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Research article

Codewords generated by UP-valued functions

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Abstract: The concept of a UP-valued function on a nonempty set was introduced by Ansari et al. [3]. Codewords in a binary block-code generated by a UP-valued function are established and some interesting results are obtained. Finally, we prove that every finite UP-algebra A which has the order less than or equal to the order of a finite set X determines a binary block-code V such that $(A, \leq)$ is isomorphic to $(V, \leq)$.

Keywords: UP-algebra; UP-valued function; cut function; codeword; binary block-code
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1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [9], BCI-algebras [10], BE-algebras [12], UP-algebras [6], extension of KU/UP-algebras [18] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [10] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [9, 10] in 1966 and have been extensively investigated by many researchers.

Jun and Song [11] said the following: In computer science, a block code is a type of channel coding. It adds redundancy to a message so that, at the receiver, one can decode with minimal (theoretically zero) errors, provided that the information rate would not exceed the channel capacity. The main characterization of a block code is that it is a fixed length channel code (unlike source coding schemes such as Huffman coding, and unlike channel coding methods like convolutional encoding). Typically, a block code takes a k-digit information word, and transforms this into an n-digit codeword. Block coding is the primary type of channel coding used in earlier mobile communication systems. A block code is a code which encodes strings formed an alphabet set S into code words by encoding each letter of S separately.

Coding theory was applied to BCK-algebras in 2011 by Jun and Song [11] and in 2015 by Flaut [5]. They proved that every finite BCK-algebra determines a binary block-code. In 2015, Mostafa et al. [15] applied coding theory to KU-algebras and gave some relation and connection between binary block-codes and KU-algebras. They proved that every finite KU-algebra determines a binary block-code which is isomorphic to it. In 2020, Koam et al. [13] defined and investigated KU-valued generalized cut functions and their properties. They proved that for each n-ary block code K we can associate a KU-algebra X, such that the constructed n-ary block codes generated by X, and proved that for every n-ary block code K, there exists a KU-valued function on a KU-algebra which determines K. Moreover, they have introduced and studied UP-valued functions in [3]. For many studies of KU-algebras, see [14, 16, 23, 25].

In this paper, we establish binary block-codes by using the concept of UP-valued functions, introduced by Ansari et al. [3]. We show that every finite UP-algebra A which has the order less than or equal to the order of a finite set X determines a binary block-code V such that $(A, \leq)$ is isomorphic to $(V, \leq)$.

Before we begin our study, let's review the definition of UP-algebras.

Definition 1.1. [6] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra*, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

$$(\text{for all } x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0), \tag{1.1}$$

$$(\text{for all } x \in A)(0 \cdot x = x), \tag{1.2}$$

$$(\text{for all } x \in A)(x \cdot 0 = 0), \text{ and} \tag{1.3}$$

$$(\text{for all } x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y). \tag{1.4}$$

From [6], we know that the concept of UP-algebras is a generalization of KU-algebras (see [17]). The binary relation $\leq$ on a UP-algebra $A = (A, \cdot, 0)$ defined as follows:

$$(\text{for all } x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0) \tag{1.5}$$

and the following assertions are valid (see [6, 7]).

$$(\text{for all } x \in A)(x \leq x), \tag{1.6}$$

$$(\text{for all } x, y, z \in A)(x \leq y, y \leq z \Rightarrow x \leq z), \tag{1.7}$$

$$(\text{for all } x, y, z \in A)(x \leq y \Rightarrow z \cdot x \leq z \cdot y), \tag{1.8}$$

$$(\text{for all } x, y, z \in A)(x \leq y \Rightarrow y \cdot z \leq x \cdot z), \tag{1.9}$$

$$(\text{for all } x, y, z \in A)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)), \tag{1.10}$$

$$(\text{for all } x, y \in A)(y \cdot x \leq x \Leftrightarrow x = y \cdot x), \tag{1.11}$$

$$(\text{for all } x, y \in A)(x \leq y \cdot y), \tag{1.12}$$

$$(\text{for all } a, x, y, z \in A)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z))), \tag{1.13}$$

$$(\text{for all } a, x, y, z \in A)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z), \tag{1.14}$$

$$(\text{for all } x, y, z \in A)((x \cdot y) \cdot z \leq y \cdot z), \tag{1.15}$$

$$(\text{for all } x, y, z \in A)(x \leq y \Rightarrow x \leq z \cdot y), \tag{1.16}$$

$$(\text{for all } x, y, z \in A)((x \cdot y) \cdot z \leq x \cdot (y \cdot z)), \text{ and} \tag{1.17}$$

$$(\text{for all } a, x, y, z \in A)((x \cdot y) \cdot z \leq y \cdot (a \cdot z)). \tag{1.18}$$

Example 1.2. [20] Let U be a nonempty set and let $X \in \mathcal{P}(U)$ where $\mathcal{P}(U)$ means the power set of U. Let $\mathcal{P}_X(U) = \{A \in \mathcal{P}(U) \mid X \subseteq A\}$. Define a binary operation $\vartriangle$ on $\mathcal{P}_X(U)$ by putting $A \vartriangle B = B \cap (A^C \cup X)$ for all $A, B \in \mathcal{P}_X(U)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_X(U), \vartriangle, X)$ is a UP-algebra. Let $\mathcal{P}^X(U) = \{A \in \mathcal{P}(U) \mid A \subseteq X\}$. Define a binary operation $\blacktriangle$ on $\mathcal{P}^X(U)$ by putting $A \blacktriangle B = B \cup (A^C \cap X)$ for all $A, B \in \mathcal{P}^X(U)$. Then $(\mathcal{P}^X(U), \blacktriangle, X)$ is a UP-algebra.

Example 1.3. [4] Let $\mathbb{Z}^*$ be the set of all nonnegative integers. Define two binary operations $\circ$ and $\star$ on $\mathbb{Z}^*$ by:

$$(\text{for all } m, n \in \mathbb{Z}^*)\left(m \circ n = \begin{cases} n & \text{if } m < n, \\ 0 & \text{otherwise} \end{cases}\right)$$

and

$$(\text{for all } m, n \in \mathbb{Z}^*)\left(m \star n = \begin{cases} n & \text{if } m > n \text{ or } m = 0, \\ 0 & \text{otherwise} \end{cases}\right).$$

Then $(\mathbb{Z}^*, \circ, 0)$ and $(\mathbb{Z}^*, \star, 0)$ are UP-algebras.

For more examples of UP-algebras, see [1, 2, 7, 8, 19–22, 24].

2. UP-valued functions

First of all, we recall the definition of a UP-valued function on a nonempty set, which is introduced by Ansari et al. [3]. In what follows let X and A denote a nonempty set and a UP-algebra respectively, unless otherwise specified.

Definition 2.1. A mapping $\widetilde{X} : X \to A$ is called a *UP-valued function* on X.

Definition 2.2. A cut function of $\widetilde{X}$, for $a \in A$ is defined to be a mapping $\widetilde{X}_a : X \to \{0, 1\}$ such that

$$(\text{for all } x \in X)\left(\widetilde{X}_a(x) = \begin{cases} 1 & \text{if } \widetilde{X}(x) \cdot a = 0, \\ 0 & \text{otherwise} \end{cases}\right). \tag{2.1}$$

Equivalently,

$$(\text{for all } x \in X)\left(\widetilde{X}_a(x) = \begin{cases} 1 & \text{if } \widetilde{X}(x) \leq a, \\ 0 & \text{otherwise} \end{cases}\right). \tag{2.2}$$

Obviously, $\widetilde{X}_a$ is the characteristic function of the following subset of X, called a *cut subset* or an *a-cut* of $\widetilde{X}$:

$$X_a = \{x \in X \mid \widetilde{X}(x) \cdot a = 0\} = \{x \in X \mid \widetilde{X}(x) \leq a\}. \tag{2.3}$$

Then

$$X_0 = X \tag{2.4}$$

and

$$(\text{for all } x \in X)(x \in X_{\widetilde{X}(x)}). \tag{2.5}$$

By (2.1) and (2.3), we note that

$$X_a = \{x \in X \mid \widetilde{X}_a(x) = 1\}. \tag{2.6}$$

Example 2.3. Let $A = \{0, 1, 2, 3, 4, 5, 6\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table, as Figure 1:

$\cdot$	0	1	2	3	4	5	6
$\widetilde{X}(x) = 0$	0	1	2	3	4	5	6
1	0	0	2	3	2	3	6
$\widetilde{X}(y) = 2$	0	1	0	3	1	5	3
3	0	1	2	0	4	1	2
4	0	0	0	3	0	3	3
5	0	0	2	0	2	0	2
$\widetilde{X}(z) = 6$	0	1	0	0	1	1	0

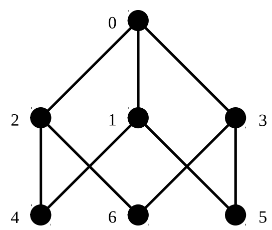

Figure 1. $(A, \leq)$.

Let $X = \{x, y, z\}$ and we define a UP-valued function $\widetilde{X}\colon X \to A$ on X by:

$$\widetilde{X} = \begin{pmatrix} x & y & z \\ 0 & 2 & 6 \end{pmatrix}.$$

Then all cut subsets of $\widetilde{X}$ are as follows:

$$X_0 = X, X_1 = \emptyset, X_2 = \{y, z\}, X_3 = \{z\}, X_4 = \emptyset, X_5 = \emptyset, \text{ and } X_6 = \{z\}.$$

Proposition 2.4. *Every UP-valued function $\widetilde{X}\colon X \to A$ on X is represented by the minimum of the set $\{q \in A \mid \widetilde{X}_q(x) = 1\}$ for all $x \in X$, that is,*

$$(\textit{for all } x \in X)(\widetilde{X}(x) = \min\{q \in A \mid \widetilde{X}_q(x) = 1\}). \tag{2.7}$$

Proof. Let $x \in X$. Then $\widetilde{X}(x) = r$ for some $r \in A$. By (1.6), we have $\widetilde{X}(x) \cdot r = 0$ and so $\widetilde{X}_r(x) = 1$. Thus $r \in \{q \in A \mid \widetilde{X}_q(x) = 1\}$. Let $q \in A$ be such that $\widetilde{X}_q(x) = 1$. Then $r \cdot q = \widetilde{X}(x) \cdot q = 0$, so $r \leq q$. Hence,

$$\widetilde{X}(x) = r = \min\{q \in A \mid \widetilde{X}_q(x) = 1\}.$$

□

Proposition 2.5. *Let $\widetilde{X}\colon X \to A$ be a UP-valued function on X. Then*

$$(\textit{for all } q, r \in A)(q \leq r \Rightarrow X_q \subseteq X_r). \tag{2.8}$$

Proof. Let $q, r \in A$ be such that $q \leq r$. Then $q \cdot r = 0$. Let $x \in X_q$. Then $\widetilde{X}(x) \cdot q = 0$. By (1.9) and (1.2), we have $0 = (q \cdot r) \cdot (\widetilde{X}(x) \cdot r) = 0 \cdot (\widetilde{X}(x) \cdot r) = \widetilde{X}(x) \cdot r$, that is, $x \in X_r$. Hence, $X_q \subseteq X_r$. □

The following example shows that the converse of (2.8) of Proposition 2.5 is not true in general.

Example 2.6. From Example 2.3, we have $X_5 = \emptyset \subseteq \{z\} = X_6$ but $5 \nleq 6$.

Corollary 2.7. *Let $\widetilde{X}\colon X \to A$ be a UP-valued function on X. Then*

$$(\textit{for all } x, y \in X)(\widetilde{X}(x) = \widetilde{X}(y) \Leftrightarrow X_{\widetilde{X}(x)} = X_{\widetilde{X}(y)}). \tag{2.9}$$

Proof. It is straightforward by Proposition 2.5, (1.6), (2.5), and (1.4). □

Corollary 2.8. *Let $\widetilde{X}\colon X \to A$ be a UP-valued function on X. Then*

$$(\textit{for all } x, y \in X)(\widetilde{X}(x) \leq \widetilde{X}(y) \Leftrightarrow X_{\widetilde{X}(x)} \subseteq X_{\widetilde{X}(y)}). \tag{2.10}$$

Proof. It is straightforward by Proposition 2.5 and (2.5). □

For a UP-valued function $\widetilde{X}\colon X \to A$ on X, consider the following sets:

$$X_A = \{X_a \mid a \in A\}$$

and

$$\widetilde{X}_A = \{\widetilde{X}_a \mid a \in A\}.$$

Proposition 2.9. *Let $\widetilde{X}\colon X \to A$ be a UP-valued function on X. Then*

$$(\textit{for all } Y \subseteq A, \inf Y \textit{ exists})(X_{\inf Y} = \bigcap_{y \in Y} X_y). \tag{2.11}$$

Proof. Let $Y \subseteq A$ be such that $\inf Y$ exists and let $x \in X$. Then

$$\begin{aligned} x \in X_{\inf Y} &\Leftrightarrow \widetilde{X}(x) \cdot \inf Y = 0 \\ &\Leftrightarrow (\text{for all } y \in Y)(\widetilde{X}(x) \cdot y = 0) && ((1.7)) \\ &\Leftrightarrow (\text{for all } y \in Y)(x \in X_y) \\ &\Leftrightarrow x \in \bigcap_{y \in Y} X_y. \end{aligned}$$

Hence, $X_{\inf Y} = \bigcap_{y \in Y} X_y$. □

Corollary 2.10. *Let $\widetilde{X}\colon X \to A$ be a UP-valued function on X. Then*

$$(\textit{for all } Y \subseteq A, \inf Y \textit{ exists})(\bigcap_{y\in Y} X_y \in X_A). \tag{2.12}$$

Proof. It is straightforward by Proposition 2.9. □

The following example shows that the result of Corollary 2.10 is not true in case of union operation.

Example 2.11. From Example 2.3, we define a new UP-valued function $\widetilde{X}\colon X \to A$ on X by:

$$\widetilde{X} = \begin{pmatrix} x & y & z \\ 1 & 2 & 3 \end{pmatrix}.$$

Then cut subsets of $\widetilde{X}$ are

$$X_0 = X, X_1 = \{x\}, X_2 = \{y\}, X_3 = \{z\}, X_4 = \emptyset, X_5 = \emptyset, \text{ and } X_6 = \emptyset.$$

Let $Y = \{1, 2\}$. Then $\inf Y$ exists and equal 4 but $X_1 \cup X_2 = \{x, y\} \notin X_A$.

Proposition 2.12. *Let $\widetilde{X}\colon X \to A$ be a UP-valued function on X. Then*

$$\bigcup_{a\in A} X_a = X. \tag{2.13}$$

$$(\textit{for all } x \in X)(\bigcup_{a\in A}\{X_a \mid x \in X_a\} = X). \tag{2.14}$$

Proof. It is straightforward by (2.4). □

For a UP-valued function $\widetilde{X}\colon X \to A$ on X, define the binary relation Θ on A by:

$$(\text{for all } a, b \in A)(a\Theta b \Leftrightarrow X_a = X_b). \tag{2.15}$$

Theorem 2.13. *Let $\widetilde{X}\colon X \to A$ be a UP-valued function on X. Then the binary relation Θ which is defined in* (2.15) *is an equivalence relation on A.*

Proof. Straightforward. □

If $x \in A$, then the Θ-class of x is the set $(x)_\Theta$ defined as follows:

$$(x)_\Theta = \{y \in A \mid x\Theta y\}.$$

We define two subsets of A by:

$$\operatorname{Im}(\widetilde{X}) = \widetilde{X}(X) = \{a \in A \mid \widetilde{X}(x) = a \text{ for some } x \in X\} \tag{2.16}$$

and

$$(\text{for all } b \in A)((b] = \{a \in A \mid a \cdot b = 0\} = \{a \in A \mid a \leq b\}). \tag{2.17}$$

By (1.4), we have the following assertions:

$$(\text{for all } a, b \in A)((a] = (b] \Leftrightarrow a = b). \tag{2.18}$$

Proposition 2.14. *Let* $\widetilde{X}\colon X \to A$ *be a UP-valued function on* X*. Then*

$$(\textit{for all } a, b \in A)(a\Theta b \Leftrightarrow (a] \cap \mathrm{Im}(\widetilde{X}) = (b] \cap \mathrm{Im}(\widetilde{X})). \tag{2.19}$$

In particular, if $\widetilde{X}$ *is surjective, then*

$$(\textit{for all } a, b \in A)(a\Theta b \Leftrightarrow (a] = (b] \Leftrightarrow a = b). \tag{2.20}$$

Proof. For all $a, b \in A$, we have

$$\begin{aligned} a\Theta b &\Leftrightarrow X_a = X_b \\ &\Leftrightarrow (\text{for all } x \in X)(\widetilde{X}(x) \cdot a = 0 \Leftrightarrow \widetilde{X}(x) \cdot b = 0) && ((2.3)) \\ &\Leftrightarrow \{x \in X \mid \widetilde{X}(x) \in (a]\} = \{x \in X \mid \widetilde{X}(x) \in (b]\} && ((2.17)) \\ &\Leftrightarrow (a] \cap \mathrm{Im}(\widetilde{X}) = (b] \cap \mathrm{Im}(\widetilde{X}). \end{aligned}$$

□

Example 2.15. From Example 2.3, we have all cut subsets of $\widetilde{X}$ are as follows:

$$X_0 = X, X_1 = \emptyset, X_2 = \{y, z\}, X_3 = \{z\}, X_4 = \emptyset, X_5 = \emptyset, \text{ and } X_6 = \{z\}.$$

Then all cut functions of $\widetilde{X}$ are as follows:

$\cdot$	x	y	z
$\widetilde{X_0}$	1	1	1
$\widetilde{X_1}$	0	0	0
$\widetilde{X_2}$	0	1	1
$\widetilde{X_3}$	0	0	1
$\widetilde{X_4}$	0	0	0
$\widetilde{X_5}$	0	0	0
$\widetilde{X_6}$	0	0	1

3. Codewords generated by UP-valued functions

In this section, we establish codewords in a binary block-code generated by a UP-valued function. Finally, we prove that every finite UP-algebra which has the order less than or equal to the order of a finite set determines a binary block-code which is isomorphic to it.

Lemma 3.1. *Let* $\widetilde{X}\colon X \to A$ *be a UP-valued function on* X*. Then*

$$(\textit{for all } x \in X)(\widetilde{X}(x) = \max(\widetilde{X}(x))_\Theta \cap \mathrm{Im}(\widetilde{X})). \tag{3.1}$$

In particular, if $\widetilde{X}$ *is surjective, then*

$$(\textit{for all } x \in X)(\widetilde{X}(x) = \max(\widetilde{X}(x))_\Theta). \tag{3.2}$$

Proof. Let $x \in X$. Then $\widetilde{X}(x) \in (\widetilde{X}(x))_\Theta \cap \mathrm{Im}(\widetilde{X})$. Let $a \in (\widetilde{X}(x))_\Theta \cap \mathrm{Im}(\widetilde{X})$. By Proposition 2.14, we have $a \in (a] \cap \mathrm{Im}(\widetilde{X}) = (\widetilde{X}(x)] \cap \mathrm{Im}(\widetilde{X})$. Thus $a \in (\widetilde{X}(x)]$, that is, $a \leq \widetilde{X}(x)$. Hence, $\widetilde{X}(x) = \max(\widetilde{X}(x))_\Theta \cap \mathrm{Im}(\widetilde{X})$. □

Let X be a nonempty set with n elements. We consider $X = \{1, 2, 3, \ldots, n\}$ and let A be a UP-algebra. For each UP-valued function $\widetilde{X}\colon X \to A$ on X, we can define a binary block-code V of length n in the following way: Each Θ-class $(a)_\Theta$ where $a \in A$, will corresponds to a codeword $w_a = a_1a_2a_3 \ldots a_n$ with

$$(\text{for all } i \in X, j \in \{0, 1\})(a_i = j \Leftrightarrow \widetilde{X}_a(i) = j). \tag{3.3}$$

We observe that

$$(\text{for all } a, b \in A)((a)_\Theta = (b)_\Theta \Leftrightarrow w_a = w_b). \tag{3.4}$$

Indeed,

$$\begin{aligned}(a)_\Theta = (b)_\Theta &\Leftrightarrow X_a = X_b \\ &\Leftrightarrow \{i \in X \mid \widetilde{X}_a(i) = 1\} = \{i \in X \mid \widetilde{X}_b(i) = 1\} && ((2.6)) \\ &\Leftrightarrow (\text{for all } i \in X)(a_i = b_i) \\ &\Leftrightarrow w_a = w_b.\end{aligned}$$

Let $w_a = a_1a_2a_3 \ldots a_n$ and $w_b = b_1b_2b_3 \ldots b_n$ be two codewords belonging to a binary block-code V. Define an order relation $\preceq$ on the set of codewords belonging to a binary block-code V as follows:

$$w_a \preceq w_b \Leftrightarrow \text{for all } i \in X, a_i \leq b_i. \tag{3.5}$$

Example 3.2. From Example 2.3, we have all cut subsets of $\widetilde{X}$ are as follows:

$$X_0 = X, X_1 = \emptyset, X_2 = \{y, z\}, X_3 = \{z\}, X_4 = \emptyset, X_5 = \emptyset, \text{ and } X_6 = \{z\}.$$

Then the equivalence relation Θ on A is as follows:

$$\Theta = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (1, 5), (5, 1), (4, 5), (5, 4), (3, 6), (6, 3)\}.$$

From Example 2.15, we have all distinct codewords of the binary block-code V are as follows (see Figure 2):

$$w_0 = 111, w_1 = w_4 = w_5 = 000, w_2 = 011, \text{ and } w_3 = w_6 = 001.$$

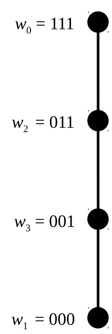

Figure 2. $(V, \preceq)$.

From Figures 1 and 2, we conclude that $(A, \leq)$ is not isomorphic to $(V, \preceq)$.

The following example will lead to the next important theorem.

Example 3.3. Let $A = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table, as Figure 3:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

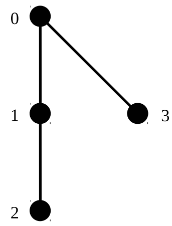

Figure 3. $(A, \leq)$.

Let $\widetilde{A}: A \to A$ be the identity UP-valued function on A. Then all cut subsets of $\widetilde{X}$ are as follows:

$$A_0 = A, A_1 = \{1, 2\}, A_2 = \{2\}, \text{ and } A_3 = \{3\}.$$

Thus all cut functions of $\widetilde{A}$ are as follows:

	0	1	2	3
$\widetilde{A_0}$	1	1	1	1
$\widetilde{A_1}$	0	1	1	0
$\widetilde{A_2}$	0	0	1	0
$\widetilde{A_3}$	0	0	0	1

and the equivalence relation Θ on A is as follows:

$$\Theta = \{(0, 0), (1, 1), (2, 2), (3, 3)\}.$$

Hence, all distinct codewords of the binary block-code V are as follows (see Figure 4):

$$w_0 = 1111, w_1 = 0110, w_2 = 0010, \text{ and } w_3 = 0001.$$

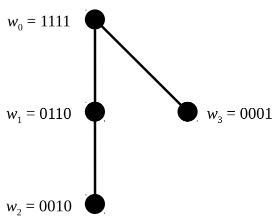

Figure 4. $(V, \leq)$.

From Figures 3 and 4, we conclude that $(A, \leq)$ is isomorphic to $(V, \leq)$ under the isomorphism sending $a \mapsto w_a$. In addition, the error pattern $e = 1000$ can be detected because $w_0 + e = 1111 + 1000 = 0111 \notin V, w_1 + e = 0110 + 1000 = 1110 \notin V, w_2 + e = 0010 + 1000 = 1010 \notin V$, and $w_3 + e = 0001 + 1000 = 1001 \notin V$. Hence, V detects e.

Theorem 3.4. *Every finite UP-algebra A which is equipotent to a nonempty set X determines a binary block-code V such that $(A, \leq)$ is isomorphic to $(V, \leq)$.*

Proof. Let $A = \{0, 1, 2, \ldots, n\}$ be a finite UP-algebra in which 0 is the maximum element, $X = \{x_0, x_1, x_2 \ldots, x_n\}$ and let $\widetilde{X}: X \to A$ be a bijective UP-valued function on X sending $x_a \mapsto a$. By (2.20) of Proposition 2.14 and (2.18), we have

$$(\text{for all } a \in A)((a)_\Theta = \{b \in A \mid (a] = (b]\} = \{a\}). \tag{3.6}$$

Thus $\Theta = \{(a, a) \mid a \in A\}$. By (3.4), we have all codewords w_a of the binary block-code V are distinct. Let $f: A \to V$ be a function defined by:

$$(\text{for all } a \in A)(f(a) = w_a).$$

Clearly, f is surjective. By (3.4) and (3.6), we have f is injective. Thus f is bijective. Let $a, b \in A$ be such that $a \leq b$. By (2.8) of Proposition 2.5, we have $X_a \subseteq X_b$. This means that $w_a \leq w_b$, that is, $f(a) \leq f(b)$. Conversely, let $a, b \in A$ be such that $f(a) \leq f(b)$. Then $w_a \leq w_b$, so $X_a \subseteq X_b$. By (2.5), we have $x_a \in X_{\widetilde{X}(x_a)} = X_a \subseteq X_b$, that is, $a = \widetilde{X}(x_a) \leq b$. Hence, $(A, \leq)$ is isomorphic to $(V, \leq)$. □

Corollary 3.5. *Every finite UP-algebra A determines a binary block-code V such that $(A, \leq)$ is isomorphic to $(V, \leq)$.*

Corollary 3.6. *Every finite UP-algebra A which has the order less than or equal to the order of a finite set X determines a binary block-code V such that $(A, \leq)$ is isomorphic to $(V, \leq)$.*

Proof. Let $A = \{0, 1, 2, \ldots, n\}$ be a finite UP-algebra in which 0 is the maximum element, $X = \{x_0, x_1, x_2, \ldots, x_m\}$ for $m \geq n$ and let $\widetilde{X}: X \to A$ be a UP-valued function on X defined by:

$$\widetilde{X} = \begin{pmatrix} x_0 & x_1 & x_2 & \ldots & x_n & x_{n+1} & x_{n+2} & & x_m \\ 0 & 1 & 2 & \ldots & n & n & n & & n \end{pmatrix}.$$

The proof is also given in a similar way of the proof of Theorem 3.4. Hence, $(A, \leq)$ is isomorphic to $(V, \leq)$. □

It is not necessary for $(A, \leq)$ and $(V, \leq)$ to be isomorphic under the identity UP-valued function on A, which shown by the following example.

Example 3.7. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a UP-algebra with a fixed element 0 and a binary operation · defined by the following Cayley table, as Figure 5:

$\cdot$	0	1	2	3	4	5	6	7
$\widetilde{A}(1) = 0$	0	1	2	3	4	5	6	7
$\widetilde{A}(0) = 1$	0	0	0	0	0	0	0	0
$\widetilde{A}(7) = 2$	0	7	0	7	7	0	0	7
$\widetilde{A}(6) = 3$	0	6	6	0	6	0	6	0
$\widetilde{A}(5) = 4$	0	5	5	5	0	5	0	0
$\widetilde{A}(4) = 5$	0	4	6	7	4	0	6	7
$\widetilde{A}(3) = 6$	0	3	5	3	7	5	0	7
$\widetilde{A}(2) = 7$	0	2	2	5	6	5	6	0

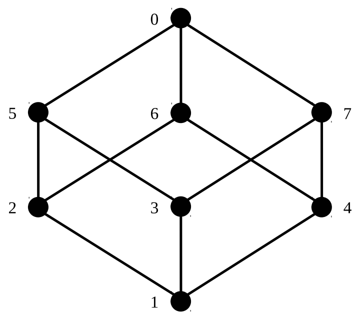

Figure 5. $(A, \leq)$.

Let $\widetilde{A}\colon A \to A$ be a UP-valued function on A defined by:

$$\widetilde{A} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 7 & 6 & 5 & 4 & 3 & 2 \end{pmatrix}.$$

Then all cut subsets of $\widetilde{A}$ are as follows:

$A_0 = A, A_1 = \{0\}, A_2 = \{0, 7\}, A_3 = \{0, 6\}, A_4 = \{0, 5\}, A_5 = \{0, 4, 6, 7\}, A_6 = \{0, 3, 5, 7\},$ and $A_7 = \{0, 2, 5, 6\}$.

Thus all cut functions of $\widetilde{A}$ are as follows:

	0	1	2	3	4	5	6	7
$\widetilde{A_0}$	1	1	1	1	1	1	1	1
$\widetilde{A_1}$	1	0	0	0	0	0	0	0
$\widetilde{A_2}$	1	0	0	0	0	0	0	1
$\widetilde{A_3}$	1	0	0	0	0	0	1	0
$\widetilde{A_4}$	1	0	0	0	0	1	0	0
$\widetilde{A_5}$	1	0	0	0	1	0	1	1
$\widetilde{A_6}$	1	0	0	1	0	1	0	1
$\widetilde{A_7}$	1	0	1	0	0	1	1	0

and the equivalence relation Θ on A is as follows:

$$\Theta = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7)\}.$$

Hence, all distinct codewords of the binary block-code V are as follows (see Figure 6):

$$w_0 = 11111111, w_1 = 10000000, w_2 = 10000001, w_3 = 10000010, w_4 = 10000100, w_5 = 10001011, w_6 = 10010101, \text{ and } w_7 = 10100110.$$

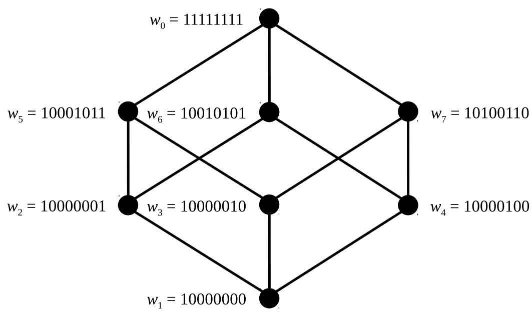

Figure 6. $(V, \leq)$.

From Figures 7 and 8, we conclude that $(A, \leq)$ is isomorphic to $(V, \leq)$ under the isomorphism sending $a \mapsto w_a$.

The following last example supports Corollary 3.8.

Example 3.8. Let $A = \{0, 1, 2, 3\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table, as Figure 7:

$\cdot$	0	1	2	3
$\widetilde{X}(u) = 0$	0	1	2	3
$\widetilde{X}(v) = 1$	0	0	2	3
$\widetilde{X}(w) = \widetilde{X}(x) = 2$	0	1	0	3
$\widetilde{X}(y) = \widetilde{X}(z) = 3$	0	1	2	0

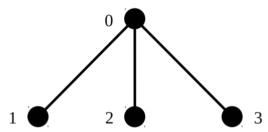

Figure 7. $(A, \leq)$.

Let $X = \{u, v, w, x, y, z\}$ and we define a UP-valued function $\widetilde{X} \colon X \to A$ on X by:

$$\widetilde{X} = \begin{pmatrix} u & v & w & x & y & z \\ 0 & 1 & 2 & 2 & 3 & 3 \end{pmatrix}.$$

Then all cut subsets of $\widetilde{X}$ are as follows:

$$X_0 = X, X_1 = \{v\}, X_2 = \{w, x\}, \text{ and } X_3 = \{y, z\}.$$

Thus all cut functions of $\widetilde{X}$ are as follows:

	u	v	w	x	y	z
$\widetilde{X_0}$	1	1	1	1	1	1
$\widetilde{X_1}$	0	1	0	0	0	0
$\widetilde{X_2}$	0	0	1	1	0	0
$\widetilde{X_3}$	0	0	0	0	1	1

and the equivalence relation Θ on A is as follows:

$$\Theta = \{(0,0),(1,1),(2,2),(3,3)\}.$$

Hence, all distinct codewords of the binary block-code V are as follows (see Figure 8):

$$w_0 = 111111, w_1 = 010000, w_2 = 001100, \text{ and } w_3 = 000011.$$

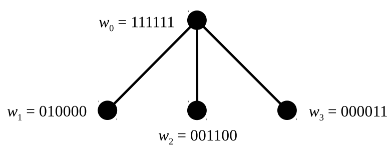

Figure 8. $(V, \leq)$.

From Figures 7 and 8, we conclude that $(A, \leq)$ is isomorphic to $(V, \leq)$ under the isomorphism sending $a \mapsto w_a$. In addition, V has the minimum distance 3. This means that can correct at most 1-error. For example, if $w_3 = 000011$ is sent and 000111 is received, then 000111 will be decoded to $w_3 = 000011$. If $w_3 = 000011$ is sent and 010111 is received, then 010111 will be decoded to $w_1 = 010000$ using the minimum distance decoding rule.

4. Conclusions

Codewords in a binary block-code generated by a UP-valued function are established and some interesting results are obtained. The main result is proved that every finite UP-algebra A which has the order less than or equal to the order of a finite set X determines a binary block-code V such that $(A, \leq)$ is isomorphic to $(V, \leq)$. Many examples were provided to support the results.

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Conflict of interest

The authors declare no conflict of interest.

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