Research article

Codewords generated by UP-valued functions

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Abstract: The concept of a UP-valued function on a nonempty set was introduced by Ansari et al. [3]. Codewords in a binary block-code generated by a UP-valued function are established and some interesting results are obtained. Finally, we prove that every finite UP-algebra $A$ which has the order less than or equal to the order of a finite set $X$ determines a binary block-code $V$ such that $(A, \leq)$ is isomorphic to $(V, \preceq)$.

Keywords: UP-algebra; UP-valued function; cut function; codeword; binary block-code
Mathematics Subject Classification: 06F35, 03G25, 94B05

1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [9], BCI-algebras [10], BE-algebras [12], UP-algebras [6], extension of KU/UP-algebras [18] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [10] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [9, 10] in 1966 and have been extensively investigated by many researchers.

Jun and Song [11] said the following: In computer science, a block code is a type of channel coding. It adds redundancy to a message so that, at the receiver, one can decode with minimal (theoretically zero) errors, provided that the information rate would not exceed the channel capacity. The main characterization of a block code is that it is a fixed length channel code (unlike source coding schemes such as Huffman coding, and unlike channel coding methods like convolutional encoding). Typically, a block code takes a $k$-digit information word, and transforms this into an $n$-digit codeword. Block
Coding is the primary type of channel coding used in earlier mobile communication systems. A block code is a code which encodes strings formed an alphabet set $S$ into code words by encoding each letter of $S$ separately.

Coding theory was applied to BCK-algebras in 2011 by Jun and Song [11] and in 2015 by Flaut [5]. They proved that every finite BCK-algebra determines a binary block-code. In 2015, Mostafa et al. [15] applied coding theory to KU-algebras and gave some relation and connection between binary block-codes and KU-algebras. They proved that every finite KU-algebra determines a binary block-code. In 2015, Mostafa et al. [15] applied coding theory to KU-algebras and gave some relation and connection between binary block-codes and KU-algebras. They proved that every finite KU-algebra determines a binary block-code. Moreover, they have introduced and studied UP-valued functions in [3]. For many studies of KU-algebras, see [14, 16, 23, 25].

In this paper, we establish binary block-codes by using the concept of UP-valued functions, introduced by Ansari et al. [3]. We show that every finite UP-algebra $A$ which has the order less than or equal to the order of a finite set $X$ determines a binary block-code $V$ such that $(A, \leq)$ is isomorphic to $(V, \leq)$.

Before we begin our study, let’s review the definition of UP-algebras.

**Definition 1.1.** [6] An algebra $A = (A, \cdot, 0)$ of type $(2,0)$ is called a **UP-algebra**, where $A$ is a nonempty set, $\cdot$ is a binary operation on $A$, and 0 is a fixed element of $A$ (i.e., a nullary operation) if it satisfies the following axioms:

\[
\begin{align*}
&(\text{for all } x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z))) = 0), \\
&(\text{for all } x \in A)(0 \cdot x = x), \\
&(\text{for all } x \in A)(x \cdot 0 = 0), \text{ and} \\
&(\text{for all } x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y).
\end{align*}
\]

From [6], we know that the concept of UP-algebras is a generalization of KU-algebras (see [17]). The binary relation $\leq$ on a UP-algebra $A = (A, \cdot, 0)$ defined as follows:

\[(\text{for all } x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0)\]

and the following assertions are valid (see [6, 7]).

\[
\begin{align*}
&(\text{for all } x \in A)(x \leq x), \\
&(\text{for all } x, y, z \in A)(x \leq y, y \leq z \Rightarrow x \leq z), \\
&(\text{for all } x, y, z \in A)(x \leq y \Rightarrow z \cdot x \leq z \cdot y), \\
&(\text{for all } x, y, z \in A)(x \leq y \Rightarrow y \cdot z \leq x \cdot z), \\
&(\text{for all } x, y, z \in A)(x \leq y \cdot x, \text{ in particular, } y \cdot z \leq x \cdot (y \cdot z)), \\
&(\text{for all } x, y \in A)(y \cdot x \leq x \Leftrightarrow x = y \cdot x), \\
&(\text{for all } x, y \in A)(x \leq y \cdot y),
\end{align*}
\]
(for all $a, x, y, z \in A)(x \cdot (y \cdot z) \leq x \cdot ((a \cdot y) \cdot (a \cdot z)))$, \hspace{1cm} (1.13)

(for all $a, x, y, z \in A)(((a \cdot x) \cdot (a \cdot y)) \cdot z \leq (x \cdot y) \cdot z)$, \hspace{1cm} (1.14)

(for all $x, y, z \in A)((x \cdot y) \cdot z \leq y \cdot z)$, \hspace{1cm} (1.15)

(for all $x, y, z \in A)(x \leq y \Rightarrow x \leq z \cdot y)$, \hspace{1cm} (1.16)

(for all $x, y, z \in A)((x \cdot y) \cdot z \leq y \cdot (a \cdot z))$, \hspace{1cm} (1.17)

(1.18)

**Example 1.2.** [20] Let $U$ be a nonempty set and let $X \in \mathcal{P}(U)$ where $\mathcal{P}(U)$ means the power set of $U$. Let $\mathcal{P}_X(U) = \{A \in \mathcal{P}(U) \mid X \subseteq A\}$. Define a binary operation $\triangle$ on $\mathcal{P}_X(U)$ by putting $A \triangle B = B \cap (A^C \cup X)$ for all $A, B \in \mathcal{P}_X(U)$ where $A^C$ means the complement of a subset $A$. Then $(\mathcal{P}_X(U), \triangle, X)$ is a UP-algebra. Let $\mathcal{P}^X(U) = \{A \in \mathcal{P}(U) \mid A \subseteq X\}$. Define a binary operation $\blacklozenge$ on $\mathcal{P}^X(U)$ by putting $A\blacklozenge B = B \cup (A^C \cap X)$ for all $A, B \in \mathcal{P}^X(U)$. Then $(\mathcal{P}^X(U), \blacklozenge, X)$ is a UP-algebra.

**Example 1.3.** [4] Let $Z^*$ be the set of all nonnegative integers. Define two binary operations $\circ$ and $\star$ on $Z^*$ by:

\[
(m \circ n) = \begin{cases} 
 n & \text{if } m < n, \\
 0 & \text{otherwise}
\end{cases}
\]

and

\[
(m \star n) = \begin{cases} 
 n & \text{if } m > n \text{ or } m = 0, \\
 0 & \text{otherwise}
\end{cases}
\]

Then $(Z^*, \circ, 0)$ and $(Z^*, \star, 0)$ are UP-algebras.

For more examples of UP-algebras, see [1, 2, 7, 8, 19–22, 24].

2. **UP-valued functions**

First of all, we recall the definition of a UP-valued function on a nonempty set, which is introduced by Ansari et al. [3]. In what follows let $X$ and $A$ denote a nonempty set and a UP-algebra respectively, unless otherwise specified.

**Definition 2.1.** A mapping $\overline{X} : X \to A$ is called a *UP-valued function* on $X$.

**Definition 2.2.** A cut function of $\overline{X}$, for $a \in A$ is defined to be a mapping $\overline{X}_a : X \to \{0, 1\}$ such that

\[
(\text{for all } x \in X) \begin{cases} 
 1 & \text{if } \overline{X}(x) \cdot a = 0, \\
 0 & \text{otherwise}
\end{cases}
\]

Equivalently,

\[
(\text{for all } x \in X) \begin{cases} 
 1 & \text{if } \overline{X}(x) \leq a, \\
 0 & \text{otherwise}
\end{cases}
\]
Obviously, \( \widetilde{X}_a \) is the characteristic function of the following subset of \( X \), called a cut subset or an \( a \)-cut of \( \widetilde{X} \):

\[
X_a = \{ x \in X \mid \widetilde{X}(x) \cdot a = 0 \} = \{ x \in X \mid \widetilde{X}(x) \leq a \}.
\] (2.3)

Then

\[
X_0 = X
\] (2.4)

and

\[
(\text{for all } x \in X)(x \in X_{\widetilde{X}(x)}).
\] (2.5)

By (2.1) and (2.3), we note that

\[
X_a = \{ x \in X \mid \widetilde{X}_a(x) = 1 \}.
\] (2.6)

**Example 2.3.** Let \( A = \{0, 1, 2, 3, 4, 5, 6\} \) be a UP-algebra with a fixed element 0 and a binary operation \( \cdot \) defined by the following Cayley table, as Figure 1:

\[
\begin{array}{c|ccccccc}
\cdot & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\widetilde{X}(x) = 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0 & 2 & 3 & 2 & 3 & 6 \\
\widetilde{X}(y) = 2 & 0 & 1 & 0 & 3 & 1 & 5 & 3 \\
3 & 0 & 1 & 2 & 0 & 4 & 1 & 2 \\
4 & 0 & 0 & 0 & 3 & 0 & 3 & 3 \\
5 & 0 & 0 & 2 & 0 & 2 & 0 & 2 \\
\widetilde{X}(z) = 6 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\end{array}
\]

![Figure 1. \((A, \leq)\).](image)

Let \( X = \{x, y, z\} \) and we define a UP-valued function \( \widetilde{X}: X \rightarrow A \) on \( X \) by:

\[
\widetilde{X} = \begin{pmatrix}
x & y & z \\
0 & 2 & 6
\end{pmatrix}.
\]

Then all cut subsets of \( \widetilde{X} \) are as follows:

\[
X_0 = X, X_1 = \emptyset, X_2 = \{y, z\}, X_3 = \{z\}, X_4 = \emptyset, X_5 = \emptyset, \text{ and } X_6 = \{z\}.
\]

**Proposition 2.4.** Every UP-valued function \( \widetilde{X}: X \rightarrow A \) on \( X \) is represented by the minimum of the set \( \{q \in A \mid \widetilde{X}_q(x) = 1\} \) for all \( x \in X \), that is,

\[
(\text{for all } x \in X)(\widetilde{X}(x) = \min\{q \in A \mid \widetilde{X}_q(x) = 1\}).
\] (2.7)
Proof. Let \( x \in X \). Then \( \tilde{X}(x) = r \) for some \( r \in A \). By (1.6), we have \( \tilde{X}(x) \cdot r = 0 \) and so \( \tilde{X}(x) = 1 \). Thus \( r \in \{ q \in A \mid \tilde{X}_q(x) = 1 \} \). Let \( q \in A \) be such that \( \tilde{X}_q(x) = 1 \). Then \( r \cdot q = \tilde{X}(x) \cdot q = 0 \), so \( r \leq q \). Hence,
\[
\tilde{X}(x) = r = \min\{ q \in A \mid \tilde{X}_q(x) = 1 \}.
\]
\[\square\]

**Proposition 2.5.** Let \( \tilde{X} : X \to A \) be a UP-valued function on \( X \). Then
\[
(\text{for all } q, r \in A)(q \leq r \Rightarrow X_q \subseteq X_r) \tag{2.8}
\]

Proof. Let \( q, r \in A \) be such that \( q \leq r \). Then \( q \cdot r = 0 \). Let \( x \in X_q \). Then \( \tilde{X}(x) \cdot q = 0 \). By (1.9) and (1.2), we have 0 = \((q \cdot r) \cdot (\tilde{X}(x) \cdot r) = 0 \cdot (\tilde{X}(x) \cdot r) = \tilde{X}(x) \cdot r \), that is, \( x \in X_r \). Hence, \( X_q \subseteq X_r \). \[\square\]

The following example shows that the converse of (2.8) of Proposition 2.5 is not true in general.

**Example 2.6.** From Example 2.3, we have \( X_5 = \emptyset \subseteq \{z\} = X_6 \) but \( 5 \not\sim 6 \).

**Corollary 2.7.** Let \( \tilde{X} : X \to A \) be a UP-valued function on \( X \). Then
\[
(\text{for all } x, y \in X)(\tilde{X}(x) = \tilde{X}(y) \Leftrightarrow X_{\tilde{X}(x)} = X_{\tilde{X}(y)}) \tag{2.9}
\]

Proof. It is straightforward by Proposition 2.5, (1.6), (2.5), and (1.4). \[\square\]

**Corollary 2.8.** Let \( \tilde{X} : X \to A \) be a UP-valued function on \( X \). Then
\[
(\text{for all } x, y \in X)(\tilde{X}(x) \leq \tilde{X}(y) \Leftrightarrow X_{\tilde{X}(x)} \subseteq X_{\tilde{X}(y)}) \tag{2.10}
\]

Proof. It is straightforward by Proposition 2.5 and (2.5). \[\square\]

For a UP-valued function \( \tilde{X} : X \to A \) on \( X \), consider the following sets:
\[
X_A = \{ X_a \mid a \in A \}
\]
and
\[
\tilde{X}_A = \{ \tilde{X}_a \mid a \in A \}.
\]

**Proposition 2.9.** Let \( \tilde{X} : X \to A \) be a UP-valued function on \( X \). Then
\[
(\text{for all } Y \subseteq A, \inf Y \exists)(X_{\inf Y} = \bigcap_{y \in Y} X_y) \tag{2.11}
\]

Proof. Let \( Y \subseteq A \) be such that \( \inf Y \exists \) and let \( x \in X \). Then
\[
x \in X_{\inf Y} \Leftrightarrow \tilde{X}(x) \cdot \inf Y = 0
\]
\[
\Leftrightarrow (\text{for all } y \in Y)(\tilde{X}(x) \cdot y = 0) \tag{(1.7)}
\]
\[
\Leftrightarrow (\text{for all } y \in Y)(x \in X_y)
\]
\[
\Leftrightarrow x \in \bigcap_{y \in Y} X_y.
\]
Hence, \( X_{\inf Y} = \bigcap_{y \in Y} X_y \). \[\square\]
Corollary 2.10. Let $\tilde{X}: X \to A$ be a UP-valued function on $X$. Then
\[
(\text{for all } Y \subseteq A, \inf Y \text{ exists})\left(\bigcap_{y \in Y} X_y \in X_A\right).
\] (2.12)

Proof. It is straightforward by Proposition 2.9. \qed

The following example shows that the result of Corollary 2.10 is not true in case of union operation.

Example 2.11. From Example 2.3, we define a new UP-valued function $\tilde{X}: X \to A$ on $X$ by:
\[
\tilde{X} = \begin{pmatrix} x & y & z \\ 1 & 2 & 3 \end{pmatrix}.
\]
Then cut subsets of $\tilde{X}$ are
\[
X_0 = X, X_1 = \{x\}, X_2 = \{y\}, X_3 = \{z\}, X_4 = \emptyset, X_5 = \emptyset, \text{ and } X_6 = \emptyset.
\]
Let $Y = \{1, 2\}$. Then $\inf Y$ exists and equal 4 but $X_1 \cup X_2 = \{x, y\} \notin X_A$.

Proposition 2.12. Let $\tilde{X}: X \to A$ be a UP-valued function on $X$. Then
\[
\bigcup_{a \in A} X_a = X.
\] (2.13)

\[
(\text{for all } x \in X)\left(\bigcup_{a \in A} \{X_a \mid x \in X_a\} = X\right).
\] (2.14)

Proof. It is straightforward by (2.4). \qed

For a UP-valued function $\tilde{X}: X \to A$ on $X$, define the binary relation $\Theta$ on $A$ by:
\[
(\text{for all } a, b \in A)(a \Theta b \iff X_a = X_b).
\] (2.15)

Theorem 2.13. Let $\tilde{X}: X \to A$ be a UP-valued function on $X$. Then the binary relation $\Theta$ which is defined in (2.15) is an equivalence relation on $A$.

Proof. Straightforward. \qed

If $x \in A$, then the $\Theta$-class of $x$ is the set $(x)_\Theta$ defined as follows:
\[
(x)_\Theta = \{y \in A \mid x \Theta y\}.
\]

We define two subsets of $A$ by:
\[
\text{Im}(\tilde{X}) = \tilde{X}(X) = \{a \in A \mid \tilde{X}(x) = a \text{ for some } x \in X\}
\] (2.16)

and
\[
(\text{for all } b \in A)(b) = \{a \in A \mid a \cdot b = 0\} = \{a \in A \mid a \leq b\}.
\] (2.17)

By (1.4), we have the following assertions:
\[
(\text{for all } a, b \in A)((a) = (b) \iff a = b).
\] (2.18)
Proposition 2.14. Let \( \tilde{X} : X \to A \) be a UP-valued function on \( X \). Then

\[
\text{(for all } a, b \in A \text{)} (a \Theta b \Leftrightarrow (a] \cap \text{Im}(\tilde{X})) = (b] \cap \text{Im}(\tilde{X})).
\]

(2.19)

In particular, if \( \tilde{X} \) is surjective, then

\[
\text{(for all } a, b \in A \text{)} (a \Theta b \Leftrightarrow (a] = (b] \Leftrightarrow a = b).
\]

(2.20)

Proof. For all \( a, b \in A \), we have

\[
a \Theta b \Leftrightarrow X_a = X_b
\]

\[
\Leftrightarrow (\text{for all } x \in X)(\tilde{X}(x) \cdot a = 0 \Leftrightarrow \tilde{X}(x) \cdot b = 0)
\]

(2.3)

\[
\Leftrightarrow \{x \in X \mid \tilde{X}(x) \in (a]\} = \{x \in X \mid \tilde{X}(x) \in (b]\}
\]

(2.17)

\[
\Leftrightarrow (a] \cap \text{Im}(\tilde{X}) = (b] \cap \text{Im}(\tilde{X}).
\]

\[\square\]

Example 2.15. From Example 2.3, we have all cut subsets of \( \tilde{X} \) are as follows:

\[
X_0 = X, X_1 = \emptyset, X_2 = \{y, z\}, X_3 = \{z\}, X_4 = \emptyset, X_5 = \emptyset, \text{ and } X_6 = \{z\}.
\]

Then all cut functions of \( \tilde{X} \) are as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{X}_0 )</td>
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<td>1</td>
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<tr>
<td>( \tilde{X}_1 )</td>
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<td>0</td>
</tr>
<tr>
<td>( \tilde{X}_2 )</td>
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<tr>
<td>( \tilde{X}_3 )</td>
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<tr>
<td>( \tilde{X}_4 )</td>
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<tr>
<td>( \tilde{X}_5 )</td>
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<td>0</td>
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<tr>
<td>( \tilde{X}_6 )</td>
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<td>0</td>
</tr>
</tbody>
</table>

3. Codewords generated by UP-valued functions

In this section, we establish codewords in a binary block-code generated by a UP-valued function. Finally, we prove that every finite UP-algebra which has the order less than or equal to the order of a finite set determines a binary block-code which is isomorphic to it.

Lemma 3.1. Let \( \tilde{X} : X \to A \) be a UP-valued function on \( X \). Then

\[
\text{(for all } x \in X \text{)} (\tilde{X}(x) = \max(\tilde{X}(x))_\Theta \cap \text{Im}(\tilde{X})).
\]

(3.1)

In particular, if \( \tilde{X} \) is surjective, then

\[
\text{(for all } x \in X \text{)} (\tilde{X}(x) = \max(\tilde{X}(x))_\Theta).
\]

(3.2)
Proof. Let \( x \in X \). Then \( \tilde{X}(x) \in (\tilde{X}(x))_\Theta \cap \text{Im}(\tilde{X}) \). Let \( a \in (\tilde{X}(x))_\Theta \cap \text{Im}(\tilde{X}) \). By Proposition 2.14, we have \( a \in (a) \cap \text{Im}(\tilde{X}) = (\tilde{X}(x)] \cap \text{Im}(\tilde{X}) \). Thus \( a \in (\tilde{X}(x)) \), that is, \( a \leq \tilde{X}(x) \). Hence, \( \tilde{X}(x) = \max(\tilde{X}(x))_\Theta \cap \text{Im}(\tilde{X}) \).

Let \( X \) be a nonempty set with \( n \) elements. We consider \( X = \{1, 2, 3, \ldots, n\} \) and let \( A \) be a UP-algebra. For each UP-valued function \( \tilde{X} : X \to A \) on \( X \), we can define a binary block-code \( V \) of length \( n \) in the following way: Each \( \Theta \)-class \( (a)_\Theta \) where \( a \in A \), will corresponds to a codeword \( w_a = a_1a_2a_3 \ldots a_n \) with

\[
\begin{align*}
\text{(for all } i \in X, j \in \{0, 1\}(a_i = j \iff \tilde{X}_a(i) = j). \quad (3.3)
\end{align*}
\]

We observe that

\[
\begin{align*}
\text{(for all } a, b \in A)((a)_\Theta = (b)_\Theta \iff w_a = w_b). \quad (3.4)
\end{align*}
\]

Indeed,

\[
\begin{align*}
(a)_\Theta = (b)_\Theta \iff X_a = X_b
\iff [i \in X | \tilde{X}_a(i) = 1] = [i \in X | \tilde{X}_b(i) = 1]
\iff (\text{for all } i \in X)(a_i = b_i)
\iff w_a = w_b.
\end{align*}
\]

Let \( w_a = a_1a_2a_3 \ldots a_n \) and \( w_b = b_1b_2b_3 \ldots b_n \) be two codewords belonging to a binary block-code \( V \). Define an order relation \( \leq \) on the set of codewords belonging to a binary block-code \( V \) as follows:

\[
\begin{align*}
w_a \leq w_b \iff \text{for all } i \in X, a_i \leq b_i. \quad (3.5)
\end{align*}
\]

Example 3.2. From Example 2.3, we have all cut subsets of \( \tilde{X} \) are as follows:

\[
X_0 = X, X_1 = \emptyset, X_2 = \{y, z\}, X_3 = \{z\}, X_4 = \emptyset, X_5 = \emptyset, \text{ and } X_6 = \{z\}.
\]

Then the equivalence relation \( \Theta \) on \( A \) is as follows:

\[
\Theta = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 4), (4, 1), (5, 1), (4, 5), (5, 4), (3, 6), (6, 3)\}.
\]

From Example 2.15, we have all distinct codewords of the binary block-code \( V \) are as follows (see Figure 2):

\[
w_0 = 111, w_1 = w_4 = w_5 = 000, w_2 = 011, \text{ and } w_3 = w_6 = 001.
\]

![Figure 2. (V, \leq).](image)

From Figures 1 and 2, we conclude that \((A, \leq)\) is not isomorphic to \((V, \leq)\).
The following example will lead to the next important theorem.

**Example 3.3.** Let \( A = \{0, 1, 2, 3\} \) be a UP-algebra with a fixed element 0 and a binary operation \( \cdot \) defined by the following Cayley table, as Figure 3:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

![Figure 3. (A, ≤).](image-url)

Let \( \tilde{A} : A \to A \) be the identity UP-valued function on \( A \). Then all cut subsets of \( \tilde{X} \) are as follows:

\[
\begin{align*}
A_0 &= A, \\
A_1 &= \{1, 2\}, \\
A_2 &= \{2\}, \\
A_3 &= \{3\}.
\end{align*}
\]

Thus all cut functions of \( \tilde{A} \) are as follows:

<table>
<thead>
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<th></th>
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<td>0</td>
</tr>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
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<td>( \tilde{A}_3 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

and the equivalence relation \( \Theta \) on \( A \) is as follows:

\[
\Theta = \{(0, 0), (1, 1), (2, 2), (3, 3)\}.
\]

Hence, all distinct codewords of the binary block-code \( V \) are as follows (see Figure 4):

\[
w_0 = 1111, w_1 = 0110, w_2 = 0010, \text{ and } w_3 = 0001.
\]

![Figure 4. (V, ≤).](image-url)
From Figures 3 and 4, we conclude that \((A, \leq)\) is isomorphic to \((V, \leq)\) under the isomorphism sending \(a \mapsto w_a\). In addition, the error pattern \(e = 1000\) can be detected because \(w_0 + e = 1111 + 1000 = 0111 \notin V, w_1 + e = 0110 + 1000 = 1110 \notin V, w_2 + e = 0010 + 1000 = 1010 \notin V, \) and \(w_3 + e = 0001 + 1000 = 1001 \notin V\). Hence, \(V\) detects \(e\).

**Theorem 3.4.** Every finite UP-algebra \(A\) which is equipotent to a nonempty set \(X\) determines a binary block-code \(V\) such that \((A, \leq)\) is isomorphic to \((V, \leq)\).

**Proof.** Let \(A = \{0, 1, 2, \ldots, n\}\) be a finite UP-algebra in which 0 is the maximum element, \(X = \{x_0, x_1, x_2, \ldots, x_n\}\) and let \(\bar{X}: X \to A\) be a bijective UP-valued function on \(X\) sending \(x_a \mapsto a\). By (2.20) of Proposition 2.14 and (2.18), we have

\[
\text{(for all } a \in A)(a)\circ a = \{b \in A \mid (a) = (b)\} = \{a\}.\tag{3.6}
\]

Thus \(\Theta = \{(a, a) \mid a \in A\}\). By (3.4), we have all codewords \(w_a\) of the binary block-code \(V\) are distinct. Let \(f: A \to V\) be a function defined by:

\[
\text{(for all } a \in A)(f(a) = w_a).\]

Clearly, \(f\) is surjective. By (3.4) and (3.6), we have \(f\) is injective. Thus \(f\) is bijective. Let \(a, b \in A\) be such that \(a \leq b\). By (2.8) of Proposition 2.5, we have \(X_a \subseteq X_b\). This means that \(w_a \leq w_b\), that is, \(f(a) \leq f(b)\). Conversely, let \(a, b \in A\) be such that \(f(a) \leq f(b)\). Then \(w_a \leq w_b\), so \(X_a \subseteq X_b\). By (2.5), we have \(x_a \in X_{\bar{X}(x_a)} = X_a \subseteq X_b\), that is, \(a = \bar{X}(x_a) \leq b\). Hence, \((A, \leq)\) is isomorphic to \((V, \leq)\). \(\square\)

**Corollary 3.5.** Every finite UP-algebra \(A\) determines a binary block-code \(V\) such that \((A, \leq)\) is isomorphic to \((V, \leq)\).

**Corollary 3.6.** Every finite UP-algebra \(A\) which has the order less than or equal to the order of a finite set \(X\) determines a binary block-code \(V\) such that \((A, \leq)\) is isomorphic to \((V, \leq)\).

**Proof.** Let \(A = \{0, 1, 2, \ldots, n\}\) be a finite UP-algebra in which 0 is the maximum element, \(X = \{x_0, x_1, x_2, \ldots, x_m\}\) for \(m \geq n\) and let \(\bar{X}: X \to A\) be a UP-valued function on \(X\) defined by:

\[
\bar{X} = \begin{pmatrix} x_0 & x_1 & x_2 & \ldots & x_n & x_{n+1} & x_{n+2} & \ldots & x_m \\ 0 & 1 & 2 & \ldots & n & n & n & \ldots & n \end{pmatrix}.
\]

The proof is also given in a similar way of the proof of Theorem 3.4. Hence, \((A, \leq)\) is isomorphic to \((V, \leq)\). \(\square\)

It is not necessary for \((A, \leq)\) and \((V, \leq)\) to be isomorphic under the identity UP-valued function on \(A\), which shown by the following example.

**Example 3.7.** Let \(A = \{0, 1, 2, 3, 4, 5, 6, 7\}\) be a UP-algebra with a fixed element 0 and a binary
operation defined by the following Cayley table, as Figure 5:

\[
\begin{array}{cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\tilde{A}(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{A}(1) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\tilde{A}(2) & 0 & 4 & 6 & 7 & 4 & 0 & 6 & 7 \\
\tilde{A}(3) & 0 & 3 & 5 & 3 & 7 & 5 & 0 & 7 \\
\tilde{A}(4) & 0 & 5 & 5 & 5 & 0 & 5 & 0 & 0 \\
\tilde{A}(5) & 0 & 6 & 6 & 0 & 6 & 0 & 6 & 0 \\
\tilde{A}(6) & 0 & 7 & 0 & 7 & 7 & 0 & 0 & 7 \\
\tilde{A}(7) & 0 & 2 & 5 & 6 & 5 & 6 & 0 & 0 \\
\end{array}
\]

Figure 5. \((A, \leq)\).

Let \(\tilde{A}: A \rightarrow A\) be a UP-valued function on \(A\) defined by:

\[
\tilde{A} = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 7 & 6 & 5 & 4 & 3 & 2
\end{pmatrix}.
\]

Then all cut subsets of \(\tilde{A}\) are as follows:

\(A_0 = A, A_1 = \{0\}, A_2 = \{0, 7\}, A_3 = \{0, 6\}, A_4 = \{0, 5\}, A_5 = \{0, 4, 6, 7\}, A_6 = \{0, 3, 5, 7\}, \) and \(A_7 = \{0, 2, 5, 6\}\).

Thus all cut functions of \(\tilde{A}\) are as follows:

\[
\begin{array}{cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\tilde{A}_0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\tilde{A}_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{A}_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\tilde{A}_3 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\tilde{A}_4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\tilde{A}_5 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
\tilde{A}_6 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\tilde{A}_7 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\end{array}
\]

and the equivalence relation $\Theta$ on $A$ is as follows:

$$\Theta = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)\}.$$ 

Hence, all distinct codewords of the binary block-code $V$ are as follows (see Figure 6):

$$w_0 = 11111111, w_1 = 10000000, w_2 = 10000001, w_3 = 10000010, w_4 = 10001000, w_5 = 10001011, w_6 = 10010101, \text{ and } w_7 = 10100110.$$ 

\[\text{Figure 6. } (V, \leq).\]

From Figures 7 and 8, we conclude that $(A, \leq)$ is isomorphic to $(V, \leq)$ under the isomorphism sending $a \mapsto w_a$.

The following last example supports Corollary 3.8.

**Example 3.8.** Let $A = \{0, 1, 2\}$ be a UP-algebra with a fixed element 0 and a binary operation $\cdot$ defined by the following Cayley table, as Figure 7:

$$\begin{array}{c|ccc}
\cdot & 0 & 1 & 2 \\
\hline
\bar{X}(u) & 0 & 1 & 2 \\
\bar{X}(v) & 0 & 0 & 2 \\
\bar{X}(w) & 0 & 0 & 3 \\
\bar{X}(x) & 0 & 1 & 3 \\
\bar{X}(y) & 0 & 1 & 0 \\
\bar{X}(z) & 0 & 1 & 2 \\
\end{array}$$

\[\text{Figure 7. } (A, \leq).\]

Let $X = \{u, v, w, x, y, z\}$ and we define a UP-valued function $\tilde{X}: X \rightarrow A$ on $X$ by:

$$\tilde{X} = \left( \begin{array}{ccccccc}
u & v & w & x & y & z \\
0 & 1 & 2 & 2 & 3 & 3 \\
\end{array} \right).$$
Then all cut subsets of $\widetilde{X}$ are as follows:

\[ X_0 = X, X_1 = \{v\}, X_2 = \{w, x\}, \text{ and } X_3 = \{y, z\}. \]

Thus all cut functions of $\widetilde{X}$ are as follows:

<table>
<thead>
<tr>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\widetilde{X}_0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\widetilde{X}_1$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\widetilde{X}_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\widetilde{X}_3$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

and the equivalence relation $\Theta$ on $A$ is as follows:

\[ \Theta = \{(0, 0), (1, 1), (2, 2), (3, 3)\}. \]

Hence, all distinct codewords of the binary block-code $V$ are as follows (see Figure 8):

\[ w_0 = 111111, w_1 = 010000, w_2 = 001100, \text{ and } w_3 = 000011. \]

From Figures 7 and 8, we conclude that $(A, \leq)$ is isomorphic to $(V, \preceq)$ under the isomorphism sending $a \mapsto w_a$. In addition, $V$ has the minimum distance 3. This means that can correct at most 1-error. For example, if $w_3 = 000011$ is sent and 000111 is received, then 000111 will be decoded to $w_3 = 000011$. If $w_3 = 000011$ is sent and 010111 is received, then 010111 will be decoded to $w_1 = 010000$ using the minimum distance decoding rule.

4. Conclusions

Codewords in a binary block-code generated by a UP-valued function are established and some interesting results are obtained. The main result is proved that every finite UP-algebra $A$ which has the order less than or equal to the order of a finite set $X$ determines a binary block-code $V$ such that $(A, \leq)$ is isomorphic to $(V, \preceq)$. Many examples were provided to support the results.

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Conflict of interest

The authors declare no conflict of interest.

References


