclinic solutions of dynamical systems. In the present paper, let **Z** and **R** be the set of integers and real numbers, respectively. For parameter $\lambda > 0$, we investigate the existence of two nontrivial homoclinic solutions for the following nonperiodic fourth-order difference equation

$$\Delta^4 u(t-2) + \omega\Delta^2 u(t-1) + a(t)u(t) = f(t, u(t)) + \lambda h(t)|u(t)|^{p-2}u(t), \quad t \in \mathbf{Z}. \tag{1.1}$$

Here ω is a given constant, $1 \le p < 2$. $f(t,u) : \mathbf{Z}\times\mathbf{R} \to \mathbf{R}$ is continuous in u and $\frac{\partial F(t,u)}{\partial u} = f(t,u)$. $a(t) : \mathbf{Z} \to \mathbf{R}$ and $h(t) : \mathbf{Z} \to \mathbf{R}^+$. Let $\Delta u(t) = u(t+1) - u(t)$ be the forward difference operator and define $\Delta^0 u(t) = u(t)$, $\Delta^i u(t) = \Delta(\Delta^{i-1}u(t))$ for $i \ge 1$. As usual, we say that a solution $u = \{u(t)\}$ of (1.1) is homoclinic (to 0) if $\lim\limits_{|t|\to+\infty} u(t) = 0$. In addition, if $u(t) \neq 0$, then u is called a nontrivial homoclinic solution.

Difference equations possess theoretical background and practical significance. For example, [1] proposes a Dirichlet boundary value problem of difference equation to represent the amplitude of the motion of every particle in the string, [2] uses difference equations to study the impact of dispersal of a two-patch SIR disease model and [3] studies Wolbachia infection in mosquito population based on discrete models. Consequently, difference equations have attracted many researchers' attentions and rich results are obtained. To mention a few, [4–6] establish criteria for the existence and multiplicity of solutions, [7–11] focus on sign-changing solutions and [12–18] deal with homoclinic solutions or heteroclinic solutions.

Consider (1.1), it has been put forward as a discrete mathematical model for the study of pattern formation in physics and mechanics and deeply studied. For example, [19, 20] investigate sign-changing of a special case of (1.1) by the invariant sets of descending flow. As mention to homoclinic solutions, which play an important role in analyzing the chaos of dynamical systems, there are many publications such as [21] studies (1.1) in a special form with periodic assumption and [22, 23] prove that some special kind of (1.1) admits one nontrivial homoclinic solution. To the best of our knowledge, as considering homoclinic solutions for fourth order difference equations similar to (1.1), in the most known results of the existence of one non-zero homoclinic solution usually depend on periodic conditions or on the multiplicity with assumption of odevity on nonlinear terms. Meanwhile, using a compactness lemma and variational techniques, we achieve two nontrivial homoclinic solutions for (1.1) not only with neither periodic conditions nor odd-even requirements on nonlinear terms, but also with a perturbation. In some sense that our result improves and extends some known results.

Let constants c_0 be given in Lemma 2.2 and c_s be the best constant for the embedding of a Hilbert space X, which is defined in Section 2, in L^s, $2 \le s < +\infty$ and $q(t) : \mathbf{Z} \to \mathbf{R}^+$ with $\max_{t\in\mathbf{Z}}\{q(t)\} = q > c_0 c_2^2$. Write

$$\mu^* = \inf\{\|u\|_E : u \in E, \sum_{t=-\infty}^{+\infty} q(t)u^2(t) = 1\}. \tag{1.2}$$

Remark 1.1. *μ^* defined as (1.2) is reasonable. We state the proof of it in Lemma 3.2.*

With the above notations, now we establish our main result:

Theorem 1.1. *Let $h(t) : \mathbf{Z} \to \mathbf{R}^+$ with $\max_{t\in\mathbf{Z}}\{h(t)\} = h > 0$. Assume $a(t) : \mathbf{Z} \to \mathbf{R}$ and there exists a constant a_1 such that $\omega \le 2\sqrt{a_1}$ and*

$$0 < a_1 \le a(t) \to +\infty, \qquad as \quad |t| \to +\infty. \tag{1.3}$$

Further, for $i \ge 1$, suppose $f(t,u) : \mathbf{Z}\times\mathbf{R} \to \mathbf{R}$ is continuous in u and the following assumptions hold:
(F_1) $f(t,u) \equiv 0$ for all $u < 0$, $t \in \mathbf{Z}$ and there exists $b(t) : \mathbf{Z} \to \mathbf{R}^+$ with $\max_{t\in\mathbf{Z}}\{b(t)\} = b < \frac{c_0 c_2^2}{2}$ such that

$$\lim_{u\to 0^+} \frac{f(t,u)}{u^i} = b(t) \qquad \forall t \in \mathbf{Z},$$

and

$$\frac{f(t,u)}{u^i} \ge b(t) \quad \forall u > 0, \quad t \in \mathbf{Z};$$

(F_2)

$$\lim_{u\to+\infty}\frac{f(t,u)}{u^i}=q(t)\qquad \forall t\in\mathbf{Z};$$

(F_3) *there exist two constants* $\theta>2$ *and* $\frac{c_0c_2^2(\theta-2)}{4\theta}<d_0<\frac{c_0c_2^2(\theta-2)}{2\theta}$*, such that*

$$F(t,u)-\frac{1}{\theta}f(t,u)u\le d_0u^2,\quad \forall u>0,\quad t\in\mathbf{Z}. \tag{1.4}$$

Then either (i) $i=1$ *and* $\mu^*<1$ *or* (ii) $i>1$*, there exists* $\Lambda_0>0$ *such that (1.1) admits at least two distinct homoclinic solutions for every* $\lambda\in(0,\Lambda_0)$*.*

Remark 1.2. *The assumptions in Theorem 1.1 are feasible. For example, take* $b(t)=\frac{c_0c_2^2}{4}$ *for all* $t\in\mathbf{Z}$*. It follows that* $0<\max\limits_{t\in\mathbf{Z}}\{b(t)\}=\frac{c_0c_2^2}{4}<\frac{c_0c_2^2}{2}$*. Let*

$$f(t,u)=\begin{cases}\frac{12}{\pi}b(t)u^i\arctan(u+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}}), & \text{if}\quad u\ge 0,\quad \text{for all}\quad t\in\mathbf{Z};\\ 0, & \text{if}\quad u<0,\quad \text{for all}\quad t\in\mathbf{Z};\end{cases} \tag{1.5}$$

then

$$F(t,u)=\begin{cases}\frac{12}{\pi}b(t)\int_0^u x^i\arctan(x+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}})dx, & \text{if}\quad u\ge 0,\quad \text{for all}\quad t\in\mathbf{Z};\\ 0, & \text{if}\quad u<0,\quad \text{for all}\quad t\in\mathbf{Z}.\end{cases}$$

Consequently,

$$\lim_{u\to0^+}\frac{f(t,u)}{u^i}=\lim_{u\to0^+}\frac{\frac{12}{\pi}b(t)\arctan(u+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}})u^i}{u^i}=b(t)\qquad\forall t\in\mathbf{Z},$$

and

$$\lim_{u\to+\infty}\frac{f(t,u)}{u^i}=\lim_{u\to+\infty}\frac{\frac{12}{\pi}b(t)\arctan(u+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}})u^i}{u^i}=6b(t)=\frac{3c_0c_2^2}{2}:=q(t)>c_0c_2^2\qquad\forall t\in\mathbf{Z},$$

which means the assumptions (F_1) *and* (F_2) *are satisfied.*

Moreover, from the expression of $F(t,u)$*, for all* $t\in\mathbf{Z}$ *and* $u\ge0$*, we have*

$$\begin{aligned}F(t,u)=&\frac{12}{\pi}b(t)\int_0^u x^i\arctan(x+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}})dx\\ =&\frac{12}{\pi}b(t)\frac{x^{i+1}}{i+1}\arctan(x+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}})-\frac{12}{\pi}b(t)\frac{1}{i+1}\int_0^u\frac{x^{i+1}}{1+(x+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}})^2}dx.\end{aligned}$$

Obviously, $\frac{x^{i+1}}{1+(x+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}})^2}>0$ *for* $x>0$*, which follows that* $\int_0^u\frac{x^{i+1}}{1+(x+\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}+\sqrt{2}})^2}dx\ge0$*. Then one can verify that condition* (F_3) *is satisfied.*

Remark 1.3. *Our result improves and generalizes some known results. For example, similarly, [23] gives that there exists one homoclinic solution for the following difference equation*

$$\Delta^2(\varphi_p(\Delta^2 u(t-2))) - a\Delta(\varphi_p(\Delta u(t-1))) + \lambda V(t)\varphi_p(u(t)) = f(t,u(t)). \tag{1.6}$$

(1.1) can be regarded as a special case of (1.6) when $p = 2$. Our result Theorem 1.1 points out that (1.1) has at least two nontrivial homoclinic solutions. Meanwhile, according to [23], (1.1) possesses one homoclinic solution. Therefore, our result improves the result in [23] in some sense.

The organization of the paper is as follows: After this introduction, we present some basic lemmas and establish the corresponding variational functional to (1.1) in Section 2. Section 3 provides the detailed proof of our main result.

2. Variational structure and basic lemmas

In this section, we give some notations and basic lemmas to prepare for the proof of our main result Theorem 1.1.

Denote $u = \{u(t)\}_{t\in\mathbf{Z}} = (\cdots, u(-t), \cdots, u(-1), u(0), u(1), \cdots, u(t), \cdots)$. Let the set of all two-sided sequences $S = \{u = \{u(t)\} : u(t) \in \mathbf{R}, t \in \mathbf{Z}\}$, then S is a vector space with $au + bv = \{au(t) + bv(t)\}$ for $u, v \in S$, $a, b \in \mathbf{R}$. Define a subspace E of S as

$$E = \left\{ u \in S : \sum_{t=-\infty}^{+\infty} \left[|\Delta^2 u(t-1)|^2 - \omega|\Delta u(t-1)|^2 + a(t)|u(t)|^2 \right] < +\infty \right\}.$$

For any $u, v \in E$, define

$$< u, v >_E = \sum_{t=-\infty}^{+\infty} \left[\Delta^2 u(t-1) \cdot \Delta^2 v(t-1) - \omega \Delta u(t-1) \cdot \Delta v(t-1) + a(t) u(t) \cdot v(t) \right].$$

For later use, we define another Hilbert space $(X, < u, v >_X)$, where

$$X = \left\{ u \in S : \sum_{t=-\infty}^{+\infty} \left[|\Delta^2 u(t-1)|^2 + |\Delta u(t-1)|^2 + |u(t)|^2 \right] < +\infty \right\}$$

and, for any $u, v \in X$, the inner product is given by

$$< u, v >_X = \sum_{t=-\infty}^{+\infty} \left[\Delta^2 u(t-1) \cdot \Delta^2 v(t-1) + \Delta u(t-1) \cdot \Delta v(t-1) + u(t) \cdot v(t) \right].$$

Then the corresponding norm is

$$\|u\|_X = \sqrt{< u, u >_X} = \left(\sum_{t=-\infty}^{+\infty} \left[|\Delta^2 u(t-1)|^2 + |\Delta u(t-1)|^2 + |u(t)|^2 \right] \right)^{1/2}, \quad \forall u, v \in X.$$

In what follows, let

$$L^s = \left\{ u \in S : \|u\|_{L^s} = \left(\sum_{t=-\infty}^{+\infty} |u(t)|^s \right)^{\frac{1}{s}} < +\infty \right\}$$

denote the space of functions whose s-th powers are summable on $\mathbf{Z}$ and

$$\|u\|_{L^\infty} = \sup_{t\in\mathbf{Z}} |u(t)| < +\infty.$$

Thus the following embedding between L^s spaces holds,

$$L^q \subset L^p, \quad \|u\|_{L^p} \le \|u\|_{L^q}, \quad 1 \le q \le p \le \infty.$$

Now we start to consider the variational functional of (1.1). Define a functional $J : E \to \mathbf{R}$ as

$$\begin{aligned} J(u) =& \frac{1}{2}\sum_{t=-\infty}^{+\infty}\left[\Delta^2 u(t-1)\cdot\Delta^2 u(t-1) - \omega\Delta u(t-1)\cdot\Delta u(t-1) + a(t)u(t)\cdot u(t)\right] \\ & - \sum_{t=-\infty}^{+\infty} F(t,u(t)) - \lambda\sum_{t=-\infty}^{+\infty} h(t)|u(t)|^p. \end{aligned} \tag{2.1}$$

Then the continuity of f indicates that $J \in C^1(E, \mathbf{R})$ and, for any $u, v \in E$, its derivative is expressed as

$$\begin{aligned} < J'(u), v >_E =& \sum_{t=-\infty}^{+\infty}\left[\Delta^2 u(t-1)\cdot\Delta^2 v(t-1) - \omega\Delta u(t-1)\cdot\Delta v(t-1) + a(t)u(t)\cdot v(t)\right] \\ & - \sum_{t=-\infty}^{+\infty} f(t,u(t))\cdot v(t) - \lambda\sum_{t=-\infty}^{+\infty} h(t)|u(t)|^{p-2}u(t)\cdot v(t), \end{aligned}$$

which means that $u \in E$ is a critical point of J if and only if u is a homoclinic solution of (1.1).

Recall the definition of Cerami sequence and the variant version of the mountain pass theorem from critical point theory, which are helpful for us to seek critical points of (2.1).

Definition 2.1. *Let $J \in C^1(E, \mathbf{R})$. A sequence $\{u_n\} \in E$ is called a Cerami sequence ($(C)_c$ sequence for short) for J if $J(u_n) \to c$ for some $c \in \mathbf{R}$ and $(1 + \|u_n\|)J'(u_n) \to 0$ as $n \to \infty$. If any $(C)_c$ sequence for J possesses a convergent subsequence, then J satisfies the $(C)_c$ condition.*

Lemma 2.1. *(Mountain pass theorem) ([24]) Let E be a real Banach space with its dual space E^*, and suppose that $I \in C^1(E, \mathbf{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \mu < \eta \le \inf_{\|u\|=\rho} I(u),$$

for some $\mu < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $\hat{c} \ge \eta$ be characterized by

$$\hat{c} = \inf_{\gamma\in\Gamma}\max_{0\le\tau\le1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to \hat{c} \ge \eta \quad and \quad (1 + \|u_n\|_{E^*})\|I'(u_n)\|_{E^*} \to 0, \quad as \quad n \to \infty.$$

Remark 2.1. *Similar to [23], Lemma 2.1 allows us to find a $(C)_c$ sequence for J.*

In the following, we establish some compactness conditions.

Lemma 2.2. *Suppose $a(t) \geq a_1 > 0$ and $\omega < 2\sqrt{a_1}$, then for any $u \in E$*

$$\sum_{t=-\infty}^{+\infty} \left[\Delta^2 u(t-1) \cdot \Delta^2 u(t-1) - \omega \Delta u(t-1) \cdot \Delta u(t-1) + a(t)u(t) \cdot u(t)\right] \geq c_0 \|u\|_X^2 \tag{2.2}$$

is true for some constant $c_0 > 0$.

Proof. Since $\omega < 2\sqrt{a_1}$, we divide it into two cases.
Case 1. $\omega < 0$. Take $c_0 = \min\{-\omega, a_1, 1\}$, obviously,

$$\begin{aligned}&\sum_{t=-\infty}^{+\infty} \left[|\Delta^2 u(t-1)|^2 - \omega |\Delta u(t-1)|^2 + a(t)|u(t)|^2\right]\\ \geq & c_0 \sum_{t=-\infty}^{+\infty} \left[|\Delta^2 u(t-1)|^2 + |\Delta u(t-1)|^2 + |u(t)|^2\right]\end{aligned}$$

which implies that (2.2) is true.
Case 2. $0 \leq \omega < 2\sqrt{a_1}$. In this case, it is easy to get $\omega^2 < 4a_1$, then there exists $k \in (0, 3)$ satisfying

$$3 > k \geq 1 - \omega - 2a_1 + \sqrt{(\omega + 2a_1 - 1)^2 + 3(\omega + 1)^2}.$$

Hence,

$$k^2 + 2k(\omega + 2a_1 - 1) - 3(\omega + 1)^2 \geq 0. \tag{2.3}$$

Consider

$$g(\xi) := \xi^4 + \left(1 - \frac{3(\omega + 1)}{k}\right)\xi^2 + \left(1 + \frac{3(a_1 - 1)}{k}\right) \qquad \forall \xi \in \mathbf{R}. \tag{2.4}$$

Denote

$$\triangle := \left(1 - \frac{3(\omega + 1)}{k}\right)^2 - 4\left(1 + \frac{3(a_1 - 1)}{k}\right),$$

then (2.3) ensures

$$\triangle \leq 0,$$

which indicates that

$$g(\xi) \geq 0, \qquad \forall \xi \in \mathbf{R}.$$

Therefore, for $k \in (0, 3)$, we have

$$(\omega + 1)\xi^2 - a_1 + 1 \leq \frac{k}{3}(1 + \xi^2 + \xi^4), \qquad \forall \xi \in \mathbf{R}. \tag{2.5}$$

Analogous to [25], for any $u(t) \in X$, we take

$$u(t) = \sum_{t \in \mathbf{Z}} e^{i\zeta t} \bar{u}(\zeta),$$

then

$$\Delta u(t-1) = (1-e^{i\zeta})\sum_{t\in\mathbf{Z}} e^{i\zeta t}\bar{u}(\zeta) \quad \text{and} \quad \Delta^2 u(t-1) = (1-e^{i\zeta})^2\sum_{t\in\mathbf{Z}} e^{i\zeta t}\bar{u}(\zeta).$$

Denote $\xi = 1-e^{i\zeta} \triangleq \varphi(\zeta)$, then $\zeta = \varphi^{-1}(\xi)$. Now we have

$$u(t) = \sum_{t\in\mathbf{Z}} e^{i\zeta t}\bar{u}(\zeta) = \sum_{t\in\mathbf{Z}} e^{i\varphi^{-1}(\xi)t}\bar{u}(\varphi^{-1}(\xi)) \triangleq \hat{u}(\xi)$$

and

$$\Delta u(t-1) = \xi\hat{u}(\xi), \qquad \Delta^2 u(t-1) = \xi^2\hat{u}(\xi).$$

Thanks to (2.5), there has

$$\begin{aligned}
&\sum_{t=-\infty}^{+\infty}\left[|\Delta^2 u(t-1)|^2 - \omega|\Delta u(t-1)|^2 + a(t)|u(t)|^2\right] \\
\geq &\sum_{t=-\infty}^{+\infty}\left[|\Delta^2 u(t-1)|^2 - \omega|\Delta u(t-1)|^2 + a_1|u(t)|^2\right] \\
= &\sum_{\xi=-\infty}^{+\infty}\left(\xi^4 + \xi^2 + 1 - (\omega+1)\xi^2 + a_1 - 1\right)|\hat{u}(\xi)|^2 \\
\geq &\sum_{\xi=-\infty}^{+\infty}\left(\xi^4 + \xi^2 + 1 - \frac{k}{3}(\xi^4 + \xi^2 + 1)\right)|\hat{u}(\xi)|^2 \\
= &(1-\frac{k}{3})\sum_{\xi=-\infty}^{+\infty}(\xi^4 + \xi^2 + 1)|\hat{u}(\xi)|^2 \\
= &(1-\frac{k}{3})\|u\|_X^2.
\end{aligned} \tag{2.6}$$

Choose $c_0 = \sqrt{1-\frac{k}{3}} > 0$, then (2.6) leads to $\|u\|_E^2 \geq c_0\|u\|_X^2$, that is, (2.2) holds for $0 \leq \omega < 2\sqrt{a_1}$. Therefore, Lemma 2.2 is true and the proof is completed. □

With the help of Lemma 2.2, we obtain that $< u, u >_E$ is positive for all nonzero $u \in E$ and E is a Hilbert space. Here and hereafter, we write

$$\|u\|_E^2 = \sum_{t=-\infty}^{+\infty}\left[|\Delta^2 u(t-1)|^2 - \omega|\Delta u(t-1)|^2 + a(t)|u(t)|^2\right].$$

Now we state the main compactness lemma and present its proof in detail.

Lemma 2.3. *Let (1.3) hold and $\omega < 2\sqrt{a_1}$. Then, for $2 \leq s \leq +\infty$, E is compactly embedded in L^s.*

Proof. First, we prove Lemma 2.3 holds for $s = 2$.

Define

$$\alpha(A) = \inf_{|t|>A} a(t), \qquad A \in [0, +\infty).$$

From (1.3), $\alpha(A)$ increases and $\alpha(A) \to +\infty$ as $|t| \to +\infty$.

Let K be a bounded subset of E. It follows that, if $u \in K$, there exists a constant $M > 0$ such that $\|u\|_E \leq M$. Thanks to (2.2), we have

$$\|u\|_X^2 \leq \frac{1}{c_0}\|u\|_E^2 \leq \frac{M^2}{c_0} \qquad \text{and} \qquad \|u\|_{L^2}^2 \leq \frac{M^2}{c_0}, \qquad \forall u \in K.$$

Hence, we have

$$\begin{aligned}
&\sum_{t=-\infty}^{+\infty} a(t)|u(t)|^2 \\
&= \sum_{t=-\infty}^{+\infty} \left[|\Delta^2 u(t-1)|^2 - \omega|\Delta u(t-1)|^2 + a(t)|u(t)|^2 - |\Delta^2 u(t-1)|^2 + \omega|\Delta u(t-1)|^2\right] \\
&\leq \sum_{t=-\infty}^{+\infty} \left[|\Delta^2 u(t-1)|^2 - \omega|\Delta u(t-1)|^2 + a(t)|u(t)|^2 + |\omega||\Delta u(t-1)|^2\right] \\
&\leq M^2 + |\omega| \sum_{t=-\infty}^{+\infty} |\Delta u(t-1)|^2 \\
&\leq M^2 + 4|\omega| \sum_{t=-\infty}^{+\infty} |u(t)|^2 \\
&\leq M^2 + |\omega| \cdot \frac{4M^2}{c_0}.
\end{aligned}$$

Write $\hat{b} \triangleq M^2 + |\omega| \cdot \frac{4M^2}{c_0}$. For any $\epsilon > 0$, take A_0 large enough such that

$$\frac{4\hat{b}}{\alpha(A_0)} < \frac{\epsilon^2}{2}. \tag{2.7}$$

Since $K \subset E$ is bounded by M, there are $u_1, u_2, \cdots, u_m \in K$ such that for any $u \in K$, there exists some u_l $(1 \leq l \leq m)$ satisfying

$$\sum_{t=-A_0}^{A_0} |u(t) - u_l(t)| \leq \frac{\epsilon}{\sqrt{2}} \tag{2.8}$$

Combining (2.7) with (2.8), it yields

$$\begin{aligned}
\sum_{t=-\infty}^{+\infty} |u(t) - u_l(t)|^2 &= \sum_{|t|\leq A_0} |u(t) - u_l(t)|^2 + \sum_{|t|>A_0} |u(t) - u_l(t)|^2 \qquad (2.9)\\
&\leq \frac{\epsilon^2}{2} + \sum_{|t|>A_0} \frac{a(t)}{\alpha(A_0)}|u(t) - u_l(t)|^2 \\
&< \frac{\epsilon^2}{2} + \frac{4\hat{b}}{\alpha(A_0)} \\
&< \epsilon^2
\end{aligned}$$

which implies $\|u - u_l\|_{L^2} \to 0$.

Next we verify that our claim is true for $s = +\infty$. Notice that for any $n \in \mathbf{N}$, $T \in \mathbf{Z}$, $2 \leq k \in \mathbf{N}$ and $u \in E$, thanks to the Newton-Lebnitz formula of indefinite summation and step by step summation [26], we have

$$\sum_{s=T}^{t-1}\left[\frac{(s-T)^{n+1}}{(t-T)^n}\Delta u(s)\right] = (t-T)u(t) - \sum_{s=T}^{t-1}\left[u(s+1)\Delta\frac{(s-T)^{n+1}}{(t-T)^n}\right]$$

and

$$\sum_{s=t}^{T+k-1}\left[\frac{(T+k-s)^{n+1}}{(T+k-t)^n}\Delta u(s)\right] = (T+k-t)u(t) - \sum_{s=t}^{T+k-1}\left[u(s+1)\Delta\frac{(T+k-s)^{n+1}}{(T+k-t)^n}\right].$$

Hence, for all $T \leq t \leq T+k-1$, we have

$$\begin{aligned} ku(t) = &\sum_{s=T}^{t-1}\left[\frac{(s-T)^{n+1}}{(t-T)^n}\Delta u(s) + u(s+1)\Delta\frac{(s-T)^{n+1}}{(t-T)^n}\right] \\ &+ \sum_{s=t}^{T+k-1}\left[\frac{(T+k-s)^{n+1}}{(T+k-t)^n}\Delta u(s) + u(s+1)\Delta\frac{(T+k-s)^{n+1}}{(T+k-t)^n}\right]. \end{aligned}$$

On the other hand, owing to $a^\theta + b^\theta \leq (a+b)^\theta$ holds for $a, b \geq 0$ and $\theta \geq 1$, it follows

$$\begin{aligned} &\sum_{s=T}^{t-1}\left|\frac{(s-T)^{n+1}}{(t-T)^n}\Delta u(s)\right| + \sum_{s=t}^{T+k-1}\left|\frac{(T+k-s)^{n+1}}{(T+k-t)^n}\Delta u(s)\right| \\ \leq &\frac{1}{(t-T)^n}\left(\sum_{s=T}^{t-1}|(s-T)^{n+1}|^2\right)^{1/2}\cdot\left(\sum_{s=T}^{t-1}|\Delta u(s)|^2\right)^{1/2} \\ &+ \frac{1}{(T+k-t)^n}\left(\sum_{s=t}^{T+k-1}|(T+k-s)^{n+1}|^2\right)^{1/2}\cdot\left(\sum_{s=t}^{T+k-1}|\Delta u(s)|^2\right)^{1/2} \\ = &\frac{1}{\sqrt{2n+3}}(t-T)^{3/2}\left(\sum_{s=T}^{t-1}|\Delta u(s)|^2\right)^{1/2} + \frac{1}{\sqrt{2n+3}}(T+k-t)^{3/2}\left(\sum_{s=t}^{T+k-1}|\Delta u(s)|^2\right)^{1/2} \\ \leq &\frac{1}{\sqrt{2n+3}}\left[(t-T)^{3/2} + (T+k-t)^{3/2}\right]\left(\sum_{s=T}^{T+k-1}|\Delta u(s)|^2\right)^{1/2} \\ \leq &\frac{k^{3/2}}{\sqrt{2n+3}}\left(\sum_{s=T}^{T+k-1}|\Delta u(s)|^2\right)^{1/2}. \end{aligned} \tag{2.10}$$

In view of $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}} \leq \sqrt{a+b}$ $(a, b \geq 0)$, similar to (2.10), we have

$$\begin{aligned} &\sum_{s=T}^{t-1}\left|u(s+1)\Delta\frac{(s-T)^{n+1}}{(t-T)^n}\right| + \sum_{s=t}^{T+k-1}\left|u(s+1)\Delta\frac{(T+k-s)^{n+1}}{(T+k-t)^n}\right| \\ \leq &\frac{\sqrt{2k}(n+1)}{\sqrt{2n+1}}\left(\sum_{s=T}^{T+k-1}|u(s+1)|^2\right)^{1/2}. \end{aligned} \tag{2.11}$$

Therefore, for all $T \le t \le T+k-1$ and $2 \le k \in \mathbf{N}$, with the aid of (2.10) and (2.11), we have

$$|u(t)| \le \frac{\sqrt{k}}{\sqrt{2n+3}}\left(\sum_{s=T}^{T+k-1}|\Delta u(s)|^2\right)^{1/2} + \frac{\sqrt{2}}{\sqrt{k}}\cdot\frac{n+1}{\sqrt{2n+1}}\left(\sum_{s=T}^{T+k-1}|u(s+1)|^2\right)^{1/2}, \tag{2.12}$$

which implies that

$$\begin{aligned}
&|u(t)-v(t)|\\
\le&\frac{\sqrt{k}}{\sqrt{2n+3}}\left(\sum_{|s|\ge A}|\Delta\left(u(s)-v(s)\right)|^2\right)^{1/2} + \frac{\sqrt{2}(n+1)}{\sqrt{k(2n+1)}}\left(\sum_{|s|\ge A}|u(s+1)-v(s+1)|^2\right)^{1/2}\\
\le&\frac{\sqrt{k}}{\sqrt{2n+3}}\left[\left(\sum_{s=-\infty}^{+\infty}|\Delta u(s)|^2\right)^{1/2} + \left(\sum_{s=-\infty}^{+\infty}|\Delta v(s)|^2\right)^{1/2}\right] + \frac{\sqrt{2}(n+1)}{\sqrt{k(2n+1)}}\\
&\left(\sum_{|s|>A}\frac{a(s)|u(s+1)-v(s+1)|^2}{\alpha(A)}\right)^{1/2}\\
=&\frac{4M\sqrt{k}}{\sqrt{2n+3}} + \frac{\sqrt{2}(n+1)}{\sqrt{k(2n+1)}}\cdot\frac{2\sqrt{\hat{b}}}{\sqrt{\alpha(A)}}
\end{aligned}$$

holds for any $u, v \in K$, $A > 0$ and all $|t| > A$. For any $\epsilon > 0$, take first n large enough such that

$$\frac{4M\sqrt{k}}{\sqrt{2n+3}} < \frac{\epsilon}{2}.$$

Notice that $2 \le k \in \mathbf{N}$, then, for any $\epsilon > 0$, choose A_0 large enough such that

$$\frac{\sqrt{2}(n+1)}{\sqrt{k(2n+1)}}\cdot\frac{2\sqrt{\hat{b}}}{\sqrt{\alpha(A_0)}} < \frac{\epsilon}{2}.$$

Therefore, we can draw a conclusion that

$$\max_{|t|>A_0}|u(t)-v(t)| < \epsilon, \qquad \forall u, v \in K. \tag{2.13}$$

By the same method of (2.8), it follows that

$$\max_{|t|\le A_0}|u(t)-u_l(t)| < \epsilon. \tag{2.14}$$

Combing (2.13) with (2.14), we obtain

$$\|u-u_l\|_{L^\infty} < \epsilon. \tag{2.15}$$

Finally, we accomplish the proof of Lemma 2.3 by verifying it is correct for $2 < s < +\infty$. Given an arbitrary $u \in E$, there has

$$\sum_{t=-\infty}^{+\infty}|u(t)|^s = \sum_{t=-\infty}^{+\infty}\left(|u(t)|^{s-2}\cdot|u(t)|^2\right) \le \max_{t\in\mathbf{N}}|u(t)|^{s-2}\cdot\sum_{t=-\infty}^{+\infty}|u(t)|^2 = \|u\|_{L^\infty}^{s-2}\cdot\|u\|_{L^2}^2, \tag{2.16}$$

which implies that K is precompact in L^s. Combing (2.9), (2.15) and (2.16), we complete the proof of Lemma 2.3 immediately. □

3. Proof of the main results

In this section, we intend to prove the main result at length. Now we are in the position to state the following several lemmas which guarantee that the functional J, defined by (2.1), has the mountain pass geometry at first.

Lemma 3.1. *Let $a(t)$ and ω satisfy the assumptions in Theorem 1.1 and the conditions (F_1) and (F_2) hold. Then there exist $\Lambda_0 > 0$ and constants $\rho, \eta > 0$ such that*

$$J(u)|_{\|u\|_E=\rho} \geq \eta > 0$$

for every $\lambda \in (0, \Lambda_0)$.

Proof. For every $\epsilon > 0$, notice that the condition (F_1) implies that there exists a constant $\delta > 0$ such that $f(t, s) \leq (b+\epsilon)s^i \leq (b+\epsilon)s$ holds for $0 < s < \delta$. From (F_2), $\lim_{s\to\infty} \frac{f(t,s)}{s^i} = q(t)$ leads to $\lim_{s\to\infty} \frac{f(t,s)}{s^{i+1}} = 0$, which implies that there exists a constant $M > 0$ big sufficiently such that $\frac{f(t,s)}{s^{i+1}} \leq \epsilon$, that is, $f(t, s) \leq \epsilon s^{i+1}$ with $s > M$. Further, since $f(t, s)$ is continuous, it is not difficult to choose a constant C such that $\frac{f(t,s)}{s^{i+1}} \leq C$ for $\delta \leq s \leq M$. Therefore,

$$f(t, s) \leq (b + \epsilon)s + \epsilon s^{i+1} + C s^{i+1}, \qquad \forall s \in \mathbf{R}, \tag{3.1}$$

which indicates that there exist $C_\epsilon > 0$ and $r \geq i + 2$ such that

$$F(t, s) \leq \frac{b + \epsilon}{2} s^2 + \frac{C_\epsilon}{r} |s|^r, \qquad \forall s \in \mathbf{R}. \tag{3.2}$$

For $2 \leq r < +\infty$, let c_r be the best constants for the embedded of X in L^r. With the aid of Lemma 2.2, we get

$$\|u\|_E^r \geq c_0^{\frac{r}{2}} \|u\|_X^r \geq c_0^{\frac{r}{2}} c_r^r \|u\|_{L^r}^r,$$

that is,

$$\|u\|_{L^r}^r \leq \frac{1}{c_0^{\frac{r}{2}} c_r^r} \|u\|_E^r.$$

Together with (3.2), for all $u \in E$, one can obtain

$$\begin{aligned}
\sum_{t=-\infty}^{+\infty} F(t, u(t)) &\leq \frac{b + \epsilon}{2} \sum_{t=-\infty}^{+\infty} |u(t)|^2 + \frac{C_\epsilon}{r} \sum_{t=-\infty}^{+\infty} |u(t)|^r \\
&= \frac{b + \epsilon}{2} \|u\|_{L^2}^2 + \frac{C_\epsilon}{r} \|u\|_{L^r}^r \\
&\leq \frac{b + \epsilon}{2c_0 c_2^2} \|u\|_E^2 + \frac{C_\epsilon}{r c_0^{\frac{r}{2}} c_r^r} \|u\|_E^r.
\end{aligned}$$

Therefore,

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_E^2 - \sum_{t=-\infty}^{+\infty} F(t,u(t)) - \lambda \sum_{t=-\infty}^{+\infty} h(t)|u(t)|^p \\ &\geq \frac{1}{2}\|u\|_E^2 - \frac{b+\epsilon}{2c_0c_2^2}\|u\|_E^2 - \frac{C_\epsilon}{rc_0^{\frac{r}{2}}c_r^r}\|u\|_E^r - \lambda c_0^{-\frac{p}{2}} c_p^{-p} h\|u\|_E^p \\ &= \|u\|_E^p\left[\frac{1}{2}\left(1 - \frac{b+\epsilon}{c_0c_2^2}\right)\|u\|_E^{2-p} - \frac{C_\epsilon}{rc_0^{\frac{r}{2}}c_r^r}\|u\|_E^{r-p} - \lambda c_0^{-\frac{p}{2}} c_p^{-p} h\right]. \end{aligned} \tag{3.3}$$

By the last equation in (3.3), select $\epsilon = \frac{c_0c_2^2}{2} - b > 0$ and denote $t = \|u\|_E \geq 0$, we define

$$g(t) = \frac{1}{4}t^{2-p} - \frac{C_\epsilon}{rc_0^{\frac{r}{2}}c_r^r}t^{r-p}.$$

Since $r > 2$ and $1 \leq p < 2$, it is easy to find that $g(t)$ will get its maximum value at $t = \left(\frac{rc_0^{\frac{r}{2}}c_r^r(2-p)}{4C_\epsilon(r-p)}\right)^{\frac{1}{r-2}} \triangleq \rho > 0$. Hence

$$\max_{t\geq 0} g(t) = g(\rho) = \frac{r-2}{4(r-p)}\left(\frac{(2-p)rc_0^{\frac{r}{2}}c_r^r}{4(r-p)C_\epsilon}\right)^{\frac{2-p}{r-2}} \triangleq M > 0.$$

Combing with (3.3), it yields that there exists $\Lambda_0 = \frac{Mc_0^{\frac{p}{2}}c_p^p}{h} > 0$ such that we can find a constant $\eta > 0$ which satisfies $J(u)|_{\|u\|_E=\rho} \geq \eta$ for every $\lambda \in (0, \Lambda_0)$. □

Lemma 3.2. *Let ρ and Λ_0 be defined in Lemma 3.1. Suppose that the conditions (F_1) and (F_2) hold, then for every $\lambda \in (0, \Lambda_0)$ there exists $e \in E$ with $\|e\|_E > \rho$ such that $J(e) < 0$ holds for either $i = 1$ and $\mu^* < 1$ or $i > 1$.*

Proof. We give the proof in two cases.
Case I. If $i = 1$ and $\mu^* < 1$, we first declare μ^*, defined as (1.2), is reasonable. Let $u \in E$ satisfy $\sum_{t=-\infty}^{+\infty} q(t)u^2(t) = 1$. Then

$$1 = \sum_{t=-\infty}^{+\infty} q(t)u^2(t) \leq q \sum_{t=-\infty}^{+\infty} u^2(t) = q \cdot \|u\|_{L^2}^2,$$

which means that $\|u\|_{L^2}^2 \geq \frac{1}{q}$. In view of Lemma 2.2, we get

$$\|u\|_E^2 \geq c_0\|u\|_X^2 \geq c_0c_2^2\|u\|_{L^2}^2 \geq \frac{c_0c_2^2}{q} > 0.$$

Thus $\mu^* \geq \frac{c_0c_2^2}{q} > 0$. Aim to get μ^* is attainable, let $\{u_n\} \in E$ be a minimizing sequence of (1.2), then $\{u_n\}$ is bounded and satisfies $\sum_{t=-\infty}^{+\infty} q(t)u_n^2(t) = 1$. Choose a subsequence of $\{u_n\}$, without loss of generality, still denoted by $\{u_n\}$. In view of Lemma 2.3, there exists $\phi_1 \in E$ such that $u_n \rightharpoonup \phi_1$ weakly in E and $u_n \to \phi_1$ strongly in L^2. Hence

$$\sum_{t=-\infty}^{+\infty} q(t)u_n^2(t) \to \sum_{t=-\infty}^{+\infty} q(t)\phi_1^2(t) \quad \text{as} \quad n \to \infty, \qquad \text{and} \qquad \sum_{t=-\infty}^{+\infty} q(t)\phi_1^2(t) = 1.$$

Therefore,

$$\begin{aligned}
\mu^* &\leq \sum_{t=-\infty}^{+\infty} \left[\left(\Delta^2\phi_1(t-1)\right)^2 - \omega(\Delta\phi_1(t-1))^2 + a(t)\phi_1^2(t)\right] \\
&\leq \liminf_{n\to\infty} \sum_{t=-\infty}^{+\infty} \left[\left(\Delta^2 u_n(t-1)\right)^2 - \omega(\Delta u_n(t-1))^2 + a(t)u_n^2(t)\right] \\
&= \mu^*,
\end{aligned}$$

which indicates that $\mu^* = \sum_{t=-\infty}^{+\infty} \left[\left(\Delta^2\phi_1(t-1)\right)^2 - \omega(\Delta\phi_1(t-1))^2 + a(t)\phi_1^2(t)\right] = \|\phi_1\|_E^2$.

Since $\mu^* < 1$, it is not difficult to choose $0 \leq \varphi \in E$ with $\sum_{t=-\infty}^{+\infty} q(t)\varphi^2(t) = 1$ such that $\|\varphi\|_E < 1$. Using the given condition (F_2), we have

$$\begin{aligned}
\lim_{s\to+\infty} \frac{J(s\varphi)}{s^2} &= \frac{1}{2}\|\varphi\|_E^2 - \lim_{s\to+\infty} \sum_{t=-\infty}^{+\infty} \frac{F(t, s\varphi(t))}{s^2} - \lim_{s\to+\infty} \frac{\lambda}{s^2} \sum_{t=-\infty}^{+\infty} h(t)|s\varphi(t)|^p \\
&\leq \frac{1}{2}\|\varphi\|_E^2 - \lim_{s\to+\infty} \sum_{t=-\infty}^{+\infty} \frac{F(t, s\varphi(t))}{s^2} \\
&= \frac{1}{2}\|\varphi\|_E^2 - \sum_{t=-\infty}^{+\infty} \lim_{s\to+\infty} \frac{f(t, s\varphi(t)) \cdot \varphi(t)}{2s} \\
&= \frac{1}{2}\|\varphi\|_E^2 - \sum_{t=-\infty}^{+\infty} \lim_{s\to+\infty} \frac{f(t, s\varphi(t)) \cdot \varphi(t)}{2s\varphi(t)} \cdot \varphi(t) \\
&= \frac{1}{2}\|\varphi\|_E^2 - \frac{1}{2} \sum_{t=-\infty}^{+\infty} q(t)\varphi^2(t) \\
&= \frac{1}{2}(\|\varphi\|_E^2 - 1) \\
&< 0,
\end{aligned}$$

which tells us that $J(s\varphi) \to -\infty$ as $s \to +\infty$. Then there exists $e \in E$ with $\|e\|_E > \rho$ such that $J(e) < 0$.

Case II. If $i > 1$, in view of $q(t) : \mathbf{Z} \to \mathbf{R}^+$, we find there exists $\psi \in E$ such that

$$\sum_{t=-\infty}^{+\infty} q(t)\psi^{i+1}(t) > 0.$$

In the same manner as Case I, we have

$$\begin{aligned}\lim_{s\to+\infty}\frac{J(s\psi)}{s^{i+1}} &= \lim_{s\to+\infty}\frac{\frac{1}{2}\|s\psi\|_E^2-\sum\limits_{t=-\infty}^{+\infty}F(t,s\psi(t))-\lambda\sum\limits_{t=-\infty}^{+\infty}h(t)|s\psi(t)|^p}{s^{i+1}}\\ &\le \lim_{s\to+\infty}\frac{\|\psi\|_E^2}{2s^{i-1}}-\lim_{s\to+\infty}\sum_{t=-\infty}^{+\infty}\frac{F(t,s\psi(t))}{s^{i+1}}\\ &= \lim_{s\to+\infty}\frac{\|\psi\|_E^2}{2s^{i-1}}-\frac{1}{i+1}\sum_{t=-\infty}^{+\infty}q(t)\psi^{i+1}(t)\\ &\le -\frac{1}{i+1}\sum_{t=-\infty}^{+\infty}q(t)\psi^{i+1}(t)\\ &< 0.\end{aligned}$$

Therefore, there exists $e \in E$ with $\|e\|_E > \rho$ such that $J(e) < 0$. The proof is completed. □

Notice Lemma 3.1 and Lemma 3.2 show that J meets all conditions in Lemma 2.1, hence J possesses a $(C)_c$ sequence $\{u_n\} \subset E$ for the mountain pass level β which is defined by

$$\beta = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} J(\gamma(t))$$

and $\Gamma = \{\gamma \in C([0,1],E)|\gamma(0) = 0, \gamma(1) = e\}$.

In the following, we set out to look for homoclinic solutions for (1.1). Denote $B_\rho = \{u \in E : \|u\|_E < \rho\}$, where ρ is given by Lemma 3.1. We first seek for a critical point of J by showing J attains a local minimum for small λ.

Lemma 3.3. *Let ρ and Λ_0 be defined in Lemma 3.1. Assume $a(t)$, ω and $h(t)$ satisfy Theorem 1.1 and (F_1) hold. Then, for $\lambda \in (0,\Lambda_0)$, (1.1) possesses a homoclinic solution $u_0 \in E$ such that*

$$J(u_0) = \inf\{J(u)|u \in \bar{B}_\rho\} < 0.$$

Proof. Since $h(t) : \mathbf{Z} \to \mathbf{R}^+$, it is convenient to select $\zeta \in E$ such that $\sum\limits_{t=-\infty}^{+\infty} h(t)|\zeta(t)|^p > 0$. For $\kappa > 0$ small enough, (F_1) induces $F(t,\kappa\zeta(t)) > 0$ is correct for all $t \in \mathbf{Z}$. Then for $1 \le p < 2$, one has

$$\begin{aligned}J(\kappa\zeta) &= \frac{1}{2}\|\kappa\zeta\|_E^2-\sum_{t=-\infty}^{+\infty}F(t,\kappa\zeta(t))-\lambda\sum_{t=-\infty}^{+\infty}h(t)|\kappa\zeta(t)|^p\\ &= \frac{\kappa^2}{2}\|\zeta\|_E^2-\sum_{t=-\infty}^{+\infty}F(t,\kappa\zeta(t))-\lambda\kappa^p\sum_{t=-\infty}^{+\infty}h(t)|\zeta(t)|^p\\ &\le \frac{\kappa^2}{2}\|\zeta\|_E^2-\lambda\kappa^p\sum_{t=-\infty}^{+\infty}h(t)|\zeta(t)|^p\\ &< 0\end{aligned}$$

Write $m \triangleq \inf\{J(u) : u \in \bar{B}_\rho\}$, then $m < 0$. Thus there exists a minimizing sequence $\{u_n\} \subset E$ such that $J(u_n) \to m$ and $J'(u_n) \to 0$ as $n \to \infty$. Therefore, Lemma 2.3 ensures that J admits a critical point $u_0 \in E$ which satisfies $J'(u_0) = 0$ and $J(u_0) = m < 0$. □

In view of Lemma 3.3, it is necessary for us to show that there exists another $\bar{u} \in E$ such that $J'(\bar{u}) = 0$ and $\bar{u} \neq u_0$ to accomplish the proof of Theorem 1.1.

Proof of Theorem 1.1. We complete the proof in two steps.

Step 1. The $(C)_c$ sequence $\{u_n\} \in E$ of J, defined by

$$J(u_n) \to \beta > 0 \quad \text{and} \quad (1 + \|u_n\|_E)\|J'(u_n)\|_{E^*} \to 0, \quad \text{as} \quad n \to \infty,$$

is bounded. Let n be large enough. By the condition (F_3) and Lemma 3.1, it follows that

$$\begin{aligned}
\beta + 1 &\geq J(u_n) - \frac{1}{\theta} < J'(u_n), u_n > \\
&= (\frac{1}{2} - \frac{1}{\theta})\|u_n\|_E^2 - \sum_{t=-\infty}^{+\infty}\left[F(t, u_n(t)) - \frac{1}{\theta}f(t, u_n(t))u_n(t)\right] - \lambda(1 - \frac{1}{\theta})\sum_{t=-\infty}^{+\infty} h(t)|u_n(t)|^p \\
&\geq \frac{\theta - 2}{2\theta}\|u_n\|_E^2 - d_0\sum_{t=-\infty}^{+\infty} u_n^2(t) - \lambda(1 - \frac{1}{\theta})h\sum_{t=-\infty}^{+\infty}|u_n(t)|^p \\
&\geq \frac{\theta - 2}{2\theta}\|u_n\|_E^2 - d_0\|u_n\|_{L^2}^2 - \lambda(1 - \frac{1}{\theta})h\|u_n\|_{L^p}^p \\
&\geq \frac{\theta - 2}{2\theta}\|u_n\|_E^2 - \frac{d_0}{c_0c_2^2}\|u_n\|_E^2 - \lambda(1 - \frac{1}{\theta})hc_0^{-\frac{p}{2}}c_p^{-p}\|u_n\|_E^p \\
&= \left(\frac{\theta - 2}{2\theta} - \frac{d_0}{c_0c_2^2}\right)\|u_n\|_E^2 - \lambda(1 - \frac{1}{\theta})hc_0^{-\frac{p}{2}}c_p^{-p}\|u_n\|_E^p \\
&< \left(\frac{\theta - 2}{2\theta} - \frac{\frac{c_0c_2^2(\theta-2)}{4\theta}}{c_0c_2^2}\right)\|u_n\|_E^2 - \lambda(1 - \frac{1}{\theta})hc_0^{-\frac{p}{2}}c_p^{-p}\|u_n\|_E^p \\
&= \frac{\theta - 2}{4\theta}\|u_n\|_E^2 - \lambda(1 - \frac{1}{\theta})hc_0^{-\frac{p}{2}}c_p^{-p}\|u_n\|_E^p.
\end{aligned} \tag{3.4}$$

Obviously, for $\theta > 2$ and $p < 2$, (3.4) implies $\|u_n\|_E$ is bounded for all $\lambda \in (0, \Lambda_0)$.

Step 2. Now it is time for us to verify that J has another critical point $\bar{u}$ which satisfies $J'(\bar{u}) = 0$ and $J(\bar{u}) = \beta > 0$. Since the $(C)_c$ sequence $\{u_n\} \subset E$ of J is bounded, from Lemma 2.3, there exists $\bar{u} \in E$ satisfying, up to a subsequence,

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in} \quad E, \qquad u_n \to \bar{u} \quad \text{strongly in} \quad L^2.$$

Together with the Hölder inequality, it follows that

$$\sum_{t=-\infty}^{+\infty}\left[f(t, u_n(t)) - f(t, \bar{u}(t))\right](u_n(t) - \bar{u}(t)) \to 0, \qquad \text{as} \quad n \to \infty,$$

and

$$\sum_{t=-\infty}^{+\infty}\left[h(t)\left(|u_n(t)|^{p-2}u_n(t) - |\bar{u}(t)|^{p-2}\bar{u}(t)\right)\right](u_n(t) - \bar{u}(t)) \to 0, \qquad \text{as} \quad n \to \infty.$$

On the other hand, the definition of $J(u)$ indicates that

$$\|u_n - \bar{u}\|_E^2 =< J'(u_n) - J'(\bar{u}), u_n - \bar{u} >_E - \sum_{t=-\infty}^{+\infty} \left[f(t, u_n(t)) - f(t, \bar{u}(t)) \right] (u_n(t) - \bar{u}(t))$$
$$- \lambda \sum_{t=-\infty}^{+\infty} \left[h(t) \left(|u_n(t)|^{p-2} u_n(t) - |\bar{u}(t)|^{p-2} \bar{u}(t) \right) \right] (u_n(t) - \bar{u}(t)) .$$

Hence $u_n \to \bar{u}$ strongly in E. Moreover, $J(\bar{u}) = \beta > 0$ and $\bar{u}$ is another homoclinic solution of (1.1). Consequently, u_0 and $\bar{u}$ are two distinct homoclinic solutions of (1.1). And the proof of Theorem 1.1 is finished.

Acknowledgements

The author sincerely thanks the handling editor and the anonymous referees for their valuable comments and suggestions.

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. R. P. Agarwal, *Difference equations and inequalities: Theory, methods and applications*, Marcel Dekker, 1992.
2. Y. H. Long, L. Wang, Global dynamics of a delayed two-patch discrete SIR disease model, *Commu. Nonlinear Sci.*, **83** (2020), 105117.
3. Y. T. Shi, J. S. Yu, Wolbachia infection enhancing and decaying domains in mosquito population based on discrete models, *J. Biol. Dynam.*, **14** (2020), 679–695.
4. F. M. Atici, G. S. Guseinov, Positive periodic solutions for nonlinear difference equations with periodic coefficients, *J. Math. Anal. Appl.*, **232** (1999), 166–182.
5. Z. M. Guo, J. S. Yu, The existence of periodic and subharmonic solutions of sub-quadratic second order difference equations, *J. Lond. Math. Soc.*, **68** (2003), 419–430.
6. J. Hendersona, R. Luca, Existence of positive solutions for a system of semipositone coupled discrete boundary value problems, *J. Differ. Equ. Appl.*, **25** (2019), 516–541.
7. T. S. He, Y. W. Zhou, Y. T. Xu, C. Y. Chen, Sign-changing solutions for discrete second-order periodic boundary value problems, *B. Malays. Math. Sci. So.*, **38** (2015), 181–195.
8. Y. H. Long, B. L. Zeng, Multiple and sign-changing solutions for discrete Robin boundary value problem with parameter dependence, *Open Math.*, **15** (2017), 1549–1557.
9. Y. H. Long, J. L. Chen, Existence of multiple solutions to second-order discrete Neumann boundary value problems, *Appl. Math. Lett.*, **83** (2018), 7–14.

10. Y. H. Long, S. H. Wang, Multiple solutions for nonlinear functional difference equations by the invariant sets of descending flow, *J. Differ. Equ. Appl.*, **25** (2019), 1768–1789.
11. Y. H. Long, Existence of multiple and sign-changing solutions for a second-order nonlinear functional difference equation with periodic coefficients, *J. Differ. Equ. Appl.*, **26** (2020), 966–986.
12. M. J. Ma, Z. M. Guo, Homoclinic orbits for second order self-adjoint difference equations, *J. Math. Anal. Appl.*, **232** (2006), 513–521.
13. Y. H. Long, Homoclinic orbits for a class of noncoercive discrete hamiltonian systems, *J. Appl. Math.*, **2012** (2012), 1–21.
14. Z. Zhou, D. F. Ma, Multiplicity results of breathers for the discrete nonlinear Schrödinger equations with unbounded potentials, *Sci. China Math.*, **58** (2015), 781–790.
15. Y. H. Long, Y. B. Zhang, H. P. Shi, Homoclinic solutions of $2n$th-order difference equations containing both advance and retardation, *Open Math.*, **14** (2016), 520–530.
16. L. Erbe, B. G. Jia, Q. Q. Zhang, Homoclinic solutions of discrete nonlinear systems via variational method, *J. Appl. Anal. Compt.*, **9** (2019), 271–294.
17. Q. Q. Zhang, Homoclinic orbits for discrete Hamiltonian systems with local super-quadratic conditions, *Commun. Pur. Appl. Anal.*, **18** (2019), 425–434.
18. J. H. Kuang, Z. M. Guo, Heteroclinic solutions for a class of p-Laplacian difference equations with a parameter, *Appl. Math. Lett.*, **100** (2020), 106034.
19. Y. H. Long, S. H. Wang, J. L. Chen, Multiple solutions of fourth-order difference equations with different boundary conditions, *Bound. Value Probl.*, **2019** (2019), 1–25.
20. S. H. Wang, Y. H. Long, Multiple solutions of fourth-order functional difference equation with periodic boundary conditions, *Appl. Math. Lett.*, **104** (2020), 106292.
21. G. H. Lin, Z. Zhou, Homoclinic solutions of discrete ϕ-Laplacian equations with mixed nonlinearities, *Commun. Pur. Appl. Anal.*, **17** (2018), 1723–1747.
22. H. Fang, D. P. Zhao, Existence of nontrivial homoclinic orbits for fourth-order difference equations, *Appl. Math. Compt.*, **214** (2009), 163–170.
23. N. Dimitrov, S. Tersian, Existence of homoclinic solutions for a nonlinear fourth order p-Laplacian difference equation, *Discrete Cont. Dyn. B*, **25** (2020), 555–567.
24. I. Ekeland, *Convexity methods in Hamiltonian mechanics*, Springer-Verlag Berlin Heidelberg, 1990.
25. S. Tersian, J. Chaparova, Periodic and homoclinic solutions of extended Fisher-Kolmogorov equations, *J. Math. Anal. Appl.*, **260** (2001), 490–506.
26. Y. C. Zhou, H. Cao, Y. N. Cao, *Difference equations and their applications*, Beijing, Science Press, 2014.